SOME PROPERTIES OF MAXIMAL OPEN SETS

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Some fundamental properties of maximal open sets are obtained, such as decomposition theorem for a maximal open set. Basic properties of intersections of maximal open sets are established, such as the law of radical closure.

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1. Introduction. A proper nonempty open subset U of a topological space X is said to be a *maximal open set* if any open set which contains U is X or U. In [2], we study minimal open sets. Although the definition of the maximal open set is obtained by "dualizing" the definition of the minimal open set, the properties of them are quite different, as we see in this paper, especially the results in the last two sections. The purpose of this paper is to prove some fundamental properties of maximal open sets and establish a part of the foundation of the theory of maximal open sets in topological spaces.

In Section 2, we prove some basic results which are necessary for the subsequent arguments. We obtain a relation among maximal open sets in Theorem 2.5. At the end of this section, we show that for any proper nonempty cofinite open subset V, there exists, at least, one maximal open set U which contains V (Theorem 2.7).

In Section 3, we study some relations among closure, interior, and maximal open sets. As an application, we prove a result about a preopen set (Theorem 3.11).

Let $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ be a set of some maximal open sets U_λ. Then, we refer to the intersection $\cap \mathfrak{U} = \cap_{\lambda \in \Lambda} U_\lambda$ as the *radical* of $\mathfrak{U}$. In the last two sections, we study various properties of radicals.

In Section 4, we prove fundamental properties of radicals of maximal open sets. We establish a very useful decomposition theorem for a maximal open set in Theorem 4.7. Theorem 4.7 will be applied to prove Theorem 4.8. Theorem 4.9 gives a sufficient condition for the set of all maximal open sets. In the rest of this section, we study the case when radicals are closed sets.

In Section 5, we consider the closure of the radicals of maximal open sets. We establish "The law of radical closure" in Theorem 5.4.

2. Maximal open sets. Let (X, τ) be a topological space.

DEFINITION 2.1. A proper nonempty open subset U of X is said to be a *maximal open set* if any open set which contains U is X or U.

LEMMA 2.2. (1) *Let U be a maximal open set and W an open set. Then, $U \cup W = X$ or $W \subset U$.*

(2) *Let U and V be maximal open sets. Then, $U \cup V = X$ or $U = V$.*

PROOF. (1) Let W be an open set such that $U \cup W \neq X$. Since U is a maximal open set and $U \subset U \cup W$, we have $U \cup W = U$. Therefore, $W \subset U$.

(2) If $U \cup V \neq X$, then $U \subset V$ and $V \subset U$ by (1). Therefore $U = V$. $\qquad\square$

PROPOSITION 2.3. *Let U be a maximal open set. If x is an element of U, then for any open neighborhood W of x, $W \cup U = X$ or $W \subset U$.*

PROOF. By Lemma 2.2(1), we have the result. $\qquad\square$

THEOREM 2.4. *Let U_α, U_β, and U_γ be maximal open sets such that $U_\alpha \neq U_\beta$. If $U_\alpha \cap U_\beta \subset U_\gamma$, then $U_\alpha = U_\gamma$ or $U_\beta = U_\gamma$.*

PROOF. We see that

$$
\begin{aligned}
U_\alpha \cap U_\gamma &= U_\alpha \cap (U_\gamma \cap X) \\
&= U_\alpha \cap (U_\gamma \cap (U_\alpha \cup U_\beta)) \quad \text{(by Lemma 2.2(2))} \\
&= U_\alpha \cap ((U_\gamma \cap U_\alpha) \cup (U_\gamma \cap U_\beta)) \\
&= (U_\alpha \cap U_\gamma) \cup (U_\gamma \cap U_\alpha \cap U_\beta) \\
&= (U_\alpha \cap U_\gamma) \cup (U_\alpha \cap U_\beta) \quad \text{(by } U_\alpha \cap U_\beta \subset U_\gamma) \\
&= U_\alpha \cap (U_\gamma \cup U_\beta).
\end{aligned}
\tag{2.1}
$$

Hence we have $U_\alpha \cap U_\gamma = U_\alpha \cap (U_\gamma \cup U_\beta)$. If $U_\gamma \neq U_\beta$, then $U_\gamma \cup U_\beta = X$, and hence $U_\alpha \cap U_\gamma = U_\alpha$; namely, $U_\alpha \subset U_\gamma$. Since U_α and U_γ are maximal open sets, we have $U_\alpha = U_\gamma$. $\qquad\square$

THEOREM 2.5. *Let U_α, U_β, and U_γ be maximal open sets, which are different from each other. Then,*

$$
U_\alpha \cap U_\beta \not\subset U_\alpha \cap U_\gamma.
\tag{2.2}
$$

PROOF. If $U_\alpha \cap U_\beta \subset U_\alpha \cap U_\gamma$, then we see that

$$
(U_\alpha \cap U_\beta) \cup (U_\beta \cap U_\gamma) \subset (U_\alpha \cap U_\gamma) \cup (U_\beta \cap U_\gamma)
\tag{2.3}
$$

hence,

$$
U_\beta \cap (U_\alpha \cup U_\gamma) \subset (U_\alpha \cup U_\beta) \cap U_\gamma.
\tag{2.4}
$$

Since $U_\alpha \cup U_\gamma = X = U_\alpha \cup U_\beta$, we have $U_\beta \subset U_\gamma$. It follows that $U_\beta = U_\gamma$, which contradicts our assumption. $\qquad\square$

PROPOSITION 2.6. *Let U be a maximal open set and x an element of U. Then,*

$$U = \cup\{W \mid W \text{ is an open neighborhood of } x \text{ such that } W \cup U \neq X\}. \qquad (2.5)$$

PROOF. By Proposition 2.3 and the fact that U is an open neighborhood of x, we have

$$U \subset \cup\{W \mid W \text{ is an open neighborhood of } x \text{ such that } W \cup U \neq X\} \subset U. \quad (2.6)$$

Therefore, we have the result. □

Finally, we prove an existence theorem of maximal open sets for special cases. We refer to the complement of any finite subset as a *cofinite* subset.

THEOREM 2.7. *Let V be a proper nonempty cofinite open subset. Then, there exists, at least, one (cofinite) maximal open set U such that $V \subset U$.*

PROOF. If V is a maximal open set, we may set $U = V$. If V is not a maximal open set, then there exists an (cofinite) open set V_1 such that $V \subsetneq V_1 \neq X$. If V_1 is a maximal open set, we may set $U = V_1$. If V_1 is not a maximal open set, then there exists an (cofinite) open set V_2 such that $V \subsetneq V_1 \subsetneq V_2 \neq X$. Continuing this process, we have a sequence of open sets

$$V \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_k \subsetneq \cdots. \qquad (2.7)$$

Since V is a cofinite set, this process repeats only finitely. Then, finally, we get a maximal open set $U = V_n$ for some positive integer n. □

3. Closure, interior, and maximal open sets. We begin with the following theorem.

THEOREM 3.1. *Let U be a maximal open set and x an element of $X - U$. Then, $X - U \subset W$ for any open neighborhood W of x.*

PROOF. Since $x \in X - U$, we have $W \not\subset U$ for any open neighborhood W of x. Then, $W \cup U = X$ by Lemma 2.2(1). Therefore, $X - U \subset W$. □

COROLLARY 3.2. *Let U be a maximal open set. Then, either of the following (1) and (2) holds:*

  (1) *for each $x \in X - U$ and each open neighborhood W of x, $W = X$;*

  (2) *there exists an open set W such that $X - U \subset W$ and $W \subsetneq X$.*

PROOF. If (1) does not hold, then there exists an element x of $X - U$ and an open neighborhood W of x such that $W \subsetneq X$. By Theorem 3.1, we have $X - U \subset W$. □

COROLLARY 3.3. *Let U be a maximal open set. Then, either of the following (1) and (2) holds:*

(1) *for each $x \in X - U$ and each open neighborhood W of x, we have $X - U \subsetneq W$;*

(2) *there exists an open set W such that $X - U = W \neq X$.*

PROOF. Assume that (2) does not hold. Then, by Theorem 3.1, we have $X - U \subset W$ for each $x \in X - U$ and each open neighborhood W of x. Hence, we have $X - U \subsetneq W$. □

THEOREM 3.4. *Let U be a maximal open set. Then,* $\mathrm{Cl}(U) = X$ *or* $\mathrm{Cl}(U) = U$.

PROOF. Since U is a maximal open set, only the following cases (1) and (2) occur by Corollary 3.3:

(1) for each $x \in X - U$ and each open neighborhood W of x, we have $X - U \subsetneq W$: let x be any element of $X - U$ and W any open neighborhood of x. Since $X - U \neq W$, we have $W \cap U \neq \varnothing$ for any open neighborhood W of x. Hence, $X - U \subset \mathrm{Cl}(U)$. Since $X = U \cup (X - U) \subset U \cup \mathrm{Cl}(U) = \mathrm{Cl}(U) \subset X$, we have $\mathrm{Cl}(U) = X$;

(2) there exists an open set W such that $X - U = W \neq X$: since $X - U = W$ is an open set, U is a closed set. Therefore, $U = \mathrm{Cl}(U)$. □

THEOREM 3.5. *Let U be a maximal open set. Then,* $\mathrm{Int}(X - U) = X - U$ *or* $\mathrm{Int}(X - U) = \varnothing$.

PROOF. By Corollary 3.3, we have either (1) $\mathrm{Int}(X - U) = \varnothing$ or (2) $\mathrm{Int}(X - U) = X - U$. □

THEOREM 3.6. *Let U be a maximal open set and S a nonempty subset of $X - U$. Then,* $\mathrm{Cl}(S) = X - U$.

PROOF. Since $\varnothing \neq S \subset X - U$, we have $W \cap S \neq \varnothing$ for any element x of $X - U$ and any open neighborhood W of x by Theorem 3.1. Then, $X - U \subset \mathrm{Cl}(S)$. Since $X - U$ is a closed set and $S \subset X - U$, we see that $\mathrm{Cl}(S) \subset \mathrm{Cl}(X - U) = X - U$. Therefore, $X - U = \mathrm{Cl}(S)$. □

COROLLARY 3.7. *Let U be a maximal open set and M a subset of X with $U \subsetneq M$. Then,* $\mathrm{Cl}(M) = X$.

PROOF. Since $U \subsetneq M \subset X$, there exists a nonempty subset S of $X - U$ such that $M = U \cup S$. Hence, we have $\mathrm{Cl}(M) = \mathrm{Cl}(S \cup U) = \mathrm{Cl}(S) \cup \mathrm{Cl}(U) \supset (X - U) \cup U = X$ by Theorem 3.6. Therefore, $\mathrm{Cl}(M) = X$. □

THEOREM 3.8. *Let U be a maximal open set and assume that the subset $X - U$ has two elements at least. Then,* $\mathrm{Cl}(X - \{a\}) = X$ *for any element a of $X - U$.*

PROOF. Since $U \subsetneq X - \{a\}$ by our assumption, we have the result by Corollary 3.7. □

THEOREM 3.9. *Let U be a maximal open set and N a proper subset of X with $U \subset N$. Then,* $\mathrm{Int}(N) = U$.

Proof. If $N = U$, then $\mathrm{Int}(N) = \mathrm{Int}(U) = U$. Otherwise $N \neq U$, and hence $U \subsetneq N$. It follows that $U \subset \mathrm{Int}(N)$. Since U is a maximal open set, we have also $\mathrm{Int}(N) \subset U$. Therefore, $\mathrm{Int}(N) = U$. $\qquad\square$

Theorem 3.10. *Let U be a maximal open set and S a nonempty subset of $X - U$. Then,*

$$X - \mathrm{Cl}(S) = \mathrm{Int}(X - S) = U. \tag{3.1}$$

Proof. Since $U \subset X - S \subsetneq X$ by our assumption, we have the result by Theorems 3.6 and 3.9. $\qquad\square$

A subset M of a space (X, τ) is called a *preopen* set if $M \subset \mathrm{Int}\,\mathrm{Cl}(M)$. Then, Corollary 3.7 implies the following result.

Theorem 3.11. *Let U be a maximal open set and M any subset of X with $U \subset M$. Then, M is a preopen set.*

Proof. If $M = U$, then M is an open set. Therefore, M is a preopen set. Otherwise, $U \subsetneq M$, then $\mathrm{Int}\,\mathrm{Cl}(M) = \mathrm{Int}\,X = X \supset M$ by Corollary 3.7. Therefore, M is a preopen set. $\qquad\square$

Corollary 3.12. *Let U be a maximal open set. Then, $X - \{a\}$ is a preopen set for any element a of $X - U$.*

Proof. Since $U \subset X - \{a\}$ by our assumption, we have the result by Theorem 3.11. $\qquad\square$

4. Fundamental properties of radicals

Definition 4.1. Let U_λ be a maximal open set for any element λ of Λ. Let $\mathcal{U} = \{U_\lambda \mid \lambda \in \Lambda\}$; $\cap \mathcal{U} = \cap_{\lambda \in \Lambda} U_\lambda$ is called the *radical* of $\mathcal{U}$.

The intersection of all maximal ideals of a ring $\mathcal{R}$ is called the (Jacobson) radical of $\mathcal{R}$ [1, 3]. Following this terminology in the theory of rings, we use the terminology "radical" for the intersection of maximal open sets.

The symbol $\Lambda \setminus \Gamma$ means difference of index sets; namely, $\Lambda \setminus \Gamma = \Lambda - \Gamma$, and the cardinality of a set Λ is denoted by $|\Lambda|$ in the following arguments.

Theorem 4.2. *Assume that $|\Lambda| \geq 2$. Let U_λ be a maximal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$.*
(1) Let μ be any element of Λ. Then, $X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \subset U_\mu$.
(2) Let μ be any element of Λ. Then, $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \neq \varnothing$.

Proof. Let μ be any element of Λ. (1) By Lemma 2.2(2), we have $X - U_\mu \subset U_\lambda$ for any element λ of Λ with $\lambda \neq \mu$. Then, $X - U_\mu \subset \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$. Therefore, we have $X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \subset U_\mu$.

(2) If $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda = \varnothing$, we have $X = U_\mu$ by (1). This contradicts our assumption that U_μ is a maximal open set. Therefore, we have $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \neq \varnothing$. $\qquad\square$

COROLLARY 4.3. *Let U_λ be a maximal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $|\Lambda| \geq 3$, then $U_\lambda \cap U_\mu \neq \varnothing$ for any elements λ and μ of Λ with $\lambda \neq \mu$.*

PROOF. By Theorem 4.2(2), we have the result. $\qquad\square$

THEOREM 4.4. *Let U_λ be a maximal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Assume that $|\Lambda| \geq 2$. Let μ be any element of Λ. Then, $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \not\subset U_\mu \not\subset \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$.*

PROOF. Let μ be any element of Λ. If $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \subset U_\mu$, then we see that $X = (X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda) \cup \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \subset U_\mu$ by Theorem 4.2(1). This contradicts our assumption. If $U_\mu \subset \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$, then we have $U_\mu \subset U_\lambda$, and hence $U_\mu = U_\lambda$ for any element λ of $\Lambda \setminus \{\mu\}$. This contradicts our assumption that $U_\mu \neq U_\lambda$ when $\lambda \neq \mu$. $\qquad\square$

COROLLARY 4.5. *Let U_λ be a maximal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $\cap_{\lambda \in \Lambda \setminus \Gamma} U_\lambda \not\subset \cap_{\gamma \in \Gamma} U_\gamma \not\subset \cap_{\lambda \in \Lambda \setminus \Gamma} U_\lambda$.*

PROOF. Let γ be any element of Γ. We see $\cap_{\lambda \in \Lambda \setminus \Gamma} U_\lambda = \cap_{\lambda \in ((\Lambda \setminus \Gamma) \cup \{\gamma\}) \setminus \{\gamma\}} U_\lambda \not\subset U_\gamma$ by Theorem 4.4. Therefore we see $\cap_{\lambda \in \Lambda \setminus \Gamma} U_\lambda \not\subset \cap_{\gamma \in \Gamma} U_\gamma$. On the other hand, since $\cap_{\gamma \in \Gamma} U_\gamma = \cap_{\gamma \in \Lambda \setminus (\Lambda \setminus \Gamma)} U_\gamma \not\subset \cap_{\lambda \in \Lambda \setminus \Gamma} U_\lambda$, we have $\cap_{\gamma \in \Gamma} U_\gamma \not\subset \cap_{\lambda \in \Lambda \setminus \Gamma} U_\lambda$. $\qquad\square$

THEOREM 4.6. *Let U_λ be a maximal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $\cap_{\lambda \in \Lambda} U_\lambda \subsetneqq \cap_{\gamma \in \Gamma} U_\gamma$.*

PROOF. By Corollary 4.5, we have $\cap_{\lambda \in \Lambda} U_\lambda = (\cap_{\lambda \in \Lambda \setminus \Gamma} U_\lambda) \cap (\cap_{\gamma \in \Gamma} U_\gamma) \subsetneqq \cap_{\gamma \in \Gamma} U_\gamma$. $\qquad\square$

THEOREM 4.7 (a decomposition theorem for maximal open set). *Assume that $|\Lambda| \geq 2$. Let U_λ be a maximal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Then, for any element μ of Λ,*

$$U_\mu = \left(\cap_{\lambda \in \Lambda} U_\lambda\right) \cup \left(X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda\right). \tag{4.1}$$

PROOF. Let μ be an element of Λ. By Theorem 4.2(1), we have

$$
\begin{aligned}
\left(\cap_{\lambda \in \Lambda} U_\lambda\right) \cup \left(X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda\right) &= \left(\left(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda\right) \cap U_\mu\right) \cup \left(X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda\right) \\
&= \left(\left(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda\right) \cup \left(X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda\right)\right) \\
&\quad \cap \left(U_\mu \cup \left(X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda\right)\right) \\
&= U_\mu \cup \left(X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda\right) = U_\mu.
\end{aligned}
\tag{4.2}
$$

Therefore, we have $U_\mu = (\cap_{\lambda \in \Lambda} U_\lambda) \cup (X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda)$. $\qquad\square$

THEOREM 4.8. *Let U_λ be a maximal open set for any element λ of a finite set Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\cap_{\lambda \in \Lambda} U_\lambda$ is a closed set, then U_λ is a closed set for any element λ of Λ.*

PROOF. By Theorem 4.7, we have $U_\mu = (\cap_{\lambda \in \Lambda} U_\lambda) \cup (X - \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda) = (\cap_{\lambda \in \Lambda} U_\lambda) \cup (\cup_{\lambda \in \Lambda \setminus \{\mu\}} (X - U_\lambda))$. Since Λ is a finite set, we see that $\cup_{\lambda \in \Lambda \setminus \{\mu\}} (X - U_\lambda)$ is a closed set. Hence, U_μ is a closed set by our assumption. $\square$

As an application of Theorem 4.7, we give another proof of Theorem 4.6.

ANOTHER PROOF OF THEOREM 4.6. Since $\Lambda \supsetneq \Gamma \neq \varnothing$, there exists an element ν of Λ such that $\nu \notin \Gamma$ and an element μ of Γ. If $|\Gamma| = 1$, then we have $\cap_{\lambda \in \Lambda} U_\lambda \subset U_\mu$. If $\cap_{\lambda \in \Lambda} U_\lambda = U_\mu$, then we have $U_\mu \subset U_\lambda$ for any element λ of Λ. Since U_λ is a maximal open set for any element λ of Λ, we have $U_\mu = U_\lambda$, which contradicts our assumption. Hence, we have $\cap_{\lambda \in \Lambda} U_\lambda \subsetneq U_\mu$. If $|\Gamma| \geq 2$, then by Theorem 4.7, we have

$$U_\nu = (\cap_{\lambda \in \Lambda} U_\lambda) \cup (X - \cap_{\lambda \in \Lambda \setminus \{\nu\}} U_\lambda),$$
$$U_\mu = (\cap_{\gamma \in \Gamma} U_\gamma) \cup (X - \cap_{\gamma \in \Gamma \setminus \{\mu\}} U_\gamma). \tag{4.3}$$

If $\cap_{\lambda \in \Lambda} U_\lambda = \cap_{\gamma \in \Gamma} U_\gamma$, then $\cap_{\gamma \in \Gamma} U_\gamma = \cap_{\lambda \in \Lambda} U_\lambda \subset \cap_{\lambda \in \Lambda \setminus \{\nu\}} U_\lambda \subset \cap_{\gamma \in \Gamma} U_\gamma$. Hence, we have $\cap_{\lambda \in \Lambda \setminus \{\nu\}} U_\lambda = \cap_{\gamma \in \Gamma} U_\gamma$. Therefore, $\cap_{\lambda \in \Lambda \setminus \{\nu\}} U_\lambda = \cap_{\gamma \in \Gamma} U_\gamma \subset \cap_{\gamma \in \Gamma \setminus \{\mu\}} U_\gamma$. Hence, we see that $U_\nu \supset U_\mu$. It follows that $U_\nu = U_\mu$ with $\nu \neq \mu$. This contradicts our assumption. $\square$

THEOREM 4.9. *Assume that $|\Lambda| \geq 2$. Let U_λ be a maximal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\cap_{\lambda \in \Lambda} U_\lambda = \varnothing$, then $\{U_\lambda \mid \lambda \in \Lambda\}$ is the set of all maximal open sets of X.*

PROOF. If there exists another maximal open set U_ν of X, which is not equal to U_λ for any element λ of Λ, then $\varnothing = \cap_{\lambda \in \Lambda} U_\lambda = \cap_{\lambda \in (\Lambda \cup \{\nu\}) \setminus \{\nu\}} U_\lambda$. By Theorem 4.2(2), we see that $\cap_{\lambda \in (\Lambda \cup \{\nu\}) \setminus \{\nu\}} U_\lambda \neq \varnothing$. This contradicts our assumption. $\square$

EXAMPLE 4.10. If each point $\{x\}$ is closed (e.g., X is a Hausdorff space or a cofinite space or a cocountable space), then $X - \{a\}$ is a maximal open set for any element a of X. Moreover, we see that $\{X - \{a\} \mid a \in X\}$ is the set of all maximal open sets of X by Theorem 4.9, since $\cap_{a \in X} (X - \{a\}) = \varnothing$.

PROPOSITION 4.11. *Let A and B be subsets of X. If $A \cup B = X$, $A \cap B$ is a closed set, and A is an open set, then B is a closed set.*

PROOF. Since $X - A \subset B$, then we see that

$$(A \cap B) \cup (X - A) = (A \cup (X - A)) \cap (B \cup (X - A)) = B \cup (X - A) = B. \tag{4.4}$$

Since $A \cap B$ and $X - A$ are closed sets, we see that B is a closed set. $\square$

PROPOSITION 4.12. *Let U_λ be an open set for any element λ of Λ and $U_\lambda \cup U_\mu = X$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\cap_{\lambda \in \Lambda} U_\lambda$ is a closed set, then $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$ is a closed set for any element μ of Λ.*

PROOF. Let μ be any element of Λ. Since $U_\lambda \cup U_\mu = X$ for any element λ of Λ with $\lambda \neq \mu$, we have

$$U_\mu \cup \left(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \right) = \cap_{\lambda \in \Lambda \setminus \{\mu\}} \left(U_\mu \cup U_\lambda \right) = X. \tag{4.5}$$

Since $U_\mu \cap \left(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \right) = \cap_{\lambda \in \Lambda} U_\lambda$ is a closed set by our assumption, $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$ is a closed set by Proposition 4.11. $\qquad \square$

THEOREM 4.13. *Let U_λ be a maximal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\cap_{\lambda \in \Lambda} U_\lambda$ is a closed set, then $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$ is a closed set for any element μ of Λ.*

PROOF. By Lemma 2.2(2), we have $U_\lambda \cup U_\mu = X$ for any elements λ and μ of Λ with $\lambda \neq \mu$. By Proposition 4.12, we have that $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$ is a closed set. $\quad \square$

If the assumption of Proposition 4.12 does not hold, then the condition that $\cap_{\lambda \in \Lambda} U_\lambda$ is a closed set does not always imply that $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$ is closed. The following is an example.

EXAMPLE 4.14. Let $X = \{a, b, c, d, e\}$ with topology $\theta = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$, $U_1 = \{a\}$, $U_2 = \{a, b, d\}$, and $U_3 = \{a, c, d\}$. Then, $U_1 \cap U_2 \cap U_3 = U_1$ is a closed set. It follows that $U_1 \cup U_2 = U_2 \neq X$, $U_1 \cup U_3 = U_3 \neq X$, $U_2 \cup U_3 = \{a, b, c, d\} \neq X$. We see that $U_2 \cap U_3 = \{a, d\}$ is not a closed set.

5. More about radicals of maximal open sets. In this section, we study the closure of radicals. We begin with a proposition.

PROPOSITION 5.1. *Let U_λ be a set for any element λ of Λ. If $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) = X$, then $\mathrm{Cl}(U_\lambda) = X$ for any element λ of Λ.*

PROOF. We see that $X = \mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) \subset \mathrm{Cl}(U_\lambda)$. It follows that $\mathrm{Cl}(U_\lambda) = X$ for any element λ of Λ. $\qquad \square$

THEOREM 5.2. *Let U_λ be a maximal open set for any element λ of a finite set Λ. If $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) \neq X$, then there exists an element λ of Λ such that $\mathrm{Cl}(U_\lambda) = U_\lambda$.*

PROOF. Assume that $\mathrm{Cl}(U_\lambda) = X$ for any element λ of Λ. Let μ be an element of Λ. Since $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$ is an open set, we have

$$\begin{aligned}
\mathrm{Cl}\left(\cap_{\lambda \in \Lambda} U_\lambda \right) &= \mathrm{Cl}\left(\left(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \right) \cap U_\mu \right) \supset \left(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \right) \cap \mathrm{Cl}\left(U_\mu \right) \\
&= \left(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \right) \cap X \\
&= \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda.
\end{aligned} \tag{5.1}$$

Hence, $\mathrm{Cl}(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda) \subset \mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda)$. On the other hand, we see that $\cap_{\lambda \in \Lambda} U_\lambda \subset \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda$, and hence $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) \subset \mathrm{Cl}(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda)$. It follows that $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda)$ $= \mathrm{Cl}(\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda)$. Then, by induction on the element of Λ, we see that $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) = \mathrm{Cl}(U_\lambda) = X$ for an element λ of Λ. This contradicts our assumption that $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) \neq X$. Therefore, we see that there exists an element λ of Λ such that $\mathrm{Cl}(U_\lambda) = U_\lambda$. $\square$

Theorem 5.2 is not true when Λ is not a finite set, as we see by the following example. This example also shows that if Λ is not a finite set, then Theorem 4.8 is not always true.

EXAMPLE 5.3. Let $X = \mathbb{R}^n$, the n-dimensional Euclidean space. Let $U_x = X - \{x\}$ for any element $x \in X$. Then, U_x is a maximal open set and we have

$$\mathrm{Cl}\left(\cap_{x \in X} U_x\right) = \mathrm{Cl}(\varnothing) = \varnothing \neq X. \tag{5.2}$$

However, $\mathrm{Cl}(U_x) = X$ for any element x of X.

The radicals of maximal open sets have the following outstanding property.

THEOREM 5.4 (the law of radical closure). *Let Λ be a finite set and U_λ a maximal open set for each element λ of Λ. Let Γ be a subset of Λ such that*

$$\begin{aligned} \mathrm{Cl}(U_\lambda) &= U_\lambda \quad \text{for any } \lambda \in \Gamma, \\ \mathrm{Cl}(U_\lambda) &= X \quad \text{for any } \lambda \in \Lambda \setminus \Gamma. \end{aligned} \tag{5.3}$$

Then, $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) = \cap_{\lambda \in \Gamma} U_\lambda$ *($= X$ if $\Gamma = \varnothing$).*

PROOF. If $\Gamma = \varnothing$, then we have the result by Theorem 5.2. Otherwise $\Gamma \neq \varnothing$, and hence we see that

$$\begin{aligned} \mathrm{Cl}\left(\cap_{\lambda \in \Lambda} U_\lambda\right) &= \mathrm{Cl}\left(\left(\cap_{\lambda \in \Gamma} U_\lambda\right) \cap \left(\cap_{\lambda \in \Lambda \setminus \Gamma} U_\lambda\right)\right) \supset \left(\cap_{\lambda \in \Gamma} U_\lambda\right) \cap \mathrm{Cl}\left(\cap_{\lambda \in \Lambda \setminus \Gamma} U_\lambda\right) \\ &= \left(\cap_{\lambda \in \Gamma} U_\lambda\right) \cap X = \cap_{\lambda \in \Gamma} U_\lambda \end{aligned} \tag{5.4}$$

by Theorem 5.2 and the fact that $\cap_{\lambda \in \Gamma} U_\lambda$ is an open set. It follows that $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) = \mathrm{Cl}(\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda)) \supset \mathrm{Cl}(\cap_{\lambda \in \Gamma} U_\lambda)$. On the other hand, we see that $\cap_{\lambda \in \Lambda} U_\lambda \subset \cap_{\lambda \in \Gamma} U_\lambda$, and hence $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) \subset \mathrm{Cl}(\cap_{\lambda \in \Gamma} U_\lambda)$. It follows that $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) = \mathrm{Cl}(\cap_{\lambda \in \Gamma} U_\lambda)$. The radical $\cap_{\lambda \in \Gamma} U_\lambda$ is a closed set since U_λ is a closed set for any $\lambda \in \Gamma$ by our assumption. Therefore, we see that $\mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) = \cap_{\lambda \in \Gamma} U_\lambda$. $\square$

As an application of Theorem 5.4, we give another proof of Theorem 4.8.

ANOTHER PROOF OF THEOREM 4.8. Let Γ be a subset of Λ such that

$$\begin{aligned} \mathrm{Cl}(U_\lambda) &= U_\lambda \quad \text{for any } \lambda \in \Gamma, \\ \mathrm{Cl}(U_\lambda) &= X \quad \text{for any } \lambda \in \Lambda \setminus \Gamma. \end{aligned} \tag{5.5}$$

We suppose that the radical $\cap_{\lambda \in \Lambda} U_\lambda$ is a closed set. We see that $\Gamma \neq \varnothing$ by Theorem 5.4. Then, $\cap_{\lambda \in \Lambda} U_\lambda = \mathrm{Cl}(\cap_{\lambda \in \Lambda} U_\lambda) = \cap_{\lambda \in \Gamma} U_\lambda$ for the subset Γ of Λ by Theorem 5.4. Then, we see that $\Lambda = \Gamma$ by Theorem 4.6. $\qquad \square$

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