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A RATIONALITY CONDITION FOR THE EXISTENCE OF ODD PERFECT NUMBERS

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A rationality condition for the existence of odd perfect numbers is used to derive an upper bound for the density of odd integers such that $\sigma(N)$ could be equal to $2N$, where N belongs to a fixed interval with a lower limit greater than 10^{300}. The rationality of the square root expression consisting of a product of repunits multiplied by twice the base of one of the repunits depends on the characteristics of the prime divisors, and it is shown that the arithmetic primitive factors of the repunits with different prime bases can be equal only when the exponents are different, with possible exceptions derived from solutions of a prime equation. This equation is one example of a more general prime equation, $(q_j^n - 1)/(q_i^n - 1) = p^h$, and the demonstration of the nonexistence of solutions when $h \geq 2$ requires the proof of a special case of Catalan's conjecture. General theorems on the nonexistence of prime divisors satisfying the rationality condition and odd perfect numbers N subject to a condition on the repunits in factorization of $\sigma(N)$ are proven.

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1. Introduction. The algorithm for demonstrating the nonexistence of odd perfect numbers with fewer than nine different prime divisors requires the expansion of the ratio $\sigma(N)/N$ and strict inequalities imposed on the sums of powers of the reciprocal of each prime divisor [20, 55]. Although it is possible to establish that $\sigma(N)/N \neq 2$ when N is divisible by certain primes, there are odd integers with a given number of prime divisors such that $\sigma(N)/N > 2$, while $\sigma(N)/N < 2$ for other integers with the same number of distinct prime factors. Moreover, the range of the inequality for $|\sigma(N)/N - 2|$ can be made very small even when N has a few prime factors. Examples of odd integers with only five distinct prime factors have been found, which produce a ratio nearly equal to 2: $|\sigma(N)/N - 2| < 10^{-12}$ [28]. Since it becomes progressively more difficult to establish the inequalities as the number of prime factors increases, a proof by method of induction based on this algorithm cannot be easily constructed.

In Section 2, it is shown that there is a rationality condition for the existence of odd perfect numbers. Setting $\sigma(N)/N$ equal to 2 is equivalent to equating the square root of a product, $2(4k+1)\prod_{i=1}^{\ell}((q_i^{n_i} - 1)/(q_i - 1))(((4k+1)^{4m+2} - 1)/4k)$, which contains a sequence of repunits, with a rational number. This relation provides both an upper bound for the density of odd perfect numbers in any fixed interval in $\mathbb{N}$ with a lower limit greater than 10^{300} and a direct analytical method for verifying their nonexistence, since it is based on the irrationality of the square root of any unmatched prime divisors in the product. This condition is used in Section 3 to demonstrate the nonexistence of a special category of odd perfect numbers. The properties of prime divisors of Lucas sequences required for the study of the square root of the product of the repunits are described in Sections 4 and 5. An induction argument is constructed in Section 6, which proves that the square root expression is not rational for generic sets of prime divisors, each containing a large number of elements. This is first established for odd integers with four distinct prime divisors and then by induction using the properties of the divisors of the repunits.

2. Rationality condition for the existence of odd perfect numbers. Since the nonexistence of odd perfect numbers implies that all integers of the type $N = (4k+1)^{4m+1}s^2 = (4k+1)^{4m+1}q_1^{2\alpha_1}\cdots q_\ell^{2\alpha_\ell}$, $\gcd(4k+1,s) = 1$, $\alpha_i \geq 1$, $k,m \geq 1$, q_i prime, $q_i \geq 3$ [15, 16, 56] will have the property

$$\begin{aligned}\frac{\sigma(N)}{N} &= \left[\frac{(4k+1)^{4m+2}-1}{4k(4k+1)^{4m+1}}\right]\frac{\sigma(s^2)}{s^2}\\ &= \left[\frac{(4k+1)^{4m+2}-1}{4k(4k+1)^{4m+1}}\right]\left[\frac{\sigma(s)^2}{s^2}\right]\left[\frac{\sigma(s^2)}{\sigma(s)^2}\right]\\ &\neq 2,\end{aligned} \tag{2.1}$$

it follows that

$$\frac{\sigma(s)}{s} \neq \sqrt{2}\prod_{i=1}^{\ell}\frac{(q_i^{\alpha+1}-1)}{(q_i-1)^{1/2}(q_i^{2\alpha_i+1}-1)^{1/2}}\left[\frac{4k(4k+1)^{4m+1}}{(4k+1)^{4m+2}-1}\right]^{1/2}, \tag{2.2}$$

$$\begin{aligned}\prod_{i=1}^{\ell}\frac{1}{q_i^{\alpha_i+1}-1}\frac{\sigma(s)}{s} &\neq \sqrt{2}\prod_{i=1}^{\ell}\frac{1}{(q_i-1)^{1/2}}\frac{1}{(q_i^{2\alpha_i+1}-1)^{1/2}}\\ &\quad\cdot\left[\frac{4k(4k+1)^{4m+1}}{(4k+1)^{4m+2}-1}\right]^{1/2}.\end{aligned} \tag{2.3}$$

Irrationality of the entire square root expression for all sets of primes $\{q_i;\ 4k+1 \mid q_i \geq 3,\ k \geq 1,\ q_i \neq 4k+1\}$ is therefore a sufficient condition for the proof of the nonexistence of odd perfect numbers.

The known integer solutions to $(x^n-1)/(x-1) = y^2$ [32, 38, 39, 42] do not include the pairs $(x,n) = (4k+1, 4m+2)$, implying that $[((4k+1)^{4m+2} - 1)/4k]^{1/2}$ is not a rational number. The number $[1+q_i+q_i^2+\cdots+q_i^{2\alpha_i}]^{1/2}$ is only rational when $q_i = 3$, $\alpha_i = 2$, so that if 3 is a prime factor of s, then

$$\begin{aligned}
&\prod_{i=1}^{\ell} \frac{1}{(q_i^{2\alpha_i+1}-1)^{1/2}} \frac{1}{(q_i-1)^{1/2}} \\
&\quad = \prod_{i=1}^{\ell} \frac{1}{(q_i^{2\alpha_i+1}-1)} \left[1+q_i+q_i^2+\cdots+q_i^{2\alpha_i}\right]^{1/2} \\
&\quad = \left(\frac{11}{242}\right)^{\delta_{q_i,3}\delta_{\alpha_i,2}} \times \prod_{(q_i,2\alpha_i+1)\neq(3,5)} \frac{1}{(q_i^{2\alpha_i+1}-1)} \\
&\qquad\qquad \times \left[1+q_i+q_i^2+\cdots+q_i^{2\alpha_i}\right]^{1/2}.
\end{aligned} \tag{2.4}$$

From (2.3), the nonexistence of odd perfect numbers can be deduced only if

$$\begin{aligned}
&2(4k+1)\prod_i \left(1+q_i+q_i^2+\cdots+q_i^{2\alpha_i}\right) \\
&\quad \cdot \left(1+(4k+1)+(4k+1)^2+\cdots+(4k+1)^{(4m+1)}\right)
\end{aligned} \tag{2.5}$$

is not the square of an integer, with $q_i \neq 3$ or $\alpha_i \neq 2$. This condition also can be deduced directly from the form of the integer N and the multiplicative property of $\sigma(n)$ as $\sigma(N) \neq 2N$ if

$$\begin{aligned}
&\sqrt{2(4k+1)}\left[\sigma\left((4k+1)^{4m+1}\right)\prod_{i=1}^{\ell}\sigma\left(q_i^{2\alpha_i}\right)\right]^{1/2} \\
&\quad \neq \sqrt{2(4k+1)}\left[2\left((4k+1)^{4m+1}\right)\prod_{i=1}^{\ell} q_i^{2\alpha_i}\right]^{1/2} \\
&\quad = 2(4k+1)^{2m+1}\prod_{i=1}^{\ell} q_i^{\alpha_i}.
\end{aligned} \tag{2.6}$$

As the repunit $(x^n-1)/(x-1)$ is the Lucas sequence derived from a second-order recurrence relation

$$\begin{aligned}
U_{n+2}(a,b) &= aU_{n+1}(a,b) - bU_n(a,b), \\
U_n(a,b) &= \frac{\alpha^n-\beta^n}{\alpha-\beta}
\end{aligned} \tag{2.7}$$

with $\alpha = x$, $\beta = 1$, $a = \alpha+\beta = x+1$, and $b = \alpha\beta = x$, the rationality condition is being applied to the product $[2(4k+1)\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1,q_i) \cdot U_{4m+2}(4k+2,4k+1)]^{1/2}$.

The number of square-full integers up to N is $N^{1/2} - (3/2)N^{-1} + O(N^{-3/2})$. With a lower bound of 10^{300} for an odd perfect number [4], it follows that $2(4k+1) \cdot \prod_{i=1}^{\ell}(q_i^{2\alpha_i} + O(q_i^{2\alpha_i-1})) \cdot ((4k+1)^{4m+1} + O((4k+1)^{4m})) > 10^{301}$. Given a lower bound of 10^6 for the largest prime factor [22], 10^4 for the second largest prime factor, and 10^2 for the third largest prime factor of N [25, 26], the density of prime products $(4k+1)\prod_{i=1}^{\ell} q_i$, given by $\prod_{i=1}^{\ell}(1/\ln q_i)\times(1/\ln(4k+1))$, is bounded above by 8.032×10^{-5} when there are eight different prime factors [20] and 1.004×10^{-6} when there are eleven different prime factors not including 3 [21, 29]. Given that the probability of an integer being a square is independent of it being expressible in terms of a product of repunits, the density of square-full numbers having the form $2(4k+1)\sigma((4k+1)^{4m+1}\prod_i q_i^{2\alpha_i})$ in the interval $[N^*, N^*+N_0]$, where $N^* > 10^{301}$ and N_0 is a fixed number, is bounded above by 3.28×10^{-159} when there are at least eight different prime factors and 5.13×10^{-163} when N is relatively prime to 3 and has more than ten different prime factors.

3. Proof of the nonexistence of odd perfect numbers for a special class of integers. The even repunit $((4k+1)^{4m+2}-1)/4k$ contains only a single power of 2 since $1+(4k+1)+(4k+1)^2+\cdots+(4k+1)^{4m+1}\equiv 4m+2\equiv 2(\operatorname{mod} 4)$. Thus, the rationality condition can be applied to a product of odd numbers $[(4k+1)\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1,q_i)(1/2)U_{4m+2}(4k+2,4k+1)]^{1/2}$. Suppose

$$\begin{gathered}\prod_{i=1}^{\ell}\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\cdot\left[\frac{8k(4k+1)}{(4k+1)^{4m+2}-1}\right]=\frac{r^2}{t^2}\\ \prod_{i=1}^{\ell}\frac{q_i^{2\alpha_i+1}-1}{q_i-1}(4k+1)t^2=\frac{(4k+1)^{4m+2}-1}{8k}r^2\end{gathered}\tag{3.1}$$

with $\gcd(r,t)=1$. If $\gcd(((4k+1)^{4m+2}-1)/8k,(q_i^{2\alpha_i+1}-1)/(q_i-1))=1$ for all i, the relation (3.1) requires $((4k+1)^{4m+2}-1)/8k\mid t^2$ or equivalently $((4k+1)^{4m+2}-1)/8k=\sigma_\ell\tau_\ell^2$ where $\sigma_\ell\tau_\ell\mid t$. The substitution $t=\sigma_\ell\tau_\ell u$ gives

$$\begin{gathered}(4k+1)\prod_{i=1}^{\ell}\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\cdot(\sigma_\ell\tau_\ell u)^2=\sigma_\ell\tau_\ell^2\cdot r^2\\ (4k+1)\prod_{i=1}^{\ell}\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\sigma_l u^2=r^2\end{gathered}\tag{3.2}$$

which, in turn, requires that $(4k+1)\prod_{i=1}^{\ell}((q_i^{2\alpha_i+1}-1)/(q_i-1))=\sigma_l v^2$ and $r=\sigma_l v u$, so that $\sigma_l u\mid r$ and $\sigma_l u\mid t$, contrary to the original assumption that r and t are relatively prime unless $\sigma_\ell=u=1$. The rationality condition reduces to the existence of solutions to the equation

$$\frac{x^n-1}{x-1}=2y^2,\quad x\equiv 1(\operatorname{mod} 4),\ n\equiv 2(\operatorname{mod} 4).\tag{3.3}$$

This relation is equivalent to the two conditions $(x^{2m+1}-1)/(x-1)=y_1^2$ and $(x^{2m+1}+1)/2=y_2^2$, $y=y_1y_2$, $(y_1,y_2)=1$ since $\gcd(x^{2m+1}-1,x^{2m+1}+1)=2$. It can be verified that there are no integer solutions to these simultaneous Diophantine equations, implying that when $((4k+1)^{4m+2}-1)/8k$ satisfies the gcd condition given above, the square root of $[(4k+1)\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1,q_i)(1/2)U_{4m+2}(4k+2,4k+1)]$ is not a rational number and there is no odd perfect number of the form with this constraint on the pair $(4k+1,4m+2)$.

4. Lucas terms with index 3 and the matching of prime divisors. When the index is 3, generally, $(x^3-1)/(x-1)$ will be a multiple of the square of an integer. Since the solution to $x^2+x+1=y^2/a$ is

$$x=\frac{-1\pm\sqrt{4y^2/a-3}}{2}, \tag{4.1}$$

it will be an integer only if

$$y=\frac{\sqrt{a(z^2+3)}}{2}, \quad z\in\mathbb{Z} \tag{4.2}$$

is an integer. If $z>1$, $(z+1)^2-z^2=2z+1>3$ and $\sqrt{z^2+3}$ is not rational, confirming that there are no integer solutions to the original equation when $a=1$, except when $x=0$ or $x=-1$. Integer solutions to (4.2) are determined by solutions to the quadratic equation

$$z^2-Dr^2=-3. \tag{4.3}$$

This equation has been investigated using the continued fraction expansion of $\sqrt{D}$, and ordering the integer solutions of this equation by the magnitude of $z+r\sqrt{D}$, the fundamental solutions, given by the smallest value of this expression, will be denoted by the pair of integers (z_1,r_1). For any solution (x,y) of the Pell equation $x^2-Dy^2=1$, an infinite number of solutions of (4.3) are generated by the identity

$$(z_1+r_1\sqrt{D})(x+y\sqrt{D})=z_1x+r_1yD+(z_1y+r_1x)\sqrt{D} \tag{4.4}$$

as the pairs of integers $\{(z_1x+r_1yD,z_1y+r_1x)\mid x^2-Dy^2=1\}$ define a class of solutions to (4.3). If D is a multiple of 3 but not a perfect square, there is one class of solutions, whereas, if D is not a multiple of 3, then there may be one or two classes of solutions [34].

Given any two solutions to (4.3), (z_1,r_1) and (z_2,r_2), it follows that

$$\frac{x_1^3-1}{x_1-1}\frac{x_2^3-1}{x_2-1}=\frac{(z_1^2+3)}{4}\cdot\frac{(z_2^2+3)}{4}=\frac{Dr_1^2}{4}\cdot\frac{Dr_2^2}{4}. \tag{4.5}$$

Although the repunits $(x^3-1)/(x-1)$ are not perfect squares, the extra factors may be matched in a product of quotients of this type.

A table of the square-free factors of repunits with exponent 3 and prime basis reveals that only a selected set of coefficients occur so that the elimination of unmatched prime divisors becomes more problematical. However, consider the following choices for the primes $4k+1$ and q_i and the exponents $4m+1$ and $2\alpha_i$ and the product of the prime powers:

$$\begin{aligned} 4k+1 &= 37, & 4m+1 &= 5, \\ q_1 &= 3, & 2\alpha_1 &= 2, \\ q_2 &= 5, & 2\alpha_2 &= 2, \\ q_3 &= 29, & 2\alpha_3 &= 2, \\ q_4 &= 79, & 2\alpha_4 &= 2, \\ q_5 &= 83, & 2\alpha_5 &= 2, \\ q_6 &= 137, & 2\alpha_6 &= 2, \\ q_7 &= 283, & 2\alpha_7 &= 2, \\ q_8 &= 313, & 2\alpha_8 &= 2, \end{aligned} \tag{4.6}$$

$$(4k+1)^{4m+1}\prod_{i=1}^{8} q_i^{2\alpha_i} = 37^5\cdot 3^2\cdot 5^2\cdot 29^2\cdot 79^2\cdot 83^2\cdot 137^2\cdot 283^2\cdot 313^2,$$

the sum of divisors functions of the following prime power factors are obtained:

$$\begin{aligned} \sigma\left((4k+1)^{4m+1}\right) &= \frac{(4k+1)^{4m+2}-1}{4k} = \frac{(37)^6-1}{36} \\ &= 71270178 = 2\cdot 3\cdot 7\cdot 19\cdot 31\cdot 43\cdot 67, \\ \sigma\left(q_1^{2\alpha_1}\right) &= \frac{q_1^{2\alpha_1+1}-1}{q_1-1} = \frac{3^3-1}{2} = 13, \\ \sigma\left(q_2^{2\alpha_2}\right) &= \frac{q_2^{2\alpha_2+1}-1}{q_2-1} = \frac{5^3-1}{4} = 31, \\ \sigma\left(q_3^{2\alpha_3}\right) &= \frac{q_3^{2\alpha_3+1}-1}{q_3-1} = \frac{29^3-1}{28} = 871 = 13\cdot 67, \\ \sigma\left(q_4^{2\alpha_4}\right) &= \frac{q_4^{2\alpha_4+1}-1}{q_4-1} = \frac{79^3-1}{78} = 6321 = 3\cdot 7^2\cdot 43, \\ \sigma\left(q_5^{2\alpha_5}\right) &= \frac{q_5^{2\alpha_5+1}-1}{q_5-1} = \frac{83^3-1}{82} = 6973 = 19\cdot 367, \\ \sigma\left(q_6^{2\alpha_6}\right) &= \frac{q_6^{2\alpha_6+1}-1}{q_6-1} = \frac{137^3-1}{136} = 18907 = 7\cdot 37\cdot 73, \\ \sigma\left(q_7^{2\alpha_7}\right) &= \frac{q_7^{2\alpha_7+1}-1}{q_7-1} = \frac{283^3-1}{282} = 80373 = 3\cdot 73\cdot 367, \\ \sigma\left(q_8^{2\alpha_8}\right) &= \frac{q_8^{2\alpha_8+1}-1}{q_8-1} = \frac{313^3-1}{311} = 98283 = 3\cdot 181^2, \end{aligned} \tag{4.7}$$

so that the prime divisors match in the rationality condition. However, for this integer, $\sigma(N)/N \neq 2$, a result which is consistent with the nonexistence of odd perfect numbers with $2\alpha_i+1 \equiv 0 (\bmod 3)$, $i = 1,\dots,\ell$ [33].

Other sets of primes include $\{q_i\} = \{3,29,67,79,83,137,283\}$ and $\{q_i\} = \{3,7,11,29,79,83,137,191,283\}$ with $4k+1 = 37$. The odd integers formed from the products of these powers of primes

$$37^5 \cdot 3^2 \cdot 29^2 \cdot 67^2 \cdot 79^2 \cdot 83^2 \cdot 137^2 \cdot 283^2$$
$$37^5 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 29^2 \cdot 79^2 \cdot 83^2 \cdot 137^2 \cdot 191^2 \cdot 283^2 \tag{4.8}$$

also do not satisfy the constraint $\sigma(N)/N = 2$. The decomposition of the repunit $((4k+1)^{4m+2}-1)/4k$ for different choices of $4k+1$ includes factors which cannot be matched without introducing successively higher prime divisors. For example, the prime factors $\{157;307;271;547,1723;409,919;523\}$ occur in the decomposition of the repunits with prime bases $\{13;17;29;41;53;61\}$ and exponent 6. Matching these divisors requires repunits with sufficiently large bases or exponents, and these new terms will generally contain significantly greater prime factors. Since the integers in (4.6) and (4.8) did not satisfy the condition $\sigma(N)/N = 2$ and the rationality condition is satisfied by selected sets of primes only, the results provide support for the nonexistence of odd perfect numbers for large categories of prime divisors and exponents, which will be established in Theorem 7.1.

5. Prime power divisors of Lucas sequences and Catalan's conjecture. The number of distinct prime divisors of $(q^n-1)/(q-1)$ is bounded below by $\tau(n)-1$ if $q > 2$, where $\tau(n)$ is the number of natural divisors of n [44, 55]. The characteristics of these prime divisors can be deduced from the properties of Lucas sequences. Since the repunits $(q_i^{2\alpha_i+1}-1)/(q_i-1)$ have only odd prime divisors, the proofs in the following sections will have general validity, circumventing any exceptions corresponding to the prime $q = 2$.

For a primary recurrence relation, defined by the initial values $U_0 = 0$ and $U_1 = 1$, denoting the least positive integer n such that $U_n(a,b) \equiv 0 (\operatorname{mod} p)$, the rank of apparition, by $\alpha(a,b,p)$, it is known that $\alpha(x+1,x,p) = \operatorname{ord}_p(x)$ [49].

The extent to which the arguments a and b determine the divisibility of $U_n(a,b)$ [23, 31] can be summarized as follows.

Let p be an odd prime.

(i) If $p \mid a$, $p \mid b$, then $p \mid U_n(a,b)$ for all $n > 1$.

(ii) If $p \nmid a$, $p \mid b$, then either $p \mid U_n(a,b)$, $n \geq 1$ or $p \nmid U_n(a,b)$ for any $n \geq 1$.

(iii) If $p \mid a$ and $p \nmid b$, then $p \mid U_n(a,b)$ for all even n or all odd n or $p \nmid U_n(a,b)$ for any $n \geq 1$.

(iv) If $p \nmid a$, $p \nmid b$, $p \mid D = a^2 - 4b$, then $p \mid U_n(a,b)$ when $p \mid n$.

(v) If $p \nmid abD$, then $p \mid U_{p-(D/p)}(a,b)$.

For the Lucas sequence $U_n(q+1,q)$, there is no prime which divides both q and $q+1$, and since only q is a divisor of the second parameter, there are no prime divisors of $U_n(q+1,q)$ from this category because $(q^n-1)/(q-1) \equiv 1(\bmod q)$. If $p \mid (q+1)$, then $(q^n-1)/(q-1) \equiv (1-(-1)^n)/2 \equiv 0(\bmod p)$ when n is even. However, $p \nmid (q^n-1)/(q-1)$ with n odd, and therefore, prime divisors from this class are not relevant for the study of the product of repunits with odd exponents.

When $a = q+1$ and $b = q$, $D = (q-1)^2$ and if $p \mid (q-1)$, then $p \mid U_n(q+1,q)$ when $p \mid n$. However, $p^2 \nmid (q^p-1)/(q-1)$, and under this condition, $p^2 \nmid (q^n-1)/(q-1)$ unless $n = Cp^2$. More generally, denoting the power of p which exactly divides a by $p^{v_p(a)}$, it can be deduced that $v_p((q^n-1)/(q-1)) = v_p(n)$ if $p \mid (q-1)$ and $\alpha(q+1,q,p) = p$ [42, 44].

From the last property, it follows that $\alpha(a,b,p) \mid (p-(D/p))$ when $p \nmid (q-1)$, $(D/p) = 1$, and $\alpha(q+1,q,p) \mid (p-1)$. If $p^2 \nmid (q^{p-1}-1)/(q-1)$, then $\alpha(q+1,q,p^2) = p\alpha(q+1,q,p)$ so that $\alpha(q+1,q,p^2) \mid p(p-1)$. If $p^2 \mid (q^{p-1}-1)/(q-1)$, $\alpha(q+1,q,p^2) = \alpha(q+1,q,p) \mid p-1$ [5, 57]. Thus a repunit with primitive divisor p is also divisible by p^2 if $Q_q \equiv 0(\bmod p)$ where $Q_a = (a^{p-1}-1)/p$ is the Fermat quotient.

Since $q^n-1 = \prod_{d\mid n} \Phi_d(q)$ where $\Phi_n(q)$ is the nth cyclotomic polynomial, it can be shown that the largest arithmetic primitive factor [3, 10, 54] of q^n-1 when $q \geq 2$ and $n \geq 3$ is

$$\begin{aligned} &\Phi_n(q) \quad \text{if } \Phi_n(q) \text{ and } n \text{ are relatively prime,} \\ &\frac{\Phi_n(q)}{p} \quad \text{if a common prime factor } p \text{ of } \Phi_n(q) \text{ and } n \text{ exists.} \end{aligned} \tag{5.1}$$

In the latter case, if $n = p^f p'^{f'} p''^{f''} \cdots$ is the prime factorization of n, then $\Phi_n(q)$ is divisible by p if and only if $e = n/p^f = \mathrm{ord}_p(q)$ when $p \nmid (q-1)$, and moreover, $p \| \Phi_{ep^f}(q)$ when $f > 0$ [55].

Division by $q-1$ does not alter the arithmetic primitive factor, since it is the product of the primitive divisors of q^n-1, which are also the primitive divisors of $(q^n-1)/(q-1)$. For all primitive divisors, $p' \nmid (q-1)$, so that $(p')^h \mid (q^n-1)/(q-1)$ if $(p')^h \mid q^n-1$ and the arithmetic primitive factor again would include $(p')^h$. The imprimitive divisors would be similarly unaffected because the form of the index $n = ep^f$ prevents $q-1$ from being a divisor of $\Phi_n(q)$ when $p \nmid (q-1)$. If $p \mid (q-1)$, the rank of apparition for the Lucas sequence $\{U_n(q+1,q)\}$ is p, so that it is consistent to set $n = p^{f+1}$. Then, $p \| \Phi_{p^{f+1}}(q)$ and the arithmetic primitive factor is $\Phi_{p^{f+1}}(q)/p$.

If $(q_i-1) \nmid \Phi_{n_i}(q_i)$, the product of the arithmetic primitive factors of each repunit $(q_i^{n_i}-1)/(q_i-1)$ and $((4k+1)^{4m+2}-1)/4k$ in expression (2.5) is

$$\frac{\Phi_{n_1}(q_1)}{p_1}\frac{\Phi_{n_2}(q_2)}{p_2}\cdots\frac{\Phi_{n_\ell}(q_\ell)}{p_\ell} \times \left[\frac{\Phi_{4m+2}(4k+1)}{p_{\ell+1}}\right], \tag{5.2}$$

where the indices are odd numbers $n_i = 2\alpha_i+1$, p_i, $i = 1,\ldots,l$, represents the common factor of n_i and $\Phi_{n_i}(q_i)$, and $p_{\ell+1}$ is a common factor of $4m+2$ and $\Phi_{4m+2}(4k+1)$. Division of $\Phi_{n_i}(q_i)$ by the prime p_i is necessary only when

$\gcd(n_i, \Phi_{n_i}(q_i)) \neq 1$, and $p_i = P(n_i/\gcd(3, n_i))$, where $P(n)$ represents the largest prime factor of n [46, 51, 52, 53].

Theorem 5.1. *The arithmetic primitive factors of the repunits with different prime bases could be equal only if the exponents are different, with possible exceptions being determined by the solutions to the equation* $(q_j^n - 1)/(q_i^n - 1) = p$, $q_i \neq q_j$ *with* q_i, q_j, *and* p *prime.*

Proof. Consider the following four cases.

(I) The arithmetic primitive factors of $q_i^{n_i} - 1$ and $q_j^{n_j} - 1$ are $\Phi_{n_i}(q_i)$ and $\Phi_{n_j}(q_j)$.

Since $\Phi_n(x)$ is a strictly increasing function for $x \geq 1$ [35, 36], $\Phi_n(q_j) > \Phi_n(q_i)$ when q_j is the larger prime, and equality of $\Phi_{n_i}(q_i)$ and $\Phi_{n_j}(q_j)$ could only be achieved, if at all feasible, when $n_i \neq n_j$.

(II) The arithmetic primitive factors of $q_i^{n_i} - 1$ and $q_j^{n_j} - 1$ are $\Phi_{n_i}(q_i)$ and $\Phi_{n_j}(q_j)/p_j$.

Comparing $\Phi_n(q_i)$ and $\Phi_n(q_j)/p$, $p = p_j$ is a common factor of n and $\Phi_n(q_j)$ but it does not divide $\Phi_n(q_i)$. It follows that the relation $\Phi_n(q_i) = \Phi_n(q_j)/p$ could only hold if $p \| \Phi_n(q_j)$. The prime decomposition of e as $\rho_1 \cdots \rho_s$, $\gcd(\rho_t, p) = 1$, $t = 1, \ldots, s$, leads to the following expressions for $\Phi_n(q_i)$ and $\Phi_n(q_j)$,

$$\begin{aligned}
\Phi_n(q_i) &= \Phi_{ep^f}(q_i) = \frac{\Phi_e\left(q_i^{p^f}\right)}{\Phi_e\left(q_i^{p^{f-1}}\right)} \\
&= \frac{\prod_{\substack{k \text{ even} \\ k \geq 0}} \prod_{\substack{t_k > \cdots > t_1 \\ t_k \leq s}} \left[q_i^{ep^f/\rho_{t_1}\cdots\rho_{t_k}} - 1\right]}{\prod_{\substack{\tilde{k} \text{ odd} \\ \tilde{k} \geq 1}} \prod_{\substack{t_{\tilde{k}} > \cdots > t_1 \\ t_{\tilde{k}} \leq s}} \left[q_i^{ep^f/\rho_{t_1}\cdots\rho_{t_{\tilde{k}}}} - 1\right]} \\
&\quad \cdot \frac{\prod_{\substack{\tilde{k} \text{ odd} \\ \tilde{k} \geq 1}} \prod_{\substack{t_{\tilde{k}} > \cdots > t_1 \\ t_{\tilde{k}} \leq s}} \left[q_i^{ep^{f-1}/\rho_{t_1}\cdots\rho_{t_{\tilde{k}}}} - 1\right]}{\prod_{\substack{k \text{ even} \\ k \geq 1}} \prod_{\substack{t_k > \cdots > t_1 \\ t_k \leq s}} \left[q_i^{ep^{f-1}/\rho_{t_1}\cdots\rho_{t_k}} - 1\right]}, \\
\Phi_n(q_j) &= \Phi_{ep^f}(q_j) = \frac{\Phi_e\left(q_j^{p^f}\right)}{\Phi_e\left(q_j^{p^{f-1}}\right)} \\
&= \frac{\prod_{\substack{k \text{ even} \\ k \geq 0}} \prod_{\substack{t_k > \cdots > t_1 \\ t_k \leq s}} \left[q_j^{ep^f/\rho_{t_1}\cdots\rho_{t_k}} - 1\right]}{\prod_{\substack{\tilde{k} \text{ odd} \\ \tilde{k} \geq 1}} \prod_{\substack{t_{\tilde{k}} > \cdots > t_1 \\ t_{\tilde{k}} \leq s}} \left[q_j^{ep^f/\rho_{t_1}\cdots\rho_{t_{\tilde{k}}}} - 1\right]} \\
&\quad \cdot \frac{\prod_{\substack{\tilde{k} \text{ odd} \\ \tilde{k} \geq 1}} \prod_{\substack{t_{\tilde{k}} > \cdots > t_1 \\ t_{\tilde{k}} \leq s}} \left[q_j^{ep^{f-1}/\rho_{t_1}\cdots\rho_{t_{\tilde{k}}}} - 1\right]}{\prod_{\substack{k \text{ even} \\ k \geq 1}} \prod_{\substack{t_k > \cdots > t_1 \\ t_k \leq s}} \left[q_j^{ep^{f-1}/\rho_{t_1}\cdots\rho_{t_k}} - 1\right]}.
\end{aligned} \tag{5.3}$$

Since $e = \mathrm{ord}_p(q_j)$, it follows that $p \mid (q_j^e - 1)$, and if $q_j^e = 1 + pk_j (\mathrm{mod}\, p)$, then $(q_j^e)^{p^f} = (1+pk_j)^{p^f} \equiv 1 + p^f pk_j \equiv 1(\mathrm{mod}\, p^{f+1})$. Thus, $p^{f+1} \mid (q_j^{ep^f} - 1)$ and $p^f \mid (q_j^{ep^{f-1}} - 1)$, while $p \nmid (q_j^{e/(\rho_{t_1}\cdots\rho_{t_k})p^f} - 1)$. Let $H(f) \geq f+1$ denote the exponent such that $p^{H(f)} \| (q_j^{ep^f} - 1)$. Since $q_j^{ep^{f-1}} \equiv 1(\mathrm{mod}\, p^{H(f-1)})$, $q_j^{ep^f} = (1+p^{H(f-1)}k_j')^p \equiv 1 + p\cdot p^{H(f-1)}k_j' \equiv 1(\mathrm{mod}\, p^{H(f-1)+1})$. Consequently, $H(f) - H(f-1) = 1$, which is consistent with $\Phi_n(q_j)$ being exactly divisible by p.

Although $\Phi_n(q_i)$ and $\Phi_n(q_j)/p$ are not divisible by p, consider a primitive prime factor p' of $\Phi_n(q_i)$. It must divide some factor $q_i^{n/\rho_{t_1}\cdots\rho_{t_k}} - 1$ in the expression for $\Phi_n(q_i)$, and thus, it will also divide $q_i^{n/\rho_{t_1}\cdots\rho_{t_\ell}} - 1$, $\ell < k$. Since the exponent of $q_i^{n/\rho_{t_1}\cdots\rho_{t_\ell}} - 1$ in $\Phi_n(q_i)$ is $(-1)^l$, there will be 2^{k-1} factors in the numerator and 2^{k-1} factors in the denominator divisible by p'. When $k \geq 1$, the factors of p' are exactly canceled because each term $q_i^{n/\rho_{t_1}\cdots\rho_{t_\ell}} - 1$ is divisible by the same power of p'. The exception occurs when $p' \mid q_i^n - 1$ only; if $p_a'^{f_a} \| q_i^n - 1$, then $p_a'^{f_a} \| \Phi_n(q_i)$ [43]. Equivalence of $\Phi_n(q_i)$ and $\Phi_n(q_j)/p$ requires that the prime power divisors of these quantities are equal, so that $p_a'^{f_a} \| \Phi_n(q_j)/p$ for all primes $\{p_a'\}$. However, if $p_a'^{f_a} \| q_j^n - 1$, then $q_i^n - 1$ and $q_j^n - 1$ have the same primitive prime power divisors. The imprimitive prime divisor p which divides $q_j^n - 1$ might also divide $q_i^n - 1$, although overall cancellation of p in $\Phi_n(q_i)$ requires that $p^r \mid q_i^{n/\rho_{t_1}\cdots\rho_{t_k}} - 1$ for some $k \geq 1$ and $p^{r-1} \mid q_i^{n/(p\rho_{t_1}\cdots\rho_{t_k})} - 1$. When $p^r \| q_i^n - 1$ and $(q_j^n - 1)/(q_i^n - 1) = p^{H(f)-r}$

$$
\begin{aligned}
q_i^n - 1 &= \kappa u_1^{H(f)-r}, \\
q_j^n - 1 &= \kappa u_2^{H(f)-r}, \\
\frac{u_2}{u_1} &= p.
\end{aligned}
\tag{5.4}
$$

Integer solutions of $w = y^m$, $y \geq 2$, $m \geq 2$ can be written as $w = x^n$, $x \geq 2$ with $m \mid n$. Since $y \mid x^n$, $y \nmid (x^n - 1)$ because $y \geq 2$. The nearest integers to x^n having a similar form, $\{(x-1)^n, (x+1)^n, (x+1)^{n-1}, (x-1)^{n+1}\}$ do not provide a counterexample to the conclusion since none of them are divisible by y. Furthermore, $x^n - (x-1)^n > 1$, $(x+1)^n - x^n > 1$, $|(x+1)^{n-1} - x^n| > 1$, $x \geq 2$, $n \geq 4$; $x \geq 3$, $n \geq 3$ and $|x^n - (x-1)^{n+1}| > 1$, $x \geq 2$, $n \geq 3$ so that none of these integers will have the form $y^m \pm 1$. The exception occurring when $x = y = 2$, $m = n = 3$ is the statement of Catalan's conjecture, that $(X, Y, U, V) = (3, 2, 2, 3)$ is the only integer solution of $X^U - Y^V = 1$. Thus, if $\kappa = 1$, any nontrivial solution to (5.4) is constrained by the condition $H(f) - r = 1$, which implies that $(q_j^n - 1)/(q_i^n - 1) = p$. Since the odd primes q_i, q_j and the exponent n in the prime decomposition of N must be greater than or equal to 3, this restriction is consistent with Catalan's conjecture.

When $\kappa \neq 1$, it may be noted that for $q_i, q_j \gg 1$, $(q_j^n - 1)/(q_i^n - 1) \simeq (q_j/q_i)^n \neq p^h$. Exceptional solutions to (5.4) occur, for example, when $h = 1$; they include $\{(q_i, q_j; n; p) = (3,5;2;3), (5,7;2;2), (5,11;2;5), (5,13;2;7), (11,19;2;3), (7,23;2,11), (11,29;2;7), (29,41;2;2)\}$. Since $q_i \neq q_j$, with the exception of the nontrivial solutions to (5.4), it would be necessary to set $n_i \neq n_j$ to obtain equality between $\Phi_{n_i}(q_i)$ and $\Phi_{n_j}(q_j)/p$.

(III) The arithmetic primitive factors of $q_i^{n_i} - 1$ and $q_j^{n_j} - 1$ are $\Phi_{n_i}(q_i)/p_i$ and $\Phi_{n_j}(q_j)$.

The proof of the necessity of $n_i \neq n_j$ for any equality between the arithmetic primitive factors is similar to that given in case (II) with the roles of i and j interchanged.

(IV) The arithmetic primitive factors of $q_i^{n_i} - 1$ and $q_j^{n_j} - 1$ are $\Phi_{n_i}(q_i)/p_i$ and $\Phi_{n_j}(q_j)/p_j$.

Since $p_i = \gcd(n_i, \Phi_{n_i}(q_i))$ and $p_j = \gcd(n_j, \Phi_{n_j}(q_j))$, $\Phi_{n_i}(q_i)$ and $\Phi_{n_j}(q_j)$ share a common factor if $n_i = n_j$. Thus, the primes p_i and p_j must be equal, and a comparison can be made between $\Phi_n(q_i)/p$ and $\Phi_n(q_j)/p$. Again, by the monotonicity of $\Phi_n(x)$, it follows that these quantities are not equal when q_i and q_j are different primes. Equality of the arithmetic prime factors could only occur if $n_i \neq n_j$. $\square$

6. The exponent of prime divisors of repunit factors in the rationality condition. Since all primitive divisors of $U_n(a,b)$ have the form $p = nk + 1$, it follows that $p \mid (q^{(p-1)/\iota(p)} - 1)/(q - 1)$. If $\iota(p)$ is odd, where $\iota(p)$ is the residue index, the exponent $(p-1)/\iota(p)$ will be even for all odd primes p, whereas if $\iota(p)$ is even, the exponent $(p-1)/\iota(p)$ may be even or odd. Given that $p \mid U_{2\alpha_i+1}(q_i + 1, q_i)$, $\iota(p)$ is even and $p \mid (q_i^{(p-1)/2} - 1)/(q_i - 1)$ implying $q_i^{(p-1)/2} \equiv 1 (\operatorname{mod} p)$ and $(q_i/p) = 1$. Moreover, if $(q_i/p) = (q_j/p) = 1$, $(q_i q_j/p) = 1$ implying that $p \mid (q_i q_j)^{(p-1)/2} - 1$. Thus, the Fermat quotient is $Q_{q_i q_j} = (((q_i q_j)^{(p-1)/2} - 1)/p)((q_i q_j)^{(p-1)/2} + 1) = \mathcal{N}_{q_i q_j}(\mathcal{N}_{q_i q_j} p + 2)$ where $\mathcal{N}_q$ can be defined to be $(q^{(p-1)/2} - 1)/p$. By the logarithmic rule for Fermat quotients, $Q_{qq'} \equiv Q_q + Q_{q'} (\operatorname{mod} p)$ [13], so that $\mathcal{N}_{q_i q_j} \equiv \mathcal{N}_{q_i} + \mathcal{N}_{q_j} (\operatorname{mod} p)$.

Recalling that $\alpha(q_i + 1, q_i, p^2) \neq \alpha(q_i + 1, q_i, p)$ only when $p^2 \nmid (q_i^{p-1} - 1)/(q_i - 1)$, it is sufficient to prove that the Fermat quotient $Q_{q_i} \neq 0 (\operatorname{mod} p)$ to show that p^2 is not a divisor of the repunit $(q_i^{2\alpha_i+1} - 1)/(q_i - 1)$. It has been established that $q^{p-1} - 1 \equiv p(\mu_1 + \mu_2/2 + \cdots + \mu_{p-1}/(p-1)) (\operatorname{mod} p^2)$, where $\mu_i \equiv [-i/p] (\operatorname{mod} q)$ [11, 18, 19]. Since $\mu_i \neq 0$ in general, except when $i = q$, it follows that $q^{p-1} - 1 \neq 0 (\operatorname{mod} p^2)$ except for $p - 1$ values of q between 1 and $p^2 - 1$.

By Hensel's lemma [24, 30], each of the integers between 1 and $p - 1$, which satisfy $x^{p-1} - 1 \equiv 0 (\operatorname{mod} p)$, generate the $p - 1$ solutions to the congruence equation

$$(x')^{p-1} - 1 \equiv 0 (\operatorname{mod} p^2) \tag{6.1}$$

through the formula

$$x' = x + \left(\frac{-g_1(x)p}{(p-1)q^{p-2}}\right)(\bmod p^2) \tag{6.2}$$

with $x^{p-1}-1 \equiv g_1(x)p(\bmod p^2)$. Since $\varphi(p^2) = p(p-1)$, a set of $p-1$ solutions to (6.1) can also be labelled as $c^p(\bmod p^2)$, $1 \le c \le p-1$, since $(c^p)^{p-1} = c^{p(p-1)} = c^{\varphi(p^2)} \equiv 1(\bmod p^2)$. Each power c^p is different, because $c_1^p \equiv c_2^p(\bmod p^2)$ implies $c_1 = c_2$ since $p^2 \nmid (c_3^p - 1)$ for any c_3 between 1 and $p-1$.

Theorems concerning the Fermat quotient $(q^r-1)/(q-1)$ can be extended to quotients of the type $(q^{nr}-1)/(q^n-1)$. It has been proven, for example, that $p \| (q^{nr}-1)/(q^n-1)$, $p \nmid r$, $p \nmid q^n - 1$, then $Q_q = (q^{p-1}-1)/p \not\equiv 0(\bmod p)$ [27], and more generally, if $p^h \| (q^{nr}-1)/(q^n-1)$, $p \nmid r$, $p \nmid q^n-1$, then $q^{p-1} \nmid 1(\bmod p^{h+1})$. When $p \mid (q^r-1)$, the following lemma is obtained.

Lemma 6.1. *For any prime p which is a primitive divisor of $U_{2\alpha_i+1}(q_i+1,q_i)$, $p \nmid (q_i^{p-1}-1)/(q_i^{(2\alpha_i+1)}-1)$, and if $p^h \| U_{2\alpha_i+1}(q_i+1,q_i)$, then $p^h \| (q_i^{2\alpha_i+1}-1)/(q_i^{(2\alpha_i+1)/s}-1)$ for any nontrivial divisor s of $2\alpha_i+1$.*

Proof. Defining the residue index $\iota_i(p)$ by $p-1 = (2\alpha_i+1)\iota_i(p)$, then

$$p \;\Big|\; \frac{q_i^{p-1}-1}{q_i-1} = \left[\frac{q^{(2\alpha_i+1)\iota_i(p)}-1}{q_i^{(2\alpha_i+1)}-1}\right] \cdot \left[\frac{q^{(2\alpha_i+1)}-1}{q_i-1}\right]. \tag{6.3}$$

Suppose that $p \mid (q_i^{(2\alpha_i+1)\iota_i(p)}-1)/(q_i^{2\alpha_i+1}-1)$. Then, by (6.3), $p^2 \mid (q_i^{p-1}-1)/(q-1)$. By a lemma on congruences, if $q^e \equiv 1(\bmod p)$, where $e \mid (p-1)$ and $q^{p-1} \equiv 1(\bmod p^2)$, then $q^e \equiv 1(\bmod p^2)$ [57], so that $p^2 \mid (q_i^{2\alpha_i+1}-1)/(q_i-1)$. Consequently, $p^3 \mid (q_i^{p-1}-1)/(q_i-1)$. This lemma can be extended to larger prime powers: $q^e \equiv 1(\bmod p^n)$ and $q^{p-1} \equiv 1(\bmod p^{n+1})$, then $q^e \equiv 1(\bmod p^{n+1})$. From the first congruence relation, $q^e = 1 + k'p^n$ for some integer k'. Raising this quantity to the power $(p-1)/e$, it follows that

$$1 \equiv q^{p-1} = (q^e)^{(p-1)/e} = (1+k'p^n)^{(p-1)/e} \equiv 1 + k'p^n \frac{p-1}{e}(\bmod p^{n+1}). \tag{6.4}$$

Since $(p-1)/e < p$, the integer k' must be a multiple of p. Thus, $q^e = 1 + k''p^{n+1} \equiv 1(\bmod p^{n+1})$. By the generalized congruence theorem, $p^3 \mid (q_i^{2\alpha_i+1}-1)/(q_i-1)$ and (6.4) in turn implies that $p^4 \mid (q_i^{p-1}-1)/(q_i-1)$. Since this process can be continued indefinitely to arbitrarily high powers of the prime p, a contradiction is obtained once the maximum exponent is greater than h, where $p^h \mid (q_i^{p-1}-1)/(q_i-1)$. Therefore, $p \nmid (q_i^{(2\alpha_i+1)\iota_i(p)}-1)/(q_i^{2\alpha_i+1}-1)$.

Similarly,

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1}=\left[\frac{q_i^{2\alpha_i+1}-1}{q_i^{(2\alpha_i+1)/s}-1}\right]\cdot\left[\frac{q_i^{(2\alpha_i+1)/s}-1}{q_i-1}\right]. \tag{6.5}$$

If s is a nontrivial divisor of $2\alpha_i+1$, then $p\nmid(q_i^{(2\alpha_i+1)/s}-1)/(q_i-1)$, because it is a primitive divisor of $U_{2\alpha_i+1}(q_i+1,q_i)$. Given that $p^h\|U_{2\alpha_i+1}(q_i+1,q_i)$, by (6.5), $p^h\|(q_i^{(2\alpha_i+1)}-1)/(q_i^{(2\alpha_i+1)/s}-1)$. □

Imprimitive prime divisors of $U_n(a,b)$ are characterized by the property that $p\mid U_d(a,b)$ for some $d\mid n$. The exponent of the imprimitive prime power divisor exactly dividing $(q^n-1)/(q-1)$ can be determined by a further lemma: if $p^h\mid(q^n-1)/(q-1)$, then either $\gcd(n,p-1)=1$, $q\equiv 1(\operatorname{mod}p)$, $p^h\mid n(\operatorname{mod}p)$ or $e=\gcd(n,p-1)>1$, $p^k\mid\Phi_e(q)$, $p^{h-k}\|n$ [44]. Since $v_p(\Phi_e(q))=v_p(q^e-1)$ if $p\nmid q-1$, the general formula [25, 26] for the exponent of a prime divisor of a repunit is

$$v_p\left(\frac{q^n-1}{q-1}\right)=\begin{cases}v_p(q^e-1)+v_p(n), & e=\operatorname{ord}_p(q)\mid n,\ e>1,\\ v_p(n), & p\mid q-1,\\ 0, & \text{otherwise.}\end{cases} \tag{6.6}$$

The exponent also can be deduced from the congruence properties of q-numbers $[n]=(q^n-1)/(q-1)$ and q-binomial coefficients [17], as it is equal to $s=\epsilon_0h+\epsilon_1+\cdots+\epsilon_{k-1}$ where $p^h\|q^e-1$ and

$$\begin{aligned}n-1&=a_0+e(a_1+a_2p+\cdots+a_kp^{k-1})\\ n&=b_0+e(b_1+b_2p+\cdots+b_kp^{k-1})\\ a_0+1&=\epsilon_0e+b_0\\ \epsilon_0+a_1&=\epsilon_1p+b_1\\ &\vdots\\ \epsilon_{k-2}+a_{k-1}&=\epsilon_{k-1}p+b_{k-1}\\ \epsilon_{k-1}+a_k&=b_k,\end{aligned} \tag{6.7}$$

with ϵ_i equal to 0 or 1, which is consistent with (6.6) because $\epsilon_0=1$ and $v_p(n)=\epsilon_1+\cdots+\epsilon_{k-1}$.

Specializing to the case of $h=2$, it follows that if the quotient $(q^n-1)/(q-1)$ is exactly divisible by p^2, then

(i) $\gcd(n,p-1)=1$, $p\mid(q-1)$ or $p\nmid q^n-1$, $p^2\|n$,

(ii) $p \| \Phi_e(q)$, where $e = \alpha(q+1,q,p)$ is the rank of apparition of p, $p \| n$,

(iii) $p^2 \| \Phi_e(q)$, $p \nmid n$,

and the only indices n_i which allow for exact divisibility of $(q_i^{n_i}-1)/(q_i-1)$ by p^2 are $n_i = \mu p^2$, when $p \mid (q_i-1)$ or $e_i \nmid n_i$, $n_i = \mu e_i p$ when $p \| \Phi_{e_i}(q_i)$ and $n_i = \mu e_i$ when $p^2 \| \Phi_{e_i}(q_i)$. Since n_i is odd, the three categories can be defined by the conditions: (i) $n_i = \mu p^2$, (ii) $n_i = \mu e_i p$, p is a primitive divisor of $(q_i^{e_i}-1)/(q_i-1)$, $Q_{q_i} \not\equiv 0 (\operatorname{mod} p)$ (iii) p is a primitive divisor of $(q_i^{e_i}-1)/(q_i-1)$, $Q_{q_i} \equiv 0 (\operatorname{mod} p)$.

7. A proof by the method of induction of the nonexistence of a generic set of primes satisfying the rationality condition. The equation

$$a\frac{x^m-1}{x-1} = b\frac{y^n-1}{y-1} \tag{7.1}$$

is known to have finitely many integer solutions for m, n, x, y, given a and b such that $\gcd(a,b) = 1$, $a(y-1) \neq b(x-1)$, and $\max(m,n,x,y) < C$ where C is an effectively computable number depending on a, b, and F where $|x-y| < F(z/(\log z)^2(\log\log z)^3)$ with $z = \max(x,y)$ [1, 48]. Using this relation to re-express $(q_i^{2\alpha_i+1}-1)/(q_i-1)$ in terms of $((4k+1)^{4m+2}-1)/4k$, it can be established that there are unmatched primes in the product of the repunits (2.5) and that the square root of this expression is irrational for several different categories of prime divisors $\{q_i,\ i = 1,\dots,\ell; 4k+1\}$.

Theorem 7.1. *The square-root expressions* $\sqrt{2(4k+1)}[(q_1^{2\alpha_1+1}-1)/(q_i-1) \cdots (q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)]^{1/2} \cdot (((4k+1)^{4m+2}-1)/4k)^{1/2}$ *are not rational numbers for the following sets of primes* $\{q_i,\ i = 1,\dots,\ell; 4k+1\}$ *and exponents* $2\alpha_i+1$.

(i) *For sets of primes with the number of elements given by consecutive integers,* $\{q_i,\ i = 1,\dots,\ell-1, 4k+1\}$ *and* $\{q'_j,\ j = 1,\dots,\ell, 4k'+1\}$, *there cannot be odd integers of the form* $N_1 = (4k+1)^{4m+1} q_1^{2\alpha_1} \cdots q_{\ell-1}^{2\alpha_{\ell-1}}$ *and* $N_2 = (4k'+1)^{4m'+1}(q'_1)^{2\alpha'_1} \cdots (q'_\ell)^{2\alpha'_\ell}$ *such that both* $\sigma(N_1)/N_1 = 2$ *and* $\sigma(N_2)/N_2 = 2$.

(ii) *Setting* $\alpha_j = \alpha_\ell$, *extra prime divisors* p *of the repunits* $(q_j^{2\alpha_j+1}-1)/(q_j-1)$, $j < \ell$, *and* $(q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)$, *where* $p \mid (q_j-1)$ *but* $p \nmid (q_\ell-1)$, *cannot be absorbed into the square factors if* $Q_{q_\ell} \not\equiv 0 (\operatorname{mod} p)$ *or* $p^{h'_\ell} \| (q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)$ *with* h'_ℓ *odd. Similarly, if* $p \nmid (q_j-1)$ *but* $p \mid (q_\ell-1)$, *then an odd power of* p *divides the product of the two repunits if* $Q_{q_j} \not\equiv 0 (\operatorname{mod} p)$ *or* $Q_{q_j} \equiv 0 (\operatorname{mod} p^{h'_j-1})$, $Q_{q_j} \not\equiv 0 (\operatorname{mod} p^{h'_j})$, *with* h'_j *odd, and* p *remains an unmatched prime divisor.*

(iii) *When* $n_j = 2\alpha_j+1$ *is set equal to* $n_\ell = 2\alpha_\ell+1$, *the primitive prime divisors of* $(q_j^{2\alpha_j+1}-1)/(q_j-1)$ *and* $(q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)$ *cannot be matched to produce the square of a rational number if* $(q_\ell^{n_\ell}-1)/(q_j^{n_\ell}-1) \neq y_2^2/y_1^2$, $y_1, y_2 \in \mathbb{Z}$. *This property is valid, for example, when* $q_\ell^{n_\ell/2} < \gcd(q_j^{n_j}-1, q_\ell^{n_\ell}-1)$.

Proof. Suppose $\{a_i\}$ and $\{b_i\}$ are defined by

$$
\begin{aligned}
a_1 \frac{q_1^{2\alpha_1+1}-1}{q_1-1} &= b_1 \frac{(4k+1)^{4m+2}-1}{4k} \\
a_2 \frac{q_2^{2\alpha_2+1}-1}{q_2-1} &= b_2 \frac{(4k+1)^{4m+2}-1}{4k} \\
&\vdots \\
a_\ell \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} &= b_\ell \frac{(4k+1)^{4m+2}-1}{4k},
\end{aligned}
\tag{7.2}
$$

then

$$
\begin{aligned}
&\sqrt{2(4k+1)}\left[\frac{q_1^{2\alpha_1+1}-1}{q_1-1}\frac{q_2^{2\alpha_2+1}-1}{q_2-1}\cdots\frac{q_\ell{}^{2\alpha_\ell+1}-1}{q_\ell-1}\frac{(4k+1)^{4m+2}-1}{4k}\right]^{1/2} \\
&\quad= \sqrt{2(4k+1)}\,\frac{(b_1b_2\cdots b_\ell)^{1/2}}{(a_1a_2\cdots a_\ell)^{1/2}}\cdot\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{(\ell+1)/2}.
\end{aligned}
\tag{7.3}
$$

If

$$
a_{ij}\frac{q_i^{2\alpha_i+1}-1}{q_i-1} = b_{ij}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}
\tag{7.4}
$$

define $\{a_{ij}\}$ and $\{b_{ij}\}$ with $\gcd(a_{ij},b_{ij}) = 1$,

$$
\frac{b_1b_2b_3}{a_1a_2a_3} = \frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\times\left(\frac{b_2b_3}{a_2a_3}\right)^2,
\tag{7.5}
$$

$$
\frac{b_1b_2\cdots b_\ell}{a_1a_2\cdots a_\ell} = \frac{b_{1\ell}}{a_{1\ell}}\frac{a_2}{b_2}\cdots\frac{a_{\ell-1}}{b_{\ell-1}}\left(\frac{b_2b_3\cdots b_\ell}{a_2a_3\cdots a_\ell}\right)^2.
\tag{7.6}
$$

Since the fraction b_1/a_1 can be expressed in terms of b_2/a_2

$$
\frac{b_1}{a_1} = \frac{b_2}{a_2}\frac{\rho_{12}^{r_{12}}}{\chi_{12}^{s_{12}}} = \frac{b_2}{a_2}\frac{\rho_{12}^{(r_{12})_0}}{\chi_{12}^{(s_{12})_0}}\frac{\rho_{12}^{(r_{12}-(r_{12})_0)}}{\chi_{12}^{(s_{12}-(s_{12})_0)}},
\tag{7.7}
$$

where $\rho_{12}^{r_{12}}$ and $\chi_{12}^{s_{12}}$ denote products of various powers of different primes, with r_{12} and s_{12} representing the sets of exponents, $(r_{12})_0$ and $(s_{12})_0$ labelling a collection of exponents consisting of 0 or 1, and ρ_{12} and χ_{12} being products of these primes with all of the exponents equal to 1. The sets $(r_{12})_0$ and $(s_{12})_0$ are chosen so that $r_{12}-(r_{12})_0 = 2\bar{r}_{12}$ and $s_{12}-(s_{12})_0 = 2\bar{s}_{12}$ represent even exponents. Since a similar relation exists between a_2/b_2 and a_3/b_3,

$$
\frac{b_{13}}{a_{13}}\frac{a_2}{b_2} = \left[\frac{a_2}{b_2}\frac{\rho_{12}^{(r_{12})_0}}{\chi_{12}^{(s_{12})_0}}\frac{\rho_{23}^{(r_{23})_0}}{\chi_{23}^{(s_{23})_0}}\right]\left(\frac{\rho_{12}^{\bar{r}_{12}}}{\chi_{12}^{\bar{s}_{12}}}\right)^2\left(\frac{\rho_{23}^{\bar{r}_{23}}}{\chi_{23}^{\bar{s}_{23}}}\right)^2.
\tag{7.8}
$$

If $(r_{12})_0 = (s_{12})_0 = (r_{23})_0 = (s_{23})_0 = \{0\}$, then $(b_{13}/a_{13})(a_2/b_2) \neq (4k+1)/2\cdot\square$ ($\square$ denotes the square of a rational number) because rationality of $\sqrt{2(4k+1)}[((q_2^{2\alpha_2+1}-1)/(q_2-1))(((4k+1)^{4m+2}-1)/4k)]^{1/2}$ would be contrary to the nonexistence of solutions to equation (3.3) and the nonexistence of multiply perfect numbers with less than four prime factors [6].

If $(r_{12})_0 = (s_{12})_0 = \{0\}$, the expression in brackets is not $(4k+1)/2$ times the square of a rational number because

$$\frac{a_2}{b_2}\frac{\rho_{23}^{(r_{23})_0}}{\chi_{23}^{(s_{23})_0}} = \frac{a_2}{b_2}\frac{\rho_{23}^{r_{23}}}{\chi_{23}^{s_{23}}}\cdot\left(\frac{\rho_{23}^{2\bar{r}_{23}}}{\chi_{23}^{2\bar{s}_{23}}}\right)^{-1} = \frac{a_3}{b_3}\left(\frac{\chi_{23}^{\bar{s}_{23}}}{\rho_{23}^{\bar{r}_{23}}}\right)^2 \tag{7.9}$$

and $a_3/b_3 \neq (4k+1)/2\cdot\square$, since $\sqrt{2(4k+1)}[((q_3^{2\alpha_3+1}-1)/(q_3-1))(((4k+1)^{4m+2}-1)/4k)]^{1/2}$ is not rational. A similar conclusion holds when $(r_{23})_0 = \{0\}$ and $(s_{23})_0 = \{0\}$.

If both fractions $\rho_{12}^{(r_{12})_0}/\chi_{12}^{(s_{12})_0}$ and $\rho_{23}^{(s_{23})_0}/\chi_{23}^{(s_{23})_0}$ are nontrivial, at least one of the pair of exponents $((r_{12})_0,(s_{12})_0)$, and at least one of the pair of exponents $((r_{23})_0,(s_{23})_0)$, must be equal to one. Under these conditions, the argument is not essentially changed when all of the exponents are set equal to one, because replacement of the prime factors in any of the coefficients ρ_{12}, χ_{12}, ρ_{23}, or χ_{23} by 1 only eliminates the presence of these prime factors from the remainder of the proof. The nontriviality of both fractions, therefore, can be included by setting $(r_{12})_0 = (r_{23})_0 = \{1\}$ and $(s_{12})_0 = (s_{23})_0 = \{1\}$. Expression (7.6) then would be $(4k+1)/2$ times the square of a rational number if

$$\begin{gathered} a_2 = (4k+1)\rho_{12}\cdot\rho_{23}\cdot\frac{p^2}{2}, \qquad b_2 = \chi_{12}\cdot\chi_{23}\cdot q^2 \\ \text{or} \\ a_2 = (4k+1)\chi_{12}\cdot\chi_{23}\cdot\frac{p^2}{2}, \qquad b_2 = \rho_{12}\cdot\rho_{23}\cdot q^2, \end{gathered} \tag{7.10}$$

where $\gcd(p,q) = 1$. If $a_2 = (4k+1)\rho_{12}\rho_{23}(p^2/2)$ and $b_2 = \chi_{12}\chi_{23}q^2$,

$$\begin{gathered} \frac{a_3}{b_3} = \frac{a_2}{b_2}\frac{\rho_{23}}{\chi_{23}}, \\ \frac{a_3}{b_3}\frac{2\chi_{12}}{(4k+1)\rho_{12}} = \frac{(\rho_{23}p)^2}{(\chi_{23}q)^2}. \end{gathered} \tag{7.11}$$

Since $\gcd(a_3,b_3) = 1$, the square-free factors can be separated in the fraction $a_3/b_3 = (\hat{a}_3/\hat{b}_3)\cdot(\hat{p}^2/\hat{q}^2)$,

$$\frac{2\chi_{12}}{(4k+1)\rho_{12}}\frac{\hat{a}_3}{\hat{b}_3} = \frac{(\rho_{23}p\hat{q})^2}{(\chi_{23}q\hat{p})^2}. \tag{7.12}$$

Since a_3 is even, and $\hat{a}_3$ is divisible by a single factor of 2, $\chi_{12} = \rho_{23}(p/2)\hat{q}$, and similarly, because $\hat{b}_3$ is odd, $\rho_{12} = (1/(4k+1))\chi_{23}q\hat{p}$. Since $\rho_{12}\rho_{23}/\chi_{12}\chi_{23} = (2/(4k+1))(q\hat{p}/p\hat{q})$, rationality of $[(2/(4k+1))(b_{13}/a_{13})(a_2/b_2)]^{1/2}$ also could be achieved by setting $a_2 = (4k+1)q\hat{p}(p'^2/2)$ and $b_2 = p\hat{q}q'^2$. Then

$$\frac{a_2}{b_2} \cdot \frac{\rho_{23}}{\chi_{23}} \frac{\rho_{23}}{\chi_{23}} = \frac{4k+1}{2} \frac{q\hat{p}p'^2}{p\hat{q}q'^2} \frac{\rho_{23}}{\chi_{23}} = \frac{\rho_{12}}{\chi_{12}} \cdot \frac{\left((4k+1)\rho_{23}(p'/2)\right)^2}{\left(\chi_{23}q'^2\right)^2}. \tag{7.13}$$

Separating the square factors in $a_2/b_2 = (\hat{a}_2/\hat{b}_2) \cdot (\hat{p}'^2/\hat{q}'^2)$, it follows that

$$\frac{\hat{a}_2}{\hat{b}_2} \frac{((4k+1)(p/2)\hat{q})}{(q\hat{p})} = \frac{\left((4k+1)(p'/2)\hat{q}'\right)^2}{(q'\hat{p}')^2}. \tag{7.14}$$

Either there is an overlap between the prime factors of $(4k+1)(p/2)$ and $\hat{q}$ or $\hat{a}_2 = (4k+1)(p/2)\hat{q} = (4k+1)(p'/2)\hat{q}'$, and similarly, there is either an overlap between the prime factors of q and $\hat{p}$ or $\hat{b}_2 = q\hat{p} = q'\hat{p}'$. Removing any overlap, then the remaining square factors can be separated in a_2 and b_2 obtaining the form $\hat{a}_2/\hat{b}_2$ for the square-free part of the ratio a_2/b_2. The equalities containing $\hat{a}_2$ and $\hat{b}_2$ imply that $\hat{p} > \hat{p}' \geq p' > p$ and $\hat{q} > \hat{q}' \geq q' > q$. By interchanging the roles of a_2, b_2 and a_3, b_3 in the above argument, the inequalities $p > \hat{p}$ and $q > \hat{q}$ can be derived, implying a contradiction. Thus, when $\ell = 3$, it should not be possible to find coefficients $\{a_i\}$ and $\{b_i\}$ satisfying (7.2) such that $(b_{13}/a_{13})(a_2/b_2)$ is $(4k+1)/2$ times the square of a rational number. The validity of this result is confirmed by the nonexistence of odd perfect numbers with four different prime factors.

A variation of the standard induction argument can be used to show that there cannot be different odd perfect numbers with prime decompositions $(4k+1)^{4m+1}\prod_{i=1}^{\ell-1} q_i^{2\alpha_i}$ and $(4k'+1)^{4m'+1}\prod_{i=1}^{\ell} q_i'^{2\alpha_i'}$. When ℓ is odd,

$$\frac{q_1^{2\alpha_1+1}-1}{q_1-1} \cdots \frac{q_{\ell-1}^{m_{\ell-1}}-1}{q_{\ell-1}-1} = \frac{b_1\cdots b_{\ell-1}}{a_1\cdots a_{\ell-1}}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\ell-1} = \frac{b_1\cdots b_{\ell-1}}{a_1\cdots a_{\ell-1}} \cdot \square \tag{7.15}$$

rationality of square root of the product of repunits with $\ell-1$ prime bases $\{q_i, i = 1,\dots,\ell-1\}$ would require

$$\begin{aligned} \frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= 2(4k+1)\rho_\ell \frac{b_\ell}{a_\ell} \cdot \square, \\ \frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= 2(4k+1)\rho_\ell \cdot \square. \end{aligned} \tag{7.16}$$

Since the values $q_\ell = 3$ and $\alpha_\ell = 2$ can be excluded from the product of repunits, ρ_ℓ is odd and is not equal to 1, so that $b_1\cdots b_\ell/a_1\cdots a_\ell \neq 2(4k+1)\square$. The square root of the product of repunits with ℓ prime bases $\{q_i, i = 1,\dots,\ell\}$ is therefore not rational.

When ℓ is even,

$$\frac{q_1^{2\alpha_1+1}-1}{q_1-1}\cdots\frac{q_{\ell-1}^{2\alpha_{\ell-1}+1}-1}{q_{\ell-1}-1}=\frac{b_1\cdots b_{\ell-1}}{a_1\cdots a_{\ell-1}}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\cdot\square \tag{7.17}$$

so that rationality of the square-root expression with $\ell-1$ primes $\{q_i,\ i=1,\ldots,\ell-1\}$ requires

$$\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell}=2(4k+1)\rho_\ell\cdot\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\cdot\square. \tag{7.18}$$

Again, since $\rho_\ell\neq 1$, (7.18) implies that $(b_1\cdots b_\ell/a_1\cdots a_\ell)(((4k+1)^{4m+2}-1)/4k)$ with ℓ primes $\{q_i,\ i=1,\ldots,\ell\}$ is not rational.

The proof can be continued for $\ell>3$ by assuming that there do not exist any odd primes $q_1,\ldots,q_{\ell-1}$ and $4k+1$ such that $\sqrt{2(4k+1)}[(q_1^{2\alpha_1+1}-1)/(q_1-1)\cdots(q_{\ell-1}^{2\alpha_{\ell-1}+1}-1)/(q_{\ell-1}-1)]^{1/2}(((4k+1)^{4m+2}-1)/4k)^{1/2}$ is rational and proving that the same property is valid when ℓ odd primes $q_1,\ldots,q_\ell$ arise in the prime decomposition of the integer N.

If ℓ is odd, $(((4k+1)^{4m+2}-1)/4k)^{(\ell+1)/2}$ is integer, and nonexistence of odd perfect numbers of the form $(4k+1)^{4m+1}q_1^{2\alpha_1}\cdots q_{\ell-1}^{2\alpha_{\ell-1}}$ is equivalent to the condition $b_1\cdots b_\ell/a_1\cdots a_\ell\neq 2(4k+1)\square$, as

$$\begin{aligned}2(4k+1)\frac{q_1^{m_1}-1}{q_1-1}\cdots\frac{q_{l-1}^{m_{\ell-1}-1}-1}{q_{\ell-1}-1}&\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\\ =2(4k+1)\frac{b_1\cdots b_{\ell-1}}{a_1\cdots a_{\ell-1}}&\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\cdot\square.\end{aligned} \tag{7.19}$$

Since the irrationality of the square root expression is assumed to hold generally for $\ell-1$ odd primes $\{q_i\}$ and any value of $4k+1$, the effect of the inclusion of another prime q_ℓ can be deduced. Thus, given an arbitrary set of ℓ odd primes, $q_1,\ldots,q_\ell$ and some prime of the form $4k+1$, irrationality of the square root of expression (7.19) implies that

$$\frac{b_1\cdots b_{\ell-1}}{a_1\cdots a_{\ell-1}}\neq 2(4k+1)\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\cdot\square. \tag{7.20}$$

However, by (7.2), $(((4k+1)^{4m+2}-1)/4k)=(a_\ell/b_\ell)((q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1))$, and if $(q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)\equiv\rho_\ell\chi_\ell^2$, separating the square-free factors from the factors with even exponents, it follows that

$$\begin{aligned}\frac{b_1\cdots b_{\ell-1}}{a_1\cdots a_{\ell-1}}&\neq 2(4k+1)\frac{a_\ell}{b_\ell}\rho_\ell\cdot\square,\\ \frac{b_1\cdots b_\ell}{a_1\cdots a_\ell}&\neq 2(4k+1)\rho_\ell\cdot\square.\end{aligned} \tag{7.21}$$

The form of relation (7.21) is valid for arbitrary values of b_ℓ/a_ℓ, but the choice of ρ_ℓ is specific to the repunit $(q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)$. Since $(q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)$ is the square of an integer only when $q_\ell=3$ and $\alpha_\ell=2$, it is preferable to represent the rationality condition for $\ell-1$ and ℓ primes $\{q_i\}$ as

$$\begin{aligned}\frac{b_1\cdots b_{\ell-1}}{a_1\cdots a_{\ell-1}}&=2(4k+1)\frac{a_\ell}{b_\ell}\omega_\ell\cdot\square,\\ \frac{b_1\cdots b_\ell}{a_1\cdots a_\ell}&=2(4k+1)\omega_\ell\cdot\square,\end{aligned}\tag{7.22}$$

when ℓ is odd. Irrationality of the square root expression for $\ell-1$ primes $\{q_i, i=1,\dots,\ell-1\}$, which requires that $\omega_{\ell-1}\neq 1$ is a square-free integer, implies irrationality for ℓ primes $\{q_i, i=1,\dots,\ell\}$ if $\omega_{\ell-1}\rho_\ell=\omega_\ell\neq 1$ is square free.

When ℓ is even, odd perfect numbers of the form $(4k+1)^{4m+1}q_1^{2\alpha_1}\cdots q_{\ell-1}^{2\alpha_{\ell-1}}$ do not exist if $b_1\cdots b_{\ell-1}/a_1\cdots a_{\ell-1}\neq 2(4k+1)\cdot\square$. Then $b_1\cdots b_\ell/a_1\cdots a_\ell\cdot(((4k+1)^{4m+2}-1)/4k)\neq 2(4k+1)\rho_\ell\cdot\square$. Irrationality of the square root expression with $\ell-1$ primes $\{q_i, i=1,\dots,\ell-1\}$ also can be represented as $b_1\cdots b_{\ell-1}/a_1\cdots a_{\ell-1}=2(4k+1)\omega_{\ell-1}\cdot\square$ where $\omega_{\ell-1}\neq 1$ is a square-free number. Consequently, $b_1\cdots b_\ell/a_1\cdots a_\ell=2(4k+1)\omega_{\ell-1}(b_\ell/a_\ell)\cdot\square$. Since irrationality of the square root expression with ℓ primes $\{q_i, i=1,\dots,\ell\}$ would be equivalent to

$$\begin{aligned}\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)&=2(4k+1)\omega_\ell\cdot\square,\\ \frac{b_1\cdots b_\ell}{a_1\cdots a_\ell}&=2(4k+1)\omega_\ell\rho_\ell\frac{a_\ell}{b_\ell}\cdot\square,\end{aligned}\tag{7.23}$$

this again can be achieved if $\omega_{\ell-1}\rho_\ell=\omega_\ell\cdot\square$.

For any prime divisor p

$$\begin{aligned}v_p(\omega_{\ell-1})&=\sum_{i=1}^{\ell-1}\left[v_p\left(\frac{q_i^{e_i}-1}{q_i-1}\right)\delta\left(\frac{n_i}{e_i}-\left[\frac{n_i}{e_i}\right]\right)+v_p(n_i)\right]\\ &\quad+v_p\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\pmod 2,\\ v_p(\omega_\ell)&=\sum_{i=1}^{\ell}\left[v_p\left(\frac{q_i^{e_i}-1}{q_i-1}\right)\delta\left(\frac{n_i}{e_i}-\left[\frac{n_i}{e_i}\right]\right)+v_p(n_i)\right]\\ &\quad+v_p\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\pmod 2,\end{aligned}\tag{7.24}$$

where $e_i=\mathrm{ord}_p(q_i)$. It follows that

$$v_p(\omega_\ell)=v_p(\omega_{\ell-1})+v_p\left(\frac{q_\ell^{e_\ell}-1}{q_\ell-1}\right)+v_p(n_\ell).\tag{7.25}$$

Suppose that p is one of the extra prime divisors so that $v_p(\omega_{\ell-1}) = 1$. If $e_\ell \nmid n_\ell$ or $p \nmid n_\ell$, then $p \nmid (q_\ell^{n_\ell}-1)/(q_\ell-1)$ and $v_p(\omega_\ell) = 1$.

If $p^h \| (q_\ell^{e_\ell}-1)/(q_\ell-1)$, and p is a primitive prime divisor of this repunit, then $v_p(n_\ell) = 0$ and $v_p(\omega_\ell) = 1+h (\bmod 2)$. Since $v_p(\omega_\ell) = 0 (\bmod 2)$ if $h = 1$, it would be the next category of prime divisors, with the property $v_p((q_\ell^{e_\ell} - 1)/(q_\ell-1)) = 2$ or equivalently $Q_{q_\ell} \equiv 0 (\bmod p)$, which contributes nontrivially to a square-free coefficient ω_ℓ.

Since it has been assumed that the square root expression with $\ell-1$ primes $\{q_i,\, i = 1,\dots,\ell-1\}$ is irrational, there is either an unmatched primitive divisor or an imprimitive divisor in the product $\prod_{i=1}^{\ell-1}((q_i^{2\alpha_i+1}-1)/(q_i-1)) \cdot (((4k+1)^{4m+2}-1)/4k)$. Suppose that the extra prime divisor $\hat{p}_j$ is a factor of the repunit $(q_j^{2\alpha_j+1}-1)/(q_j-1)$. By (7.4),

$$\frac{q_j^{2\alpha_j+1}-1}{q_j-1} = \rho_j \chi_j^2 = \frac{b_{j\ell}}{a_{j\ell}} \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} = \frac{b_{j\ell}}{a_{j\ell}} \rho_\ell \chi_\ell^2 \tag{7.26}$$

so that $\rho_j \rho_\ell = b_{j\ell}/a_{j\ell} \cdot \square$.

To proceed further, it is first useful to choose the exponent $2\alpha_\ell+1$ to be equal to $2\alpha_j+1$. If $p \mid (q_j-1)$, $p^{\hat{h}_j} \mid (2\alpha_j+1)$, $p \mid (q_\ell-1)$, and $p^{\hat{h}_\ell} \mid (2\alpha_\ell+1)$, then $p^{\hat{h}_j} \mid (q_j^{2\alpha_j+1}-1)/(q_j-1)$ and $p^{\hat{h}_\ell} \mid (q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)$ when $\alpha_j = \alpha_\ell$, $p^{h_j} = p^{\hat{h}_j} = p^{\hat{h}_\ell} = p^{h_\ell}$, where h_j and h_ℓ denote the exponents of p exactly dividing the repunits with bases q_j and q_ℓ, respectively, so that this prime divisor will be absorbed into the square factors.

If $p \mid (q_j-1)$ and $p \nmid (q_\ell-1)$, then $h_j = \hat{h}_j$ and $h_\ell = \hat{h}_\ell + v_p((q_\ell^{e_\ell}-1)/(q_\ell-1))$. Since $\hat{h}_j = \hat{h}_\ell$ when $\alpha_j = \alpha_\ell$, $h_\ell = h_j + v_p((q_\ell^{e_\ell}-1)/(q_\ell-1))$. Matching of the prime factors in the two repunits would require $v_p((q_\ell^{e_\ell}-1)/(q_\ell-1)) = 0 (\bmod 2)$. Because $p \mid (q_\ell^{e_\ell}-1)$, the minimum value of this exponent is 2, implying that $Q_{q_\ell} \equiv 0 (\bmod p)$. Conversely, if $Q_{q_\ell} \not\equiv 0 (\bmod p)$ or $Q_{q_\ell} \equiv 0 (\bmod p^{h'_\ell-1})$, $Q_{q_\ell} \not\equiv 0 (\bmod p^{h'_\ell})$, where h'_ℓ is odd, the prime divisor p in the product of the two repunits cannot be entirely absorbed into the square factors. Similar conclusions hold when $p \nmid (q_j-1)$ and $p \mid (q_\ell-1)$.

Let p be an imprimitive prime divisor such that $p \nmid (q_j-1)$ and $p \nmid (q_\ell-1)$, then $v_p((q_j^{n_j}-1)/(q_j-1)) = v_p(q_j^{n_j}-1)$ and $v_p((q_\ell^{n_\ell}-1)/(q_\ell-1)) = v_p(q_\ell^{n_\ell}-1)$. If $p^h \mid n_\ell$, and $n_j = n_\ell$, then $h_j = h_\ell = h$, again implying that the prime divisor can be absorbed into the square factors.

The arithmetic primitive factors of $(q_j^{n_j}-1)/(q_j-1)$ and $(q_\ell^{n_\ell}-1)/(q_\ell-1)$, $\Phi_{n_j}(q_j)/p_j$ and $\Phi_{n_\ell}(q_\ell)/p_\ell$, respectively, are different when $n_j = n_\ell$, except possibly for solutions generated by the prime equation $(q_\ell^n-1)/(q_j^n-1) = p$ required when either $p_j = \gcd(n_j, \Phi_{n_j}(q_j))$ or $p_\ell = \gcd(n_\ell, \Phi_{n_\ell}(q_\ell))$ equals 1. The algebraic primitive factors $\Phi_{n_j}(q_j)$ and $\Phi_{n_\ell}(q_\ell)$ will be necessarily different if $n_j = n_\ell$. Consider a prime divisor p' of the arithmetic primitive factors which is raised to a different power in $\Phi_{n_j}(q_j)/p_j$ and $\Phi_{n_\ell}(q_\ell)/p_\ell$. If this prime is the only factor with this property, then $(q_\ell^{n_\ell}-1)/(q_j^{n_j}-1)=(q_\ell^{n_\ell}-1)/(q_j^{n_\ell}-1)=(p')^{h_\ell-h_j}$, and the nonexistence of solutions to this equation for $h_\ell-h_j\geq 2$ has been shown in Section 5.

The error in the approximation is given by $(q_\ell^{n_\ell}/q_j^{n_\ell})[1-1/q_\ell^{n_\ell}+1/q_j^{n_j}+\mathbb{O}(1/q_j^{n_\ell}q_\ell^{n_\ell})]$, and since $|1/q_j^{n_\ell}-1/q_\ell^{n_\ell}|<\min(1/q_j^{n_\ell},1/q_\ell^{n_\ell})$, the error is less than $q_\ell^{n_\ell}/q_j^{n_\ell}(q_j^{n_\ell}-1)\simeq q_\ell^{n_\ell}/q_j^{2n_\ell}$. Given a rational number a/b, the inequality $|a/b-z_2/z_1|<1/z_1^2$ has a finite number of solutions satisfying $z_1<b$ and $\gcd(z_1,z_2)=1$ [40]. In particular, there should be finite number of solutions to

$$\left|\frac{q_\ell{}^{n_\ell}}{q_j^{n_\ell}}-\frac{z_2}{z_1}\right|=\left|\frac{q_\ell^{n_\ell}}{q_j^{n_\ell}}-\frac{y_2^2}{y_1^2}\right|<\frac{1}{y_1^4} \tag{7.27}$$

if y_1 is constrained by the inequality $y_1<q_j^{n_\ell/2}$. The condition $|q_\ell^{n_\ell}/q_j^{n_\ell}-y_2^2/y_1^2|<q_\ell^{n_\ell}/q_j^{2n_\ell}$, therefore, will be satisfied by these solutions when $q_j^{n_\ell/2}/q_\ell^{n_\ell/4}<y_1<q_j^{n_\ell/2}$.

Since it has been established that square classes of the repunits $(q^n-1)/(q-1)$ consist of only one element [41], it follows that $(q_\ell^{n_{\ell_1}}-1)(q_\ell^{n_{\ell_2}}-1)=(\kappa')^2(q_\ell-1)^2(y_1')^2(y_2')^2$ and there is only one representative from each sequence $\{q_j^{n_j}-1,\ n_j\in\mathbb{Z}\}$, $\{q_\ell^{n_\ell}-1,\ n_\ell\in\mathbb{Z}\}$ which has a specified square-free factor κ. Thus, $(q_\ell^{n_\ell}-1)/(q_j^{n_\ell}-1)\neq y_2^2/y_1^2$ unless $n_{q_j}(\kappa)$ coincides with $n_{q_\ell}(\kappa)$. If $q_\ell^{n_\ell}-1=\kappa(y_2')^2$ and $q_j^{n_\ell}-1=\kappa(y_1')^2$, and y_2^2/y_1^2 is the irreducible form of $(y_2')^2/(y_1')^2$, it follows that $y_1<q_j^{n_\ell/2}/\sqrt{\kappa\hat{\kappa}^2}$, where $\hat{\kappa}=\gcd(y_1',y_2')$. Both inequalities for y_1 cannot be satisfied if $q_\ell^{n_\ell/4}<\sqrt{\kappa\hat{\kappa}^2}$ or equivalently $q_\ell^{n_\ell/2}<\gcd(q_j^{n_\ell}-1,q_\ell^{n_\ell}-1)$. When the pair of primes (q_j,q_ℓ) satisfies the last inequality, the prime divisors in ρ_j and ρ_ℓ do not match and the product of the repunits $(q_j^{n_j}-1)/(q_j-1)$ and $(q_\ell^{n_\ell}-1)/(q_\ell-1)$, with $n_j\neq n_\ell$, is not a perfect square.

The number of solutions to the inequality $|ax^n-by^n|\leq h$ when $x\geq(2h/a^{1-\rho}\alpha)^{1/(n/2-1)}$ with $\alpha=(b/a)^{1/n}$ does not exceed $6+(1/\ln(n/2))[29+\ln\rho^{-1}+\ln(1+\ln 2h/\ln a)]$ [37]. Setting $(q_\ell^n-1)/(q_j^n-1)\simeq y_2^2/y_1^2$, it follows that $y_1^2(q_\ell^n-1)\simeq y_2^2(q_j^n-1)$ leading to consideration of the inequality $|y_2^2q_j^n-y_1^2q_\ell^n|\leq|y_2^2-y_1^2|$. The constraint placed on q_j is

$$q_j\geq\left(\frac{2|y_2^2-y_1^2|}{y_2^{2(1-\rho)}(y_2^2/y_1^2)^{1/n}}\right)^{1/(n/2-1)}. \tag{7.28}$$

Since $(q_j-1)/(q_\ell-1)\geq(y_1^2/y_2^2)(1/n)\geq q_j/q_\ell$, it is sufficient for q_j to satisfy the stronger constraint

$$q_j\geq\left(2\frac{q_\ell-1}{q_j-1}y_2^{2\rho}\right)^{1/(n/2-1)} \tag{7.29}$$

which is equivalent to an upper bound for y_2 of

$$y_2^2 \le q_j^{\rho^{-1}(n/2-1)} \cdot \left(\frac{1}{2}\frac{q_j-1}{q_\ell-1}\right)^{\rho^{-1}}. \tag{7.30}$$

The number of solutions to the inequality is not greater than

$$\begin{aligned} 6+\frac{1}{\ln(n/2)}&\left(29+\ln\rho^{-1}+\ln\left(1+\frac{\ln(2\,|\,y_2^2-y_1^2\,|)}{\ln y_2^2}\right)\right) \\ &\le 6+\frac{29+\ln(2+\ln 2)+\ln\rho^{-1}}{\ln(n/2)}. \end{aligned} \tag{7.31}$$

It has been established that the sequence q^n-1 has a primitive divisor $n>2$, $q\ne 2$, $n\ne 6$ [2, 3, 8, 58], and the same property holds for $\{(q^n-1)/(q-1)\}$. If n_ℓ is multiplied by a prime factor $\hat{p}^{r_\ell}$, where $\hat{p}\mid\rho_\ell$, then the product $\rho_\ell\hat{p}^{r_\ell}$ will contain the power $\hat{p}_\ell^{1+r_\ell}$. While the prime power can be absorbed into the product of square factors when r_ℓ is odd, the repunit $(q_\ell^{n_\ell\hat{p}^{r_\ell}}-1)/(q_\ell-1)$ now has extra primitive divisors, giving rise to a nontrivial ω_ℓ, implying irrationality of the square root expression with ℓ primes $\{q_i,\ i=1,\dots,\ell\}$. Moreover, $\gcd(\Phi_{\hat{p}^i}(q),\Phi_{\hat{p}^j}(q))=1$ when $i\ne j$ and $p\nmid(q-1)$, multiplication of the index by $\hat{p}^{r_\ell}$ will introduce new prime divisors through the decomposition of the repunit $(q_\ell^{n_\ell\hat{p}^{r_\ell}}-1)/(q_\ell-1)=\prod_{\substack{d\mid n_\ell\hat{p}^{r_\ell}\\ d>1}}\Phi_d(q_\ell)$.

The abstract argument given for $\ell=3$ could also be extended to higher values of ℓ. This approach would consist of the demonstration of the property $b_1\cdots b_\ell/a_1\cdots a_\ell\ne 2(4k+1)\cdot\Box$ if ℓ is odd, and $b_1\cdots b_\ell/a_1\cdots a_\ell\ne 2(4k+1)\cdot(((4k+1)^{4m+2}-1)/4k)\cdot\Box$ if ℓ is even, given that there are no sets of primes $\{q_i\}$ with less than ℓ elements satisfying the rationality condition. It may be noted that since

$$\begin{aligned}
\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= \left(\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right)\left(\frac{b_{46}}{a_{46}}\frac{a_5}{b_5}\right)\cdots\left(\frac{b_{\ell-2,\ell}}{a_{\ell-2,\ell}}\frac{a_{\ell-1}}{b_{\ell-1}}\right)\\
&\quad\cdot\left(\frac{b_2b_3b_5b_6\cdots b_{\ell-1}b_\ell}{a_2a_3a_5a_6\cdots a_{\ell-1}a_\ell}\right)^2 \quad \text{when } \ell\equiv 0(\operatorname{mod}3),\\
\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= \left(\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right)\left(\frac{b_{46}}{a_{46}}\frac{a_5}{b_5}\right)\cdots\left(\frac{b_{\ell-3,\ell-1}}{a_{\ell-3,\ell-1}}\frac{a_{\ell-2}}{b_{\ell-2}}\right)\frac{b_\ell}{a_\ell}\\
&\quad\cdot\left(\frac{b_2b_3b_5b_6\cdots b_{\ell-2}b_{\ell-1}}{a_2a_3a_5a_6\cdots a_{\ell-2}a_{\ell-1}}\right)^2 \quad \text{when } \ell\equiv 1(\operatorname{mod}3),\\
\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= \left(\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right)\left(\frac{b_{46}}{a_{46}}\frac{a_5}{b_5}\right)\cdots\left(\frac{b_{\ell-4,\ell-2}}{a_{\ell-4,\ell-2}}\frac{a_{\ell-3}}{b_{\ell-3}}\right)\frac{b_{\ell-1}b_\ell}{a_{\ell-1}a_\ell}\\
&\quad\cdot\left(\frac{b_2b_3b_5b_6\cdots b_{\ell-3}b_{\ell-2}}{a_2a_3a_5a_6\cdots a_{\ell-3}a_{\ell-2}}\right)^2 \quad \text{when } \ell\equiv 2(\operatorname{mod}3)
\end{aligned} \tag{7.32}$$

and $(b_{13}/a_{13})(a_2/b_2) = 2(4k+1)(\bar{\rho}_1/\bar{\chi}_1)\cdot\square, \ldots, b_{\ell-k'-2,\ell-k'}/a_{\ell-k'-2,\ell-k'} = 2(4k+1)(\bar{\rho}_{[\ell/3]}/\bar{\chi}_{[\ell/3]})\cdot\square$, where $\ell \equiv k' (\bmod 3)$, $k' = 0,1,2$, $\bar{\rho}_1,\ldots,\bar{\rho}_{[\ell/3]},\bar{\chi}_1,\ldots,\bar{\chi}_{[\ell/3]}$ are square-free factors, the quotient will be equal to $(2(4k+1))^{[\ell/3]} f_{k'}(\bar{\rho}_1/\bar{\chi}_1)\cdots(\bar{\rho}_{[\ell/3]}/\bar{\chi}_{[\ell/3]})\cdot\square$ with $f_0 = 1$, $f_1 = b_\ell/a_\ell$, and $f_2 = b_{\ell-1}b_\ell/a_{\ell-1}a_\ell$. It has been established that $b_\ell/a_\ell \neq 2(4k+1)\cdot\square$ because there is no odd integer of the form $(4k+1)^{4m+1}q_\ell^{2\alpha_\ell}$ such that $\sqrt{2(4k+1)}[((q_\ell^{2\alpha_1+1}-1)/(q_\ell-1))(((4k+1)^{4m+2}-1)/4k)]^{1/2}$ is rational and $b_{\ell-1}b_\ell/a_{\ell-1}a_\ell \neq 2(4k+1)(((4k+1)^{4m+2}-1)/4k)\cdot\square$ as $\sqrt{2(4k+1)}[((q_{\ell-1}^{2\alpha_{\ell-1}+1}-1)/(q_{\ell-1}-1))((q_\ell^{2\alpha_1+1}-1)/(q_\ell-1))\cdot(((4k+1)^{4m+2}-1)/4k)]^{1/2}$ is irrational [6, 7]. Setting $b_\ell/a_\ell = 2(4k+1)(\hat{\rho}_{\ell 1}/\hat{\chi}_{\ell 1})\cdot\square$ and $b_{\ell-1}b_\ell/a_{\ell-1}a_\ell = 2(4k+1)(\hat{\rho}_{\ell 2}/\hat{\chi}_{\ell 2})(((4k+1)^{4m+2}-1)/4k)\cdot\square$, it follows that

$$
\begin{aligned}
\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= (2(4k+1))^{\ell/3}\frac{\bar{\rho}_1}{\bar{\chi}_1}\cdots\frac{\bar{\rho}_{\ell/3}}{\bar{\chi}_{\ell/3}}\cdot\square \quad \ell\equiv 0(\bmod 3),\\
\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= (2(4k+1))^{[\ell/3]+1}\frac{\bar{\rho}_1}{\bar{\chi}_1}\cdots\frac{\bar{\rho}_{[\ell/3]}}{\bar{\chi}_{[\ell/3]}}\cdot\frac{\hat{\rho}_{\ell 1}}{\hat{\chi}_{\ell 1}}\cdot\square \quad \ell\equiv 1(\bmod 3),\\
\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= (2(4k+1))^{[\ell/3]+1}\frac{\bar{\rho}_1}{\bar{\chi}_1}\cdots\frac{\bar{\rho}_{[\ell/3]}}{\bar{\chi}_{[\ell/3]}}\\
&\quad\cdot\frac{\hat{\rho}_{\ell 2}}{\hat{\chi}_{\ell 2}}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\cdot\square \quad \ell\equiv 2(\bmod 3)
\end{aligned}
\tag{7.33}
$$

and the coefficients $\{a_i,b_i\}$ will not satisfy the rationality condition when the square-free factors $\bar{\rho}_1,\ldots,\bar{\rho}_{[\ell/3]},\hat{\rho}_{\ell 1},\hat{\rho}_{\ell 2},\bar{\chi}_1,\ldots,\bar{\chi}_{[\ell/3]},\hat{\chi}_{\ell 1},\hat{\chi}_{\ell 2}$ have prime divisors other than 2 and $4k+1$ which do not match to produce the square of a rational number.

When ℓ is odd and greater than 5, there always exists an odd integer ℓ_o and an even integer ℓ_e such that $\ell = 3\ell_o + 2\ell_e$, implying the following identity

$$
\begin{aligned}
\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= \left(\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right)\left(\frac{b_{46}}{a_{46}}\frac{a_5}{b_5}\right)\cdots\left(\frac{b_{3\ell_o-2,3\ell_o}}{a_{3\ell_0-2,3\ell_o}}\frac{a_{3\ell_o-1}}{b_{3\ell_o-1}}\right)\left(\frac{b_{3\ell_0+1}b_{3\ell_0+2}}{a_{3\ell_0+1}a_{3\ell_0+2}}\right)\\
&\quad\cdots\left(\frac{b_{\ell-1}b_\ell}{a_{\ell-1}a_\ell}\right)\cdot\square.
\end{aligned}
\tag{7.34}
$$

Consequently,

$$
\begin{aligned}
\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= (2(4k+1))^{\ell_0+\ell_e}\frac{\bar{\rho}_1}{\bar{\chi}_1}\cdots\frac{\bar{\rho}_{\ell_0}}{\bar{\chi}_{\ell_o}}\cdots\frac{\hat{\rho}_{\ell-2\ell_e+2,2}}{\hat{\chi}_{\ell-2\ell_e+1,2}}\cdots\frac{\hat{\rho}_{\ell 2}}{\hat{\chi}_{\ell 2}}\\
&\quad\cdot\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\ell_e}\cdot\square\\
&= 2(4k+1)\cdot\frac{\bar{\rho}_1}{\bar{\chi}_1}\cdots\frac{\bar{\rho}_{\ell_o}}{\bar{\chi}_{\ell_o}}\cdots\frac{\hat{\rho}_{\ell-2\ell_e+2,2}}{\hat{\chi}_{\ell-2\ell_e+2,2}}\cdots\frac{\hat{\rho}_{\ell 2}}{\hat{\chi}_{\ell 2}}\cdot\square.
\end{aligned}
\tag{7.35}
$$

Regardless of the factors of 2 and $4k+1$, the coefficients $\{a_i, b_i\}$ will produce an irrational square root expression (7.3) for odd ℓ if the product of fractions is not the square of a rational number.

If ℓ is even and greater than 4, there always exists an odd integer ℓ_o and an even integer ℓ_e such that $\ell = 2\ell_o + 3\ell_e$. From the identity

$$\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} = \left(\frac{b_1 b_2}{a_1 a_2}\right)\cdots\left(\frac{b_{2\ell_o-1}b_{2\ell_o}}{a_{2\ell_o-1}a_{2\ell_o}}\right)\cdots\left(\frac{b_{2\ell_o+1,2\ell_0+3}}{a_{2\ell_o+1}a_{2\ell_o+3}}\frac{a_{2\ell_o+2}}{b_{2\ell_o+2}}\right) \cdots\left(\frac{b_{\ell-2,\ell}}{a_{\ell-2,\ell}}\frac{a_{\ell-1}}{b_{\ell-1}}\right)\cdot\square \tag{7.36}$$

it follows that

$$\begin{aligned}\frac{b_1\cdots b_\ell}{a_1\cdots a_\ell} &= \left(2(4k+1)\right)^{\ell_o+\ell_e}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\ell_o}\cdot\frac{\hat{\rho}_{22}}{\hat{\chi}_{22}}\cdots\frac{\hat{\rho}_{2\ell_o,2}}{\hat{\chi}_{2\ell_o,2}}\frac{\bar{\rho}_{\ell-3\ell_e+1}}{\bar{\chi}_{\ell-3\ell_e+1}} \\ &\quad\cdots\frac{\bar{\rho}_{\ell-2}}{\bar{\chi}_{\ell-2}}\cdot\square \\ &= 2(4k+1)^{\ell_0+\ell_e}\cdot\frac{\hat{\rho}_{22}}{\hat{\chi}_{22}}\cdots\frac{\hat{\rho}_{2\ell_o,2}}{\hat{\chi}_{2\ell_o,2}}\frac{\bar{\rho}_{\ell-3\ell_e+1}}{\bar{\chi}_{\ell-3\ell_e+1}}\cdots\frac{\bar{\rho}_{\ell-2}}{\bar{\chi}_{\ell-2}}\cdot\square.\end{aligned} \tag{7.37}$$

Again, the factors of 2 and $4k+1$ are not relevant, and the coefficients $\{a_i, b_i\}$ give rise to an irrational square root expression (7.3) for even ℓ if

$$\prod_{i=1}^{\ell_o}\left(\frac{\hat{\rho}_{2i,2}}{\hat{\chi}_{2i,2}}\right)\prod_{j=1}^{\ell_e}\left(\frac{\bar{\rho}_{\ell-3j+1}}{\bar{\chi}_{\ell-3j+1}}\right) \tag{7.38}$$

is not the square of a rational number. □

8. On the proof of the nonexistence of odd perfect numbers. Since the condition for the existence of an odd perfect number is equality of $(((4k+1)^{4m+2}-1)/4k)\prod_{i=1}^{\ell}((q_i^{2\alpha_i+1}-1)/(q_i-1))$ with $2(4k+1)\prod_{i=1}^{\ell} q_i^{2\alpha_i}$, the number of distinct prime divisors of the product of repunits in the rationality condition must be $\ell+1$. The following lemma will be useful for obtaining an upper bound for the number of integers N which could possibly satisfy the condition $\sigma(N) = 2N$.

Lemma 8.1. *The number of integer solutions to the prime equations*

$$\begin{aligned}\frac{q_i^{n_i}-1}{q_i-1} &= n_i\cdot\frac{q_j^{n_j}-1}{q_j-1}, \\ n_j\cdot\frac{q_i^{n_i}-1}{q_i-1} &= \frac{q_j^{n_j}-1}{q_j-1} \quad q_i \neq q_j,\ n_i,\ n_j \text{ prime}, \\ n_j\cdot\frac{q_i^{n_i}-1}{q_i-1} &= n_i\cdot\frac{q_j^{n_j}-1}{q_j-1}\end{aligned} \tag{8.1}$$

is bounded by the number of integers (q_i,q_j) *and* (n_i,n_j) *which satisfy the following conditions:*

$$\begin{gathered}\frac{k\pi}{n_i} = t\pi \pm \langle p_1\rangle_d \pm \langle p_2\rangle_d \cdots \pm \langle p_n\rangle_d, \\ p_i \text{ prime for } i = 1,\dots,n, \quad k = 1,\dots,\frac{n-1}{2},\ d \in \mathbb{Z}, \\ \langle p\rangle_d = \frac{1}{s}\tan^{-1}\sqrt{\frac{db'^2}{a'^2}} \quad a'^2 + db'^2 = 4p^s \quad \text{for the minimum value of } s.\end{gathered} \tag{8.2}$$

Proof. Consider the factorization

$$\begin{aligned}\frac{x^{n_i}-1}{x-1} &= (x-\omega)(x-\omega^{-1})(x-\omega^2)(x-\omega^{-2}) \\ &\quad\cdots(x-\omega^{(n_i-1)/2})(x-\omega^{-(n_i-1)/2}), \\ \omega &= \exp\left(\frac{2\pi i}{n_i}\right).\end{aligned} \tag{8.3}$$

As the real quadratic factors have the form $x^2 - 2\cos(2k\pi/n_i)x + 1$, $k = 1,\dots,(n_i-1)/2$, the trigonometric term equals $2((1-\tan^2(k\pi/n_i))/(1+\tan^2(k\pi/n_i)))$, and $\tan\theta = (b/a)\sqrt{d}$, $a,b,d \in \mathbb{Z}$ when

$$\theta = t\pi \pm \langle p_1\rangle_d \pm \langle p_2\rangle_d \pm \cdots \pm \langle p_n\rangle_d \quad \text{for some } t \in \mathbb{Q}, \tag{8.4}$$

where the prime decomposition of a^2+db^2 is $p_1\cdots p_n$ and

$$\langle p\rangle_d = \frac{1}{s}\tan^{-1}\sqrt{\frac{db'^2}{a'^2}} \tag{8.5}$$

with s being the smallest positive integer such that $4p^s$ is expressible as $a'^2 + db'^2$ [9], quadratic factors will be rational when $k\pi/n_i = t\pi \pm \langle p_1\rangle_d \pm \langle p_2\rangle_d \pm \cdots \pm \langle p_n\rangle_d$ for all $k = 1,\dots,(n_i-1)/2$.

Except for the values of n_i and n_j for which $\cos(2k\pi/n_i)$ and $\cos(2k\pi/n_j)$ are rational, the quadratic expressions $q_i^2 - 2\cos(2k\pi/n_i)q_i + 1$ and $q_j^2 - 2\cos(2k\pi/n_j)q_j + 1$ must be unequal. Furthermore, equality of the products such as $\prod_{k=1}^{(n_i-1)/2}(q_i^2 - 2\cos(2k\pi/n_i)q_i + 1)$ and $n_i \cdot \prod_{k=1}^{(n_j-1)/2}(q_j^2 - 2\cos(2k\pi/n_j)q_j + 1)$ over the Galois extension

$$\begin{gathered}\mathbb{Q}\Bigg(\cos\left(\frac{2\pi}{n_i}\right);\cos\left(\frac{4\pi}{n_i}\right);\dots;\cos\left(\frac{2[(n_i-1)/4]\pi}{n_i}\right); \\ \cos\left(\frac{2\pi}{n_j}\right);\cos\left(\frac{4\pi}{n_j}\right);\dots;\cos\left(\frac{2[(n_j-1)/4]\pi}{n_j}\right)\Bigg)\end{gathered} \tag{8.6}$$

would require that products of selected sets of the real quadratic factors $q_i^2 - 2\cos(2k\pi/n_i)q_i + 1$ coincide with products of other sets of the quadratic factors $q_j^2 - 2\cos(2k\pi/n_i)q_j + 1$ with the exception of the extra primes n_i and n_j.

However, there will be no collection of quadratic factors $q_i^2-2\cos(2k\pi/n_i)q_i+1$ such that their product will be equal to n_i, for example, because an irrational term will still be present in a product with less than $(n_i-1)/2$ quadratic factors, since trigonometric sums of the form $\sum_{k_1<\cdots<k_m\in\{K\}}\cos(2k_1\pi/n_i)\cdots\cos(2k_m\pi/n_i)$ will be rational only if $\{K\}$ is the entire set $\{1,\ldots,(n_i-1)/2\}$. Since the inequality of $(q_i^{n_i}-1)/(q_i-1)$ and $n_i((q_j^{n_j}-1)/(q_j-1))$ holds over the Galois extension of $\mathbb{Q}$, for n_i, n_j such that $\cos(2\pi/n_i)$ or $\cos(2\pi/n_j)$ is irrational, it is also valid over $\mathbb{Q}$ and the integers $\mathbb{Z}$, implying the nonexistence of integer solutions to the first relation in (8.1), and the same conclusion is valid for the other two relations. □

Theorem 8.2. *Any odd integer N of the form $(4k+1)^{4m+1}\prod_{i=1}^{\ell}q_i^{2\alpha_i}$, $4k+1$, q_i prime, $m\geq 1$, $\alpha_i\geq 1$, subject to the condition that none of the prime divisors satisfy one of the following three equalities:*

$$\begin{aligned}\frac{q_i^{n_i}-1}{q_i-1}&=n_i\cdot\frac{q_j^{n_j}-1}{q_j-1},\\ n_j\cdot\frac{q_i^{n_i}-1}{q_i-1}&=\frac{q_j^{n_j}-1}{q_j-1},\quad n_i=2\alpha_i+1,n_i,n_j\ \textit{prime},\\ n_j\cdot\frac{q_i^{n_i}-1}{q_i-1}&=n_i\cdot\frac{q_j^{n_j}-1}{q_j-1}\end{aligned}\tag{8.7}$$

will not be a perfect number.

Proof. As the number of distinct prime divisors of $(q_i^{2\alpha_i+1}-1)/(q_i-1)$ is at least $\tau(2\alpha_i+1)-1$, since each of the cyclotomic polynomials in the factorization $(q_i^{2\alpha_i+1}-1)/(q_i-1)=\prod_{\substack{d\mid 2\alpha_i+1\\ d>1}}\Phi_d(q_i)$ has a distinct prime divisor congruent to $1(\operatorname{mod}d)$, it should be greater than or equal to 1 when $2\alpha_i+1$ is prime and at least 3 when $2\alpha_i+1$ is composite. When there is an Aurifeuillian factorization of the cyclotomic polynomial $\Phi_d(q_i)$, both factors are divisible by a primitive prime divisor of $\Phi_d(q_i)$ [50], so that if the index $2\alpha_i+1$ is a prime of the form $4k'+1$ for some $k'\in\mathbb{Z}$, the Lucas number $U_{2\alpha_i+1}(q_i+1,q_i)$ has at least two primitive prime divisors [45], whereas if the index is an odd multiple $(2\mathrm{N}+1)(4k'+1)$, $\mathrm{N}\geq 1$ there are at least five primitive prime divisors. This result can be generalized to composite indices of the form $d\delta$ with $\gcd(d,\delta)=1$ given an Aurifeuillian factorization of $\Phi_d(q_i)$, based on the formula $\Phi_{d\delta}(q_i)=\prod_{\gamma\mid\delta}\Phi_d(q_i^{\gamma})^{\mu(\delta/\gamma)}$. If the exponent is an odd multiple of a prime of the form $4k'+3$, then the repunit contains at least three distinct prime divisors, except when $2\alpha_i+1$ equals the prime itself, as the minimum number of prime divisors is attained if $\Phi_{4k'+3}(q_i)=(q_i^{4k'+3}-1)/(q_i-1)$ is prime.

If the exponent of $(q_i^{2\alpha_i+1}-1)/(q_i-1)$ is prime, the repunit will have prime divisors which divide q_i-1 or have the form $a(2\alpha_i+1)^b+1$ [47]. Primes which divide q_i-1 also must be a factor of $2\alpha_i+1$, since $(q_i^{2\alpha_i+1}-1)/(q_i-1)\equiv 2\alpha_i+1(\operatorname{mod}p)$ when $q_i\equiv 1(\operatorname{mod}p)$, and therefore they must be equal to the exponent $2\alpha_i+1$ if it is prime. Since repunits with different bases are generally different with two known exceptions, unless one of the three equations

$$\begin{aligned}\frac{q_i^{2\alpha_i+1}-1}{q_i-1}&=(2\alpha_i+1)\cdot\frac{q_j^{2\alpha_j+1}-1}{q_j-1},\\(2\alpha_j+1)\cdot\frac{q_i^{2\alpha_i+1}-1}{q_i-1}&=\frac{q_j^{2\alpha_j+1}-1}{q_j-1},\\(2\alpha_j+1)\cdot\frac{q_i^{2\alpha_i+1}-1}{q_i-1}&=(2\alpha_i+1)\cdot\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\end{aligned}\tag{8.8}$$

holds, there will be a prime divisor of the form $a(2\alpha_i+1)^b+1$ or $a'(2\alpha_j+1)^{b'}+1$ which will not be a common factor of both repunits. Each repunit $(q_i^{2\alpha_i+1}-1)/(q_i-1)$ with a prime exponent $2\alpha_i+1$ has a distinct primitive prime divisor if there is no pair (q_i,q_j) which satisfies any of the three equalities in (8.8).

Equality between $(q_i^{2\alpha_i+1}-1)/(q_i-1)$ and $(2\alpha_i+1)((q_j^{2\alpha_j+1}-1)/(q_j-1))$ would imply that the repunit $(q_j^{2\alpha_j+1}-1)/(q_j-1)$ does not introduce any additional prime divisors and $((q_i^{2\alpha_i+1}-1)/(q_i-1))((q_j^{2\alpha_j+1}-1)/(q_j-1))=(2\alpha_i+1)\cdot\square$, so that the square root of the product of the two repunits contains only one irrational factor. Since every primitive prime divisor of $(q_i^{2\alpha_i+1}-1)/(q_i-1)$ is strictly greater than $2\alpha_i+1$, this product does contribute at least two new prime divisors. In contrast to the primitive divisor $a(2\alpha_i+1)^b+1$, however, the prime $2\alpha_i+1$ might be equal to the primitive prime divisor of another repunit $(q_{i'}^{2\alpha_{i'}+1}-1)/(q_{i'}-1)$, even if this repunit does not satisfy any equalities of the type given in (8.8). It follows that the product of three repunits with prime exponents may not necessarily contain three distinct prime divisors if any two of the repunits satisfy one of these relations, and therefore, it will be assumed in the rest of the proof that the equalities in (8.8) do not hold for any of the prime divisors and exponents $\{q_i,2\alpha_i+1,\ i=1,\dots,g\}$ in the factorization of the odd integer N.

Denoting the repunits with composite exponents by $(q_{c_\iota}^{2\alpha_{c_\iota}+1}-1)/(q_{c_\iota}-1)$, the product

$$\frac{(4k+1)^{4m+2}-1}{4k}\prod_{\iota=1}^{g}\frac{q_{c_\iota}^{2\alpha_{c_\iota}+1}-1}{q_{c_\iota}-1}\tag{8.9}$$

contains the union of $g+1$ sets of at least three prime divisors. Combined with the prime divisors of the product of repunits with prime exponents,

$$\prod_{\jmath=1}^{\ell-g}\frac{q_{p_\jmath}^{2\alpha_{p_\jmath}+1}-1}{q_{p_\jmath}-1},\tag{8.10}$$

which number more than $\ell - g - 1$, it is sufficient to prove that if there are only $\ell + 2$ distinct prime divisors in the factorization of $\sigma(N)$, then $\sigma(N)/2N \neq 2$. Otherwise there are at least $\ell + 3$ distinct prime divisors of $\sigma(N)$.

For a composite exponent $2\alpha_{c_\imath} + 1 = p_{c_\imath} \cdot \delta_\imath$,

$$\frac{q_{c_\imath}^{2\alpha_{c_\imath}+1} - 1}{q_{c_\imath} - 1} = \frac{q_{c_\imath}^{p_{c_\imath}} - 1}{q_{c_\imath} - 1}\left(1 + q_{c_\imath}^{p_{c_\imath}} + \cdots + q_{c_\imath}^{p_{c_\imath} \cdot (\delta_\imath - 1)}\right) \tag{8.11}$$

so that the product of the repunits with these exponents is

$$\prod_{i=1}^{g} \frac{q_{c_\imath}^{2\alpha_{c_\imath}+1} - 1}{q_{c_\imath} - 1} = \prod_{\imath=1}^{g} \frac{q_{c_\imath}^{p_{c_\imath}} - 1}{q_{c_\imath} - 1}\left(1 + q_{c_\imath}^{p_{c_\imath}} + \cdots + q_{c_\imath}^{p_{c_\imath} \cdot (\delta_\imath - 1)}\right). \tag{8.12}$$

Since the repunits $(q_{c_\imath}^{p_{c_\imath}} - 1)/(q_{c_\imath} - 1)$ have distinct prime divisors, the minimum number dividing the product (8.9) is $g + 2$, and the entire product of repunits must contain at least $\ell + g + 2$ different prime divisors.

Since this lower bound is precisely the number necessary for equality between $\sigma(N)$ and $2N$, it is not sufficient to establish the nonexistence of odd perfect numbers. Instead, it is preferable to use the method of induction. Suppose that there are no odd perfect numbers of the form $(4k+1)^{4m+1}\prod_{i=1}^{\ell-1} q_i^{2\alpha_i}$. Then for any splitting of the set of exponents $\{2\alpha_i + 1,\ i = 1,\ldots,\ell-1\} = \{2\alpha_{c_\imath} + 1,\ \imath = 1,\ldots,g\} \cup \{2\alpha_{p_\jmath} + 1,\ \jmath = 1,\ldots,\ell - 1 - g\}$, the product $(((4k+1)^{4m+2} - 1)/4k)\prod_{i=1}^{\ell-1}((q_i^{2\alpha_i+1} - 1)/(q_i - 1))$ will be divisible by at least $\ell + 1$ distinct prime divisors and it will not be equal to $2(4k+1)^{4m+1}\prod_{i=1}^{\ell-1} q_i^{2\alpha_i}$.

Consider an odd integer N of the form $(4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$. If the last repunit $(q_\ell^{2\alpha_\ell+1} - 1)/(q_\ell - 1)$ has a composite exponent, it will introduce at least three prime divisors, including one distinct factor, derived from the repunit $(q_\ell^{p_{c_{g+1}}} - 1)/(q_\ell - 1)$, where $p_{c_{g+1}}$ would be a prime factor of $2\alpha_\ell + 1$. Since this prime divisor is not even and must be chosen from the set $\{2; 4k+1; q_1,\ldots,q_\ell\}$ and $q_\ell \nmid (q_\ell^{p_{c_{g+1}}} - 1)/(q_\ell - 1)$, it must be either $4k+1$ or q_{j_ℓ} for some $j_\ell \in \{1,\ldots,\ell-1\}$. Otherwise, equality between $\sigma(N)$ and $2N$ cannot be obtained. To proceed with full generality, it will be assumed that the prime divisor of this repunit is not equal to $4k+1$. Similarly, by interchanging the roles of q_ℓ with $q_{\bar{i}}$, $\bar{i} \in \{1,\ldots,\ell-1\}$ and $(4k+1)$, it follows that $(((4k+1)^{4m+2} - 1)/4k)\prod_{\substack{i=1 \\ i \neq \bar{i}}}^{\ell}((q_i^{2\alpha_i+1} - 1)/(q_i - 1))$ will not be divisible by $q_{j_{\bar{i}}}$ for some $j_{\bar{i}} \in \{1,\ldots,\ell\}$, $j_{\bar{i}} \neq \bar{i}$, with the exception of one value i_o, for which the product of $(((4k+1)^{4m+2} - 1)/4k)\prod_{i \neq i_o}((q_i^{2\alpha_i+1} - 1)/(q_i - 1))$ will not be divisible by $4k+1$. There will also be an index j_{i_o} such that $q_{j_{i_o}}$ is the new prime divisor contained in $((4k+1)^{4m+2} - 1)/4k$ which is not a factor of $\prod_{i=1}^{\ell}((q_i^{2\alpha_i+1} - 1)/(q_i - 1))$. Since each prime power divisor of the product of repunits $q_{j_{\bar{i}}}^{h_{j_{\bar{i}}}}$ divides $(q_{\bar{i}}^{2\alpha_{\bar{i}}+1} - 1)/(q_{\bar{i}} - 1)$, consideration of the entire set of prime power divisors implies every repunit must be equal to the power of a different prime, with the exception of the repunit $((4k+1)^{4m+2}-1)/4k$ which should also contain the factor of 2.

Solutions to the equation $(x^n-1)/(x-1)=y^m$, $m\geq 2$ include

$$\begin{aligned} &x=3, && n=5, && y=11, && m=2,\\ &x=18, && n=3, && y=7, && m=3,\\ &x=7, && n=4, && y=20, && m=2 \end{aligned} \tag{8.13}$$

and it is known that if y is prime then n must be prime and $x=\rho^b$ and $b=n^\nu$ for some odd prime ρ and $\nu\geq 0$ [12]. The finite bound on the number of solutions [14] to the equation

$$q'^a=\frac{q^p-1}{q-1}\quad q,\ q',\ p \text{ prime} \tag{8.14}$$

provides constraints on the odd primes q_i, but it is the nonexistence of solutions to the equation that must be satisfied by the prime $4k+1$

$$\frac{(4k+1)^{4m+2}-1}{4k}=2q_{j_{io}}^{h_{j_{io}}}, \tag{8.15}$$

for $h_{j_{io}}$ even, obtained by setting $y=q_{j_{io}}^{h_{j_{io}}/2}$ in (3.3), that implies that there are no sets of primes $\{q_i,\ i=1,\dots,\ell;4k+1\}$ of this type which allow for equality between $\sigma(N)$ and $2N$.

When $(q_\ell^{p_{c_{g+1}}}-1)/(q_\ell-1)$ is a prime, then it must be equal to one of the other prime divisors $q_{\hat{j}_\ell}$, so that $q_{\hat{j}_\ell}\gg q_\ell$. By repeating this process for all of the prime divisors, an ordering of the magnitudes of these factors is established. Let $q_{j_{\max}}$ represent the largest prime in this ordering and suppose that it is greater than $4k+1$. As it is a factor of the odd integer N, $\sigma(N)$ will contain the repunit $(q_{j_{\max}}^{2\alpha_{j_{\max}}+1}-1)/(q_{j_{\max}}-1)$, which, if set equal to one of the prime divisors, would be larger than $q_{j_{\max}}$. This process therefore leads to a contradiction, so that it is not possible for the inclusion of one repunit $(q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)$ in $\sigma(N)$ to provide only one additional prime divisor and obtain equality between $\sigma(N)$ and $2N$.

When the inclusion of this repunit gives rise to two additional prime factors, the total number of distinct prime divisors will be at least $(\ell-1-g)+(g+2)+2=\ell+3$, which is sufficient to establish the inequality $\sigma(N)\neq 2N$.

If the last repunit $(q_\ell^{2\alpha_\ell+1}-1)/(q_\ell-1)$ has a prime exponent, the extra factor $1+q_\ell^{p_{c_{g+1}}}+\cdots(q_\ell^{p_{c_{g+1}}})^{(\delta_{g+1}-1)}$ is no longer present, but all choices of the prime bases $\{4k+1;q_i,\ i=1,\dots,\ell\}$ and exponents $\{4m+1;2\alpha_i\}$ either lead to a contradiction when the restriction of only one additional prime factor is imposed, or at least two additional prime divisors, implying directly $\sigma(N)\neq 2N$. □

9. Conclusion. The rationality condition provides an analytic method for investigating the existence of odd perfect numbers, as it would be sufficient to demonstrate that there is an unmatched prime divisor in the product. An upper bound for the density of odd integers greater than 10^{300}, in an interval of fixed length, which could satisfy $\sigma(N)/N = 2$, may be found by considering the square root expression containing the product of repunits, combining the estimate of the density of square-full numbers in this range with the probability of an integer being expressible as the product of repunits with prime bases multiplied by $2(4k+1)$. The arithmetic primitive factors of these repunits, products of the primitive prime power divisors, can be compared for different values of the prime basis, and it has been shown that they could only be equal if the exponents differ, except possibly for pairs of divisors $(\Phi_n(q_i), \Phi_n(q_j)/p_j)$ generated by the prime equation $(q_j^n - 1)/(q_i^n - 1) = p$. In Theorem 7.1, nonexistence of the odd perfect numbers for a large set of primes $\{q_i, i = 1,\dots,\ell; 4k+1\}$, exponents $\{2\alpha_i, i = 1,\dots,\ell; 4m+1\}$, and values of ℓ using the method of induction adapted to the coefficients $\{a_i, b_i\}$ in the product of n repunits. An abstract argument is given for the nonexistence of coefficients satisfying the rationality condition when $\ell = 3$ and then various results are proven for $\ell > 3$ by using the properties of prime divisors of product of two repunits, $(q_j^{2\alpha_j+1} - 1)/(q_j - 1)$ and $(q_\ell^{2\alpha_\ell+1} - 1)/(q_\ell - 1)$, belonging to each of the four categories: (i) $p \mid (q_j - 1)$, $p \mid (q_\ell - 1)$; (ii) $p \mid (q_j - 1)$, $p \nmid (q_\ell - 1)$; (iii) $p \nmid (q_j - 1)$, $p \mid (q_\ell - 1)$; and (iv) $p \nmid (q_j - 1)$, $p \nmid (q_\ell - 1)$. Irrationality of the square root expression for any set of $\ell - 1$ primes $\{q_i, i = 1,\dots,\ell-1\}$ implies that each unmatched prime divisor in the product of repunits with bases $\{q_i, i = 1,\dots,\ell-1, 4k+1\}$ can be associated with a single repunit. Primitive prime divisors of $(q_j^{2\alpha_j+1} - 1)/(q_j - 1)$ and $(q_\ell^{2\alpha_\ell+1} - 1)/(q_\ell - 1)$, which belong to the fourth category, cannot be matched to produce a perfect square if $(q_\ell^{n_\ell} - 1)/(q_j^{n_j} - 1) \neq y_2^2/y_1^2$, which holds when $q_\ell^{n_\ell/2} < \gcd(q_j^{n_j} - 1, q_\ell^{n_\ell} - 1)$.

The set of odd integers N with a prime decomposition $(4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$, $q_i \geq 3$, $m \geq 1$, $\alpha_i \geq 1$ which does satisfy rationality condition consists of only a few elements. While the examples given in Section 4 contain at least eight prime divisors including 3, each of the exponents of the primes q_i has been set equal to 2, because of the coincidence of the prime divisors in the square-free factors of the repunits $(q_i^3 - 1)/(q_i - 1)$, and it is verified for these integers that $\sigma(N) \neq 2N$.

Combining the properties of the primitive prime divisors of the repunits in $\sigma(N)$ with the required form for equality of $\sigma(N)$ and $2N$, the nonexistence of odd perfect numbers, with the prime factors satisfying a set of three inequalities, is demonstrated in Section 8. First, a lower bound of $\ell + 1$ prime divisors is established for $\sigma(N)$ when there are ℓ primes q_i. Classifying the repunits according to whether the exponents are prime or composite, it can be shown that the inclusion of an additional factor in the prime decomposition of N has the effect of either introducing at one additional prime divisor in $\sigma(N)$

subject to constraints which imply $\sigma(N) \neq 2N$ or, at least, at two prime divisors. The nonexistence of odd perfect numbers then follows by using the method of induction to establish that integers of the form $(4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$ satisfy this inequality for all ℓ.

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