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THE CATEGORY OF LONG EXACT SEQUENCES AND THE HOMOTOPY EXACT SEQUENCE OF MODULES

C. JOANNA SU

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The relative homotopy theory of modules, including the (module) homotopy exact sequence, was developed by Peter Hilton (1965). Our thrust is to produce an alternative proof of the existence of the injective homotopy exact sequence with no reference to elements of sets, so that one can define the necessary homotopy concepts in arbitrary abelian categories with enough injectives and projectives, and obtain, automatically, the projective relative homotopy theory as the dual. Furthermore, we pursue the relative (module) homotopy theory analogously to the absolute (module) homotopy theory. For these purposes, we embed the relative category into the category of long exact sequences, as a full subcategory, in our search for suitable notions of monomorphisms and injectives in the relative category.

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1. Introduction. The relative homotopy theory of modules, including the (module) homotopy exact sequence, was developed by Peter Hilton and stated in [1, Chapter 13]. The approach in this paper produces an alternative proof of the existence of the injective homotopy exact sequence without involving any reference to elements of sets in the arguments, so that one can define the necessary homotopy concepts in arbitrary abelian categories with enough injectives and projectives, and obtain, automatically, the projective relative homotopy theory as the dual.

In addition, having established a few new examples of nontrivial (absolute) homotopy groups of modules in [2], we here pursue the relative (module) homotopy theory analogously to the absolute (module) homotopy theory. For these purposes, to find suitable notions of monomorphisms and injectives in the relative category, we embed the relative category $\mathcal{M}_r$ in the category $\mathcal{E}$ of long exact sequences as a full subcategory and say that a morphism (an object) in $\mathcal{M}_r$ is a monomorphism (an injective) if its image in $\mathcal{E}$ is a monomorphism (an injective).

2. The relative category. In the relative category, denoted $\mathcal{M}_r$, of the category $\mathcal{M}$ of, say, right Λ-modules, where Λ is a unitary ring, the objects are module homomorphisms $\phi : A \to B$ and the morphisms are pairs of module-homomorphisms $(\rho,\sigma) : \phi \to \phi'$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\rho"'] & B \arrow[d, "\sigma"] \\ A' \arrow[r, "\phi'"] & B'. \end{tikzcd} \tag{2.1}$$

We are, particularly, interested in the commutative square

$$\begin{tikzcd} \Sigma^{n-1}A \arrow[r, hook, "\iota_{n-1}"] \arrow[d, "\rho"'] & C\Sigma^{n-1}A \arrow[d, "\sigma"] \\ B_1 \arrow[r, "\beta"] & B_2, \end{tikzcd} \tag{2.2}$$

where CA is an injective container of A, ι_{n-1} is the inclusion map, and ΣA is the suspension of A (see [1, page 134]), for it represents an element of the nth (injective) relative homotopy group $\overline{\pi}_n(A,\beta)$, $n \geq 1$, $\beta : B_1 \to B_2$, which is to be discussed in Section 4.

Since we build up the relative homotopy theory of modules analogously to the absolute homotopy theory of modules (see [1, Chapter 13]), we say that a pair of maps $(\rho,\sigma) : \iota_{n-1} \to \beta$ is i-nullhomotopic if it can be extended to an injective container of ι_{n-1}. Thus, we must look for suitable notions of monomorphisms and injectives in the relative category. If, in (2.1), to say that (ρ,σ) is a monomorphism in $\mathcal{M}_r$ simply required ρ and σ to be monomorphic, one could not expect to obtain injective objects. The following diagram illustrates this situation:

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] \arrow[dd] & & A \arrow[dr, equal] \arrow[dd, hook, "\iota"' near end] & \\ & A \arrow[rr, "\phi'"] \arrow[dl, dashed] & & A \arrow[dl, dashed, hook, "\iota"] \\ 0 \arrow[rr] & & CA \text{ (injective).} & \end{tikzcd} \tag{2.3}$$

However, it is necessary, though not sufficient, that ρ and σ are both monomorphic in $\mathcal{M}$ in the search for monomorphisms in $\mathcal{M}_r$.

The next line of thought is to extend (2.1) to

$$\begin{tikzcd} \ker\phi \arrow[r, hook] \arrow[d, "\rho|"'] & A \arrow[r, "\phi"] \arrow[d, "\rho"'] & B \arrow[r, two heads] \arrow[d, "\sigma"] & \operatorname{coker}\phi \arrow[d, "\sigma'"] \\ \ker\phi' \arrow[r, hook] & A' \arrow[r, "\phi'"] & B' \arrow[r, two heads] & \operatorname{coker}\phi', \end{tikzcd} \tag{2.4}$$

where $\rho|$ is the restriction of ρ and σ' is the induced map of σ. This amounts to embedding $\mathcal{M}_r$ in $\mathcal{E}$, the category of long exact sequences in $\mathcal{M}$, and calling (ρ,σ) a monomorphism if its image in $\mathcal{E}$ is a monomorphism. In other words, we identify the object $\phi : A \to B$ in $\mathcal{M}_r$ with the exact sequence $\ker\phi \hookrightarrow A \xrightarrow{\phi} B \twoheadrightarrow \operatorname{coker}\phi$ in $\mathcal{E}$ and the morphism $(\rho,\sigma) : \phi \to \phi'$ in $\mathcal{M}_r$ with a collection of maps $(\rho|,\rho,\sigma,\sigma')$ in $\mathcal{E}$. It would, then, seem reasonable to regard (ρ,σ) as a monomorphism if $\rho|$, ρ, σ, and σ' are all monomorphisms.

3. The category of long exact sequences. In the category of long exact sequences, denoted $\mathcal{E}$, the objects are long exact sequences in $\mathcal{M}$ and the morphisms are collections of maps of $\mathcal{M}$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots .
\end{tikzcd} \tag{3.1}$$

Note that a *monomorphism* in $\mathcal{E}$ is a collection of monomorphisms; it forces the restrictions on kernel images to be monomorphic. By embedding the relative category $\mathcal{M}_r$ in $\mathcal{E}$, we say that in (2.1), (ρ,σ) is a monomorphism in $\mathcal{M}_r$ if ρ and σ, together with the induced map σ' on cokernels, are all monomorphisms. Moreover, we remark that if we embed $\mathcal{M}$ in $\mathcal{M}_r$ by identifying A with $0 \hookrightarrow A$, a monomorphism in $\mathcal{M}$ is automatically a suitable monomorphism in $\mathcal{M}_r$. Thus, this is a genuine relativization of (module) homotopy theory.

Since the purpose is to define i-nullhomotopy in the relative category, we search for injectives in $\mathcal{E}$. First, we make the following definition.

Definition 3.1. In the long exact sequence $\mathbf{A} : \cdots \to A_{n-1} \to A_n \to A_{n+1} \to \cdots$, if $A_n = 0$ for $n < r$ and $n > s$, the *span* of $\mathbf{A}$ is $[r,s]$.

Notice that we may have $r = -\infty$ or $s = \infty$.

Theorem 3.2. *In the category $\mathcal{E}$ of long exact sequences, the injectives are long exact sequences of injective modules with kernel images also injective.*

Moreover, this category has enough injectives and the span of an injective containing $\mathbf{A}$ *may be taken to be the same as that of the given sequence* $\mathbf{A}$.

We remark that the second half of the theorem is an extension of [1, Proposition 13.13].

Proof. First, let $\mathbf{I}$ be a long exact sequence of injective modules with kernel-images also injective; we show that $\mathbf{I}$ is an injective in $\mathcal{E}$. Suppose given two long exact sequences $\mathbf{A}$ and $\mathbf{B}$ with a monomorphism $\mu : \mathbf{A} \rightarrowtail \mathbf{B}$ and a map $\xi : \mathbf{A} \to \mathbf{I}$, the deduction that the map ξ extends to $\mathbf{B}$ is based on the following diagram:

$$
\begin{tikzcd}
\mathbf{A}: & \cdots \arrow[r] & A_{n-1} \arrow[rr, "\alpha_n"] \arrow[dd, "\xi_{n-1}"] \arrow[dr, "\mu_{n-1}"] & & A_n \arrow[rr, "\alpha_{n+1}"] \arrow[dd, "\xi_n"] \arrow[dr, "\mu_n"] & & A_{n+1} \arrow[r] \arrow[dd, "\xi_{n+1}"] \arrow[dr, "\mu_{n+1}"] & \cdots \\
\mathbf{B}: & \cdots \arrow[rr] & & B_{n-1} \arrow[rr, "\beta_n"] \arrow[dl, dashed, "\phi_{n-1}"] \arrow[ddll, dashed, bend right, "\eta_{n-1}"'] & & B_n \arrow[rr, "\beta_{n+1}"] \arrow[dl, dashed, "\phi_n"] \arrow[ddll, dashed, bend right, "\eta_n"'] & & B_{n+1} \arrow[r] \arrow[ddll, dashed, bend right, "\eta_{n+1}"'] & \cdots \\
\mathbf{I}: & \cdots \arrow[r, "\gamma_{n-1}"] & \underset{\text{(inj.)}}{I_{n-1}} \arrow[rr, "\gamma_n"] \arrow[dr, two heads, "\overline{\gamma}_n"] \arrow[dl, dashed, "\rho_{n-1}"] & & \underset{\text{(inj.)}}{I_n} \arrow[rr, "\gamma_{n+1}"] \arrow[dr, two heads, "\overline{\gamma}_{n+1}"] \arrow[dl, dashed, "\rho_n"] & & \underset{\text{(inj.)}}{I_{n+1}} \arrow[r] \arrow[dl, dashed, "\rho_{n+1}"] & \cdots \\
& \underset{\text{(inj.)}}{\operatorname{Im}\gamma_{n-1}} \arrow[ur, hook, "\overline{\iota}_{n-1}"] & & \underset{\text{(inj.)}}{\operatorname{Im}\gamma_n} \arrow[ul, dashed, "\sigma_{n-1}"] \arrow[ur, hook, "\overline{\iota}_n"] & & \underset{\text{(inj.)}}{\operatorname{Im}\gamma_{n+1}} \arrow[ul, dashed, "\sigma_n"] \arrow[ur, hook, "\overline{\iota}_{n+1}"] &
\end{tikzcd}
\tag{3.2}
$$

In (3.2), supplement the sequence $\mathbf{I}$ by the kernel images so that, for each $n \in \mathbb{Z}$, there is a commutative triangle

$$
\begin{tikzcd}
I_{n-1} \arrow[rr, "\gamma_n"] \arrow[dr, two heads, "\overline{\gamma}_n"'] & & I_n \\
& \operatorname{Im}\gamma_n, \arrow[ur, hook, "\overline{\iota}_n"'] &
\end{tikzcd}
\tag{3.3}
$$

which yields a splitting short exact sequence

$$
\underset{\text{(inj.)}}{\operatorname{Im}\gamma_n} \xhookrightarrow{\overline{\iota}_n} \underset{\text{(inj.)}}{I_n} \xrightarrow{\overline{\gamma}_{n+1}} \underset{\text{(inj.)}}{\operatorname{Im}\gamma_{n+1}}. \tag{3.4}
$$

This means that there exist maps $\rho_n : I_n \to \operatorname{Im}\gamma_n$ and $\sigma_n : \operatorname{Im}\gamma_{n+1} \to I_n$ such that $\rho_n\overline{\iota}_n = 1_{\operatorname{Im}\gamma_n}$, $\overline{\gamma}_{n+1}\sigma_n = 1_{\operatorname{Im}\gamma_{n+1}}$, and $\overline{\iota}_n\rho_n + \sigma_n\overline{\gamma}_{n+1} = 1_{I_n}$. In addition, the map $\rho_n\xi_n$ in the diagram

$$\begin{tikzcd} A_n \arrow[r, tail, "\mu_n"] \arrow[d, "\rho_n\xi_n"'] & B_n \\ \underset{\text{(inj.)}}{\operatorname{Im}\gamma_n} & \end{tikzcd} \tag{3.5}$$

extends to B_n, that is, there is a map $\eta_n : B_n \to \operatorname{Im}\gamma_n$ such that $\eta_n\mu_n = \rho_n\xi_n$.

Based on these, for each n, we define $\phi_n : B_n \to I_n$ via $\phi_n = \bar{\iota}_n\eta_n + \sigma_n\eta_{n+1}\beta_{n+1}$. It remains to show that $\phi_n\mu_n = \xi_n$ and $\gamma_n\phi_{n-1} = \phi_n\beta_n$:

(1)

$$\begin{aligned} \phi_n\mu_n &= (\bar{\iota}_n\eta_n + \sigma_n\eta_{n+1}\beta_{n+1})\mu_n \\ &= \bar{\iota}_n\eta_n\mu_n + \sigma_n\eta_{n+1}\beta_{n+1}\mu_n \\ &= \bar{\iota}_n\rho_n\xi_n + \sigma_n\eta_{n+1}\mu_{n+1}\alpha_{n+1} \\ &= \bar{\iota}_n\rho_n\xi_n + \sigma_n\rho_{n+1}\xi_{n+1}\alpha_{n+1} \\ &= \bar{\iota}_n\rho_n\xi_n + \sigma_n\rho_{n+1}\gamma_{n+1}\xi_n \\ &= \bar{\iota}_n\rho_n\xi_n + \sigma_n\rho_{n+1}\bar{\iota}_{n+1}\bar{\gamma}_{n+1}\xi_n \\ &= \bar{\iota}_n\rho_n\xi_n + \sigma_n\bar{\gamma}_{n+1}\xi_n \\ &= (\bar{\iota}_n\rho_n + \sigma_n\bar{\gamma}_{n+1})\xi_n \\ &= \xi_n; \end{aligned} \tag{3.6}$$

(2)

$$\begin{aligned} \gamma_n\phi_{n-1} &= \bar{\iota}_n\bar{\gamma}_n(\bar{\iota}_{n-1}\eta_{n-1} + \sigma_{n-1}\eta_n\beta_n) \\ &= \bar{\iota}_n\bar{\gamma}_n\sigma_{n-1}\eta_n\beta_n \\ &= \bar{\iota}_n\eta_n\beta_n \\ &= \bar{\iota}_n\eta_n\beta_n + \sigma_n\eta_{n+1}\beta_{n+1}\beta_n \\ &= (\bar{\iota}_n\eta_n + \sigma_n\eta_{n+1}\beta_{n+1})\beta_n \\ &= \phi_n\beta_n. \end{aligned} \tag{3.7}$$

Before deriving the converse, we prove that the category $\mathscr{E}$ has enough injectives; by this, we mean that every object in $\mathscr{E}$ can be embedded in an injective. Let $\mathbf{A} : \cdots \to A_{n-1} \xrightarrow{\alpha_n} A_n \to \cdots$ be in $\mathscr{E}$. For each $n \in \mathbb{Z}$, embed the kernel image $\operatorname{Im}\alpha_n$ in an injective module I_n, so that the long exact sequence $\mathbf{J} : \cdots \to J_{n-1} \xrightarrow{\gamma_n} J_n \to \cdots$, where $J_n = I_n \oplus I_{n+1}$ and γ_n is the expected "rotation" on the summands, is an injective in $\mathscr{E}$. To show that $\mathbf{J}$ is a container of $\mathbf{A}$, we use the facts that the inclusion map $\lambda_n : \operatorname{Im}\alpha_n \hookrightarrow I_n$, $n \in \mathbb{Z}$, extends to A_n by a map named θ_n and the map $\{\theta_n, \theta_{n+1}\alpha_{n+1}\} : A_n \to J_n$ is monomorphic to construct the desired commutative diagram

$$
\begin{tikzcd}[column sep=small]
\mathbf{A}\colon & \cdots \arrow[r] & A_{n-1} \arrow[rr, "\alpha_n"] \arrow[dr, two heads] \arrow[dd, tail, "{\{\theta_{n-1},\theta_n\alpha_n\}}"] & & A_n \arrow[rr, "\alpha_{n+1}"] \arrow[dr, two heads] \arrow[dd, tail, "{\{\theta_n,\theta_{n+1}\alpha_{n+1}\}}"] \arrow[ddl, dashed, "\theta_n"] & & A_{n+1} \arrow[r] \arrow[dd, tail] \arrow[ddl, dashed, "\theta_{n+1}"] & \cdots \\
& & & \operatorname{Im}\alpha_n \arrow[ur, hook] \arrow[dd, hook, "\lambda_n"] & & \operatorname{Im}\alpha_{n+1} \arrow[ur, hook] \arrow[dd, hook, "\lambda_{n+1}"] & & \\
\mathbf{J}\colon & \cdots \arrow[r] & I_{n-1}\oplus I_n \arrow[rr] \arrow[dr, two heads] & & I_n\oplus I_{n+1} \arrow[rr] \arrow[dr, two heads] & & I_{n+1}\oplus I_{n+2} \arrow[r] & \cdots . \\
& & & \underset{\text{(inj.)}}{I_n} \arrow[ur, hook] & & \underset{\text{(inj.)}}{I_{n+1}} \arrow[ur, hook] & &
\end{tikzcd}
\tag{3.8}
$$

About the span of an injective container, the exact sequence $\mathbf{A}$ and a suitably chosen $\mathbf{J}$ end on the left (right) simultaneously because if $A_n = 0$ for $n < r$ $(n > s)$, then $\operatorname{Im}\alpha_n = \operatorname{Im}\alpha_{n+1} = 0$, so that one makes $I_n = I_{n+1} = 0$ since it is a matter of choice.

Finally, by the following diagram, we assure that if $\mathbf{I}$ is an injective in $\mathscr{E}$, it has to be a sequence of injective modules with kernel images also injective for the diagram shows that $\operatorname{Im}\gamma_n$, as a direct summand in $\operatorname{Im}\eta_n$, is injective:

$$
\begin{tikzcd}[column sep=small]
\mathbf{I}\colon & \cdots \arrow[r] & I_{n-1} \arrow[rrrr, "\gamma_n"] \arrow[drr, two heads] \arrow[dddd, equal] \arrow[ddr, tail] & & & & I_n \arrow[r] \arrow[dddd, equal] \arrow[ddr, tail] & \cdots & \\
& & & & \operatorname{Im}\gamma_n \arrow[urr, hook] \arrow[dddd, equal] \arrow[ddr, tail] & & & & \\
\mathbf{J}\colon & \cdots \arrow[rr] & & \underset{\text{(inj.)}}{J_{n-1}} \arrow[rrrr, "\eta_n"] \arrow[dr, two heads] \arrow[ddl, dashed] & & & & \underset{\text{(inj.)}}{J_n} \arrow[r] \arrow[ddl, dashed] & \cdots \\
& & & & & \underset{\text{(inj.)}}{\operatorname{Im}\eta_n} \arrow[urr, hook] \arrow[ddl, dashed] & & & \\
\mathbf{I}\colon & \cdots \arrow[r] & I_{n-1} \arrow[rrrr, "\gamma_n"] \arrow[drr, two heads] & & & & I_n \arrow[r] & \cdots . & \\
& & & & \operatorname{Im}\gamma_n \arrow[urr, hook] & & & &
\end{tikzcd}
\tag{3.9}
$$

$\square$

Since the proof of Theorem 3.2 does not involve any reference to elements of sets, by duality, we introduce epimorphisms and projectives in $\mathscr{E}$ without further argument.

An *epimorphism* in the category of long exact sequences is a collection of epimorphisms, which forces the induced maps on kernel images to be epimorphic.

Theorem 3.3. *In $\mathscr{E}$, the projectives are long exact sequences of projective modules with kernel images also projective.*

Moreover, this category has enough projectives, and the span of a projective $\mathbf{P}$ *over* $\mathbf{A}$ *may be taken to be the same as that of the given sequence* $\mathbf{A}$*.*

In the case where an injective (a projective) in $\mathscr{E}$ ends on the left (right), we have the following corollary.

Corollary 3.4. *In the category of long exact sequences $\mathscr{E}$,*

(i) *a sequence $0 \to I_1 \to I_2 \to \cdots \to I_n \to \cdots$ is an injective if and only if each I_n is an injective module;*

(ii) *a sequence $\cdots \to P_n \to \cdots \to P_2 \to P_1 \to 0$ is a projective if and only if each P_n is a projective module.*

We remark that Corollary 3.4(i) is [1, Proposition 13.14].

4. Homotopy in the relative category. As mentioned in Section 2, we will embed the relative category $\mathcal{M}_r$ in the category of long exact sequences $\mathscr{E}$ as a full subcategory. Thus, diagram (2.1) is essentially diagram (2.4); the map $(\rho,\sigma):\phi \to \phi'$ is a monomorphism in $\mathcal{M}_r$ if ρ and σ, together with the induced map σ', are all monomorphisms, and $A \xrightarrow{\phi} B$ is an injective if A and B, together with $\ker\phi$ and $\operatorname{coker}\phi$, are all injective modules.

Notice that these extra criteria are not automatic, even for abelian groups. For instance, in the diagram

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d] & \mathbb{Z} \arrow[r, "\times 6"] \arrow[d, tail, "\times 3"'] & \mathbb{Z} \arrow[r, two heads] \arrow[d, tail, "\times 2"] & \mathbb{Z}_6 \arrow[d] \\
0 \arrow[r, hook] & \mathbb{Z} \arrow[r, "\times 4"] & \mathbb{Z} \arrow[r, two heads] & \mathbb{Z}_4,
\end{tikzcd} \tag{4.1}$$

the induced map $\mathbb{Z}_6 \to \mathbb{Z}_4$ is clearly not monomorphic; in the exact sequence $\mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$, the kernel $\mathbb{Z}$ is not injective although $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are.

Hereafter, diagram (2.2) becomes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{n-1}A \arrow[r, hook, "\iota_{n-1}"] \arrow[d, "\rho"'] & C\Sigma^{n-1}A \arrow[r, two heads, "\epsilon_n"] \arrow[d, "\sigma"] & \Sigma^n A \arrow[d, "\sigma'"] \\
\ker\beta \arrow[r, hook] & B_1 \arrow[r, "\beta"] & B_2 \arrow[r, two heads] & \operatorname{coker}\beta,
\end{tikzcd} \tag{4.2}$$

where ι_{n-1} is the inclusion map and ϵ_n is the quotient map, and Theorem 3.2 lets us pursue the relative homotopy theory analogously to the absolute theory (see [1, Chapter 13]). We first quote the definitions of i-nullhomotopy and the nth relative homotopy group (see [1, page 142]).

Definition 4.1. The map $(\sigma,\rho):\iota_{n-1}\to\beta$ is *i-nullhomotopic*, denoted by $(\rho,\sigma)\simeq_i 0$, if it can be extended to an injective container of ι_{n-1}.

Definition 4.2. Suppose given a map $\beta:B_1\to B_2$, the nth *(injective) relative homotopy group*, $n\geq 1$, is $\overline{\pi}_n(A,\beta)=\operatorname{Hom}(\iota_{n-1},\beta)/\operatorname{Hom}_0(\iota_{n-1},\beta)$, where $\operatorname{Hom}(\iota_{n-1},\beta)$ is the abelian group of maps of ι_{n-1} to β, and $\operatorname{Hom}_0(\iota_{n-1},\beta)$ is the subgroup consisting of i-nullhomotopic maps.

We remark that, by duality, given a map $\alpha:A_1\to A_2$, one can define *p-nullhomotopy* and the nth *(projective) relative homotopy group* $\underline{\pi}_n(\alpha,B)$ accordingly.

5. The homotopy exact sequence of modules. We introduce the relative homotopy groups in module theory in order to imitate the homotopy exact sequence in topology. As expected, in the injective relative (module) homotopy theory, let $\beta:B_1\to B_2$ be a map, there is then an exact sequence of the map β (see [1, Theorem 13.15]).

Theorem 5.1. *Suppose given a map $\beta:B_1\to B_2$. Then, there exists, for each A, a homotopy exact sequence*

$$\begin{aligned}\cdots\xrightarrow{\partial}\overline{\pi}_n(A,B_1)\xrightarrow{\beta_*}\overline{\pi}_n(A,B_2)\xrightarrow{J}\overline{\pi}_n(A,\beta)\xrightarrow{\partial}\overline{\pi}_{n-1}(A,B_1)\xrightarrow{\beta_*}\cdots\\ \xrightarrow{\partial}\overline{\pi}_1(A,B_1)\xrightarrow{\beta_*}\overline{\pi}_1(A,B_2)\xrightarrow{J}\overline{\pi}(A,\beta)\xrightarrow{\partial}\overline{\pi}(A,B_1)\xrightarrow{\beta_*}\overline{\pi}(A,B_2).\end{aligned}\tag{5.1}$$

We remark that a proof of this theorem is given in [1]. We produce an alternative proof without any reference to elements of sets, so that it is suitable for arbitrary abelian categories with enough injectives and projectives, and one can define the necessary homotopy concepts. Especially, the dual, the homotopy exact sequence in the projective relative homotopy theory, arises automatically.

The approach is, first, to prove the special case when the map β is a monomorphism, then to expand to Theorem 5.1 the general case by exploiting the mapping cylinder of β. We start with a couple of propositions; Proposition 5.2 is an evident relativization.

Proposition 5.2. *The following are equivalent:*

(i) $(\rho,\sigma)\simeq_i 0$;
(ii) (ρ,σ) *can be extended to every object containing ι_{n-1};*
(iii) (ρ,σ) *can be factored through some injective object.*

Proposition 5.3. *In* $\mathrm{Hom}(\iota_{n-1},\beta)$*, when* β *is monomorphic,* $(\rho,\sigma)\simeq_i 0$ *if and only if* $\sigma=\beta\theta+\chi\iota_n\epsilon_n$ *for some* $\theta : C\Sigma^{n-1}A\to B_1$ *and* $\chi : C\Sigma^n A\to B_2$*:*

$$
\begin{tikzcd}
\Sigma^{n-1}A \arrow[r, hook, "\iota_{n-1}"] \arrow[d, "\rho"'] & C\Sigma^{n-1}A \arrow[r, two heads, "\epsilon_n"] \arrow[d, "\sigma"] \arrow[dl, dashed, "\theta"'] & \Sigma^n A \arrow[r, hook, "\iota_n"] \arrow[d, "\sigma'"] & C\Sigma^n A \arrow[dll, dashed, "\chi"'] \\
B_1 \arrow[r, tail, "\beta"] & B_2 \arrow[r, two heads, "\kappa"] & B_{12}\,(=\operatorname{coker}\beta). &
\end{tikzcd}
\tag{5.2}
$$

Proof.

$$
\begin{tikzcd}
\Sigma^{n-1}A \arrow[rr, hook, "\iota_{n-1}"] \arrow[dd, "\rho"'] \arrow[dr, hook, "\iota_{n-1}"] & & C\Sigma^{n-1}A \arrow[rr, two heads, "\epsilon_n"] \arrow[dd, "\sigma"] \arrow[dr, "{\{1,\iota_n\epsilon_n\}}"] & & \Sigma^n A \arrow[dd, "\sigma'"] \arrow[dr, hook, "\iota_n"] & \\
& C\Sigma^{n-1}A \arrow[rr, hook] \arrow[dl, dashed, "\theta"] & & C\Sigma^{n-1}A\oplus C\Sigma^n A \arrow[rr, two heads] \arrow[dl, dashed, "{\langle\nu,\chi\rangle}"] & & C\Sigma^n A \arrow[dl, dashed, "\eta"] \\
B_1 \arrow[rr, tail, "\beta"] & & B_2 \arrow[rr, two heads, "\kappa"] & & B_{12}\,(=\operatorname{coker}\beta). &
\end{tikzcd}
\tag{5.3}
$$

Assume that $(\rho,\sigma)\simeq_i 0$; by Proposition 5.2(ii), (ρ,σ) must extend to the container $C\Sigma^{n-1}A\hookrightarrow C\Sigma^{n-1}A\oplus C\Sigma^n A\twoheadrightarrow C\Sigma^n A$, in which the maps are the canonical inclusion into the first factor and the projection onto the second factor, respectively. This assures the existence of the maps θ, $\langle\nu,\chi\rangle$, and η in the commutative diagram (5.3), and the facts that $\beta\theta=\nu$ and $\sigma=\nu 1+\chi\iota_n\epsilon_n$ force $\sigma=\beta\theta+\chi\iota_n\epsilon_n$.

Conversely, if $\sigma=\beta\theta+\chi\iota_n\epsilon_n$ for some $\theta : C\Sigma^{n-1}A\to B_1$ and $\chi : C\Sigma^n A\to B_2$, by letting $\nu=\beta\theta$ and $\eta=\kappa\chi$, we have $\beta\theta\iota_{n-1}=\nu\iota_{n-1}=\langle\nu,\chi\rangle\circ\{1,\iota_n\epsilon_n\}\circ\iota_{n-1}=\sigma\iota_{n-1}=\beta\rho$, so that $\theta\iota_{n-1}=\rho$. Therefore, the map (ρ,σ) factors through the injective $C\Sigma^{n-1}A\hookrightarrow C\Sigma^{n-1}A\oplus C\Sigma^n A\twoheadrightarrow C\Sigma^n A$; thus, it is i-nullhomotopic by Proposition 5.2(iii). □

Proof of Theorem 5.1. Assume, first, that the map β is monomorphic. We construct an injective resolution of A; thus,

$$
\begin{tikzcd}
\underline{C} : A \arrow[r, hook, "\iota_0"] & CA \arrow[rr, "\alpha_1"] \arrow[dr, "\epsilon_1"'] & & C\Sigma A \arrow[r, "\alpha_2"] \arrow[dr, "\epsilon_2"] & \cdots \arrow[r, "\alpha_{n-1}"] & C\Sigma^{n-1}A \arrow[rr, "\alpha_n"] \arrow[dr, "\epsilon_n"'] & & C\Sigma^n A \arrow[r, "\alpha_{n+1}"] & \cdots . \\
& & \Sigma A \arrow[ur, hook, "\iota_1"'] & & \Sigma^2 A & & \Sigma^n A \arrow[ur, hook, "\iota_n"'] & &
\end{tikzcd}
\tag{5.4}
$$

When applying the functor $\mathrm{Hom}_\Lambda(\underline{C},-)$ to β, there arises a short exact sequence of chain complexes; thus,

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(\underline{C},B_1) \arrow[d, tail, "\beta_*"] \\
\mathrm{Hom}_\Lambda(\underline{C},B_2) \arrow[d, two heads, "\text{quotient map}"] \\
\mathrm{Hom}_\Lambda(\underline{C},B_2)/\operatorname{image}\beta_*.
\end{tikzcd} \tag{5.5}$$

To conclude that, when β is monomorphic, the induced homology sequence coincides with the homotopy exact sequence (5.1); it suffices to show that in the third complex,

$$\ker\alpha_{n-1}^*/\operatorname{image}\alpha_n^* \cong \overline{\pi}_n(A,\beta), \quad n \geq 1, \text{ naturally}. \tag{5.6}$$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \mathrm{Hom}_\Lambda(C\Sigma^n A,B_1) \arrow[rr, "\alpha_n^*"] \arrow[dr, "\iota_n^*"] \arrow[dd, tail, "\beta_*"] & & \mathrm{Hom}_\Lambda(C\Sigma^{n-1} A,B_1) \arrow[rr, "\alpha_{n-1}^*"] \arrow[dr, "\iota_{n-1}^*"] \arrow[dd, tail, "\beta_*"] & & \mathrm{Hom}_\Lambda(C\Sigma^{n-2} A,B_1) \arrow[r] \arrow[dd, tail, "\beta_*"] & \cdots \\
& & \mathrm{Hom}_\Lambda(\Sigma^n A,B_1) \arrow[ur, tail, "\epsilon_n^*"] \arrow[dd, tail, "\beta_*"] & & \mathrm{Hom}_\Lambda(\Sigma^{n-1} A,B_1) \arrow[ur, tail, "\epsilon_{n-1}^*"] \arrow[dd, tail, "\beta_*"] & & \\
\cdots \arrow[r] & \mathrm{Hom}_\Lambda(C\Sigma^n A,B_2) \arrow[rr, "\alpha_n^*"] \arrow[dr, "\iota_n^*"] \arrow[dd, two heads] & & \mathrm{Hom}_\Lambda(C\Sigma^{n-1} A,B_2) \arrow[rr, "\alpha_{n-1}^*"] \arrow[dr, "\iota_{n-1}^*"] \arrow[dd, two heads] & & \mathrm{Hom}_\Lambda(C\Sigma^{n-2} A,B_2) \arrow[r] \arrow[dd, two heads] & \cdots \\
& & \mathrm{Hom}_\Lambda(\Sigma^n A,B_2) \arrow[ur, tail, "\epsilon_n^*"] \arrow[dd, two heads] & & \mathrm{Hom}_\Lambda(\Sigma^{n-1} A,B_2) \arrow[ur, tail, "\epsilon_{n-1}^*"] \arrow[dd, two heads] & & \\
\cdots \arrow[r] & \dfrac{\mathrm{Hom}_\Lambda(C\Sigma^n A,B_2)}{\operatorname{image}\beta_*} \arrow[rr, "\alpha_n^*"] \arrow[dr, dashed, "\iota_n^*"] & & \dfrac{\mathrm{Hom}_\Lambda(C\Sigma^{n-1} A,B_2)}{\operatorname{image}\beta_*} \arrow[rr, "\alpha_{n-1}^*"] \arrow[dr, dashed, "\iota_{n-1}^*"] & & \dfrac{\mathrm{Hom}_\Lambda(C\Sigma^{n-2} A,B_2)}{\operatorname{image}\beta_*} \arrow[r] & \cdots. \\
& & \dfrac{\mathrm{Hom}_\Lambda(\Sigma^n A,B_2)}{\operatorname{image}\beta_*} \arrow[ur, dashed, tail, "\epsilon_n^*"] & & \dfrac{\mathrm{Hom}_\Lambda(\Sigma^{n-1} A,B_2)}{\operatorname{image}\beta_*} \arrow[ur, dashed, tail, "\epsilon_{n-1}^*"] & &
\end{tikzcd} \tag{5.7}$$

To prove (5.6), first, we pick $\phi : C\Sigma^{n-1}A \to B_2$, for which the equivalence class $[\phi] \in \ker \alpha^*_{n-1}$. Since ϵ^*_{n-1} is monomorphic, $\ker \alpha^*_{n-1} \cong \ker \iota^*_{n-1}$ which yields $\phi\iota_{n-1} = \beta\nu$ for some $\nu : \Sigma^{n-1}A \to B_1$. In addition, since β is monomorphic, the map ν is uniquely determined; call it $\phi|$. Therefore, the map ϕ forces a commutative square

$$\begin{tikzcd} \Sigma^{n-1}A \arrow[r, hook, "\iota_{n-1}"] \arrow[d, "\phi|"'] & C\Sigma^{n-1}A \arrow[d, "\phi"] \\ B_1 \arrow[r, tail, "\beta"] & B_2 \end{tikzcd} \tag{5.8}$$

which represents an element in $\overline{\pi}_n(A,\beta)$; we thus define $\xi : \ker \alpha^*_{n-1} / \operatorname{image} \alpha^*_n \to \overline{\pi}_n(A,\beta)$ via $\xi[\phi] = [(\phi|,\phi)]$ and prove that it is an isomorphism. Suppose a given $[\phi]$ such that $(\phi|,\phi) \simeq_i 0$. Then, by Proposition 5.3, there are maps $\theta : C\Sigma^{n-1}A \to B_1$ and $\chi : C\Sigma^n A \to B_2$ such that $\phi = \beta\theta + \chi\iota_n\epsilon_n = \beta_*(\theta) + \alpha^*_n(\chi)$. Thus, $[\phi] = 0$ and ξ is a monomorphism. On the other hand, let $[(\rho,\sigma)] \in \overline{\pi}_n(A,\beta)$; then, $\xi[\sigma] = [(\sigma|,\sigma)] = [(\rho,\sigma)]$; so, ξ is epimorphic. This completes the proof if β is a monomorphism.

Now, if the map β is arbitrary, we apply the mapping cylinder of $\beta : B_1 \to B_2$; thus, $\{\iota,\beta\} : B_1 \rightarrowtail CB_1 \oplus B_2$, where CB_1 is an injective container of B_1 and $\iota : B_1 \hookrightarrow CB_1$ is the inclusion map. It yields a monomorphism $\{\iota,\beta\}$ equivalent to β and a short exact sequence

$$B_1 \xrightarrow{\{\iota,\beta\}} CB_1 \oplus B_2 \xrightarrow{\kappa} B_{12}, \tag{5.9}$$

where κ is the quotient map and $B_{12} = \operatorname{coker}\{\iota,\beta\}$, which leads to a homotopy exact sequence; thus,

$$\begin{aligned} &\cdots \xrightarrow{\partial} \overline{\pi}_n(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \overline{\pi}_n(A,CB_1 \oplus B_2) \xrightarrow{J} \overline{\pi}_n(A,\{\iota,\beta\}) \xrightarrow{\partial} \overline{\pi}_{n-1}(A,B_1) \\ &\xrightarrow{\{\iota,\beta\}_*} \cdots \xrightarrow{\partial} \overline{\pi}_1(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \overline{\pi}_1(A,CB_1 \oplus B_2) \xrightarrow{J} \overline{\pi}_1(A,\{\iota,\beta\}) \\ &\xrightarrow{\partial} \overline{\pi}(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \overline{\pi}(A,CB_1 \oplus B_2). \end{aligned} \tag{5.10}$$

Since CB_1 is injective, $\overline{\pi}_n(A,CB_1 \oplus B_2) \cong \overline{\pi}_n(A,B_2)$ for $n \geq 0$. Thus, to conclude that the homotopy exact sequence (5.10) of $\{\iota,\beta\}$ is canonically isomorphic to the homotopy exact sequence (5.1) of β, we show that $\overline{\pi}_n(A,\{\iota,\beta\}) \cong \overline{\pi}_n(A,\beta)$, $n \geq 1$.

Note that the diagrams

$$\begin{tikzcd}
\Sigma^{n-1}A \arrow[r, hook, "\iota_{n-1}"] \arrow[d, "\rho"'] & C\Sigma^{n-1}A \arrow[d, "\{\gamma,\sigma\}"] \\
B_1 \arrow[r, tail, "\{\iota,\beta\}"] & CB_1 \oplus B_2,
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^{n-1}A \arrow[r, hook, "\iota_{n-1}"] \arrow[d, "\rho"'] & C\Sigma^{n-1}A \arrow[d, "\sigma"] \\
B_1 \arrow[r, "\beta"] & B_2,
\end{tikzcd}
\tag{5.11}$$

where $\gamma : C\Sigma^{n-1}A \to CB_1$ is an arbitrary extension of $\iota\rho$, represent elements in $\overline{\pi}_n(A,\{\iota,\beta\})$ and $\overline{\pi}_n(A,\beta)$, respectively. We thus define the map $\chi : \overline{\pi}_n(A,\{\iota,\beta\}) \to \overline{\pi}_n(A,\beta)$ by assigning the equivalence class $[(\rho,\{\gamma,\sigma\})]$ to $[(\rho,\sigma)]$.

To assure that χ is well defined, suppose a given $[(\rho,\{\gamma,\sigma\})]$ such that $(\rho,\{\gamma,\sigma\}) \simeq_i 0$. By Proposition 5.3, there exist maps $\theta : C\Sigma^{n-1}A \to B_1$ and $\{\eta_1,\eta_2\} : C\Sigma^n A \to CB_1 \oplus B_2$ such that $\{\gamma,\sigma\} = \{\iota,\beta\}\theta + \{\eta_1,\eta_2\}\iota_n\epsilon_n$. The facts that $\{\iota\rho,\beta\rho\} = \{\iota,\beta\}\circ\rho = \{\gamma,\sigma\}\circ\iota_{n-1} = \{\iota\theta+\eta_1\iota_n\epsilon_n, \beta\theta+\eta_2\iota_n\epsilon_n\}\circ\iota_{n-1} = \{\iota\theta\iota_{n-1},\beta\theta\iota_{n-1}\}$ and that ι is monomorphic force $\rho = \theta\iota_{n-1}$, and thus imply the existence of the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] \arrow[dr] \arrow[dd] & & \Sigma^{n-1}A \arrow[rr, hook, "\iota_{n-1}"] \arrow[dr, hook, "\iota_{n-1}"] \arrow[dd, "\rho"' near start] & & C\Sigma^{n-1}A \arrow[rr, two heads, "\epsilon_n"] \arrow[dr, "\{1,\iota_n\epsilon_n\}"] \arrow[dd, "\sigma"' near start] & & \Sigma^n A \arrow[dr, hook, "\iota_n"] \arrow[dd, "\sigma'"' near start] & \\
& 0 \arrow[rr] \arrow[dl] & & C\Sigma^{n-1}A \arrow[rr, hook] \arrow[dl, "\theta"] & & C\Sigma^{n-1}A \oplus C\Sigma^n A \arrow[rr, two heads] \arrow[dl, "\langle\beta\theta,\eta_2\rangle"] & & C\Sigma^n A \arrow[dl, "\kappa\eta_2"] \\
\ker\beta \arrow[rr, hook] & & B_1 \arrow[rr, "\beta"] & & B_2 \arrow[rr, two heads, "\kappa"] & & \operatorname{coker}\beta. &
\end{tikzcd}
\tag{5.12}$$

Then, by Proposition 5.2(iii), $(\rho,\sigma) \simeq_i 0$.

To show that χ is monomorphic, let $[(\rho,\{\gamma,\sigma\})] \in \ker\chi$; then, $(\rho,\sigma) \simeq_i 0$ and Proposition 5.2(ii) gives us the existence of diagram (5.12). In addition, since $(\gamma-\iota\theta)\iota_{n-1} = \gamma\iota_{n-1} - \iota\theta\iota_{n-1} = \iota\rho - \iota\rho = 0$, we obtain a commutative diagram; thus,

$$\begin{tikzcd}
\Sigma^{n-1}A \arrow[r, hook, "\iota_{n-1}"] \arrow[d, "0"'] & C\Sigma^{n-1}A \arrow[r, two heads, "\epsilon_n"] \arrow[d, "\gamma-\iota\theta"] & \Sigma^n A \arrow[d] \arrow[dl, dashed, "\nu"] \\
B_1 \arrow[r, hook, "\iota"] & CB_1 \arrow[r, two heads] & \operatorname{coker}\iota
\end{tikzcd}
\tag{5.13}$$

in which the map $\gamma - \iota\theta$ factors through ϵ_n by a unique map $\nu : \Sigma^n A \to CB_1$. Furthermore, since CB_1 is injective, the map ν extends to $C\Sigma^n A$ by a map η_1. Hence, $\gamma = \iota\theta + \nu\epsilon_n = \iota\theta + \eta_1\iota_n\epsilon_n$, which yields $\{\gamma,\sigma\} = \{\iota,\beta\}\theta + \{\eta_1,\eta_2\}\iota_n\epsilon_n$, so that $(\rho,\{\gamma,\sigma\}) \simeq_i 0$ by Proposition 5.3.

Finally, if $[(\rho,\sigma)] \in \overline{\pi}_n(A,\beta)$, since CB_1 is injective, the map $\iota\rho : \Sigma^{n-1}A \to CB_1$ extends to $C\Sigma^{n-1}A$ by a map γ. Thus, there exists the equivalence class $[(\rho,\{\gamma,\sigma\})] \in \overline{\pi}_n(A,\{\iota,\beta\})$, that is, the preimage of $[(\rho,\sigma)]$. The proof of Theorem 5.1 is complete. □

One final remark is that our arguments do not involve any elements of sets, so that, by duality, one can easily proceed with the projective relative (module) homotopy theory without further argument. As an example, the statement dual to that of Theorem 5.1 is the following theorem.

Theorem 5.4. *Suppose given a map $\alpha : A_1 \to A_2$. Then, there exists, for each B, a (projective) homotopy exact sequence*

$$\begin{gathered} \cdots \xrightarrow{\partial} \underline{\pi}_n(A_2,B) \xrightarrow{\alpha^*} \underline{\pi}_n(A_1,B) \xrightarrow{J'} \underline{\pi}_n(\alpha,B) \xrightarrow{\partial} \underline{\pi}_{n-1}(A_2,B) \xrightarrow{\alpha^*} \cdots \\ \xrightarrow{\partial} \underline{\pi}_1(A_2,B) \xrightarrow{\alpha^*} \underline{\pi}_1(A_1,B) \xrightarrow{J'} \underline{\pi}_1(\alpha,B) \xrightarrow{\partial} \underline{\pi}(A_2,B) \xrightarrow{\alpha^*} \underline{\pi}(A_1,B). \end{gathered} \tag{5.14}$$

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References

[1] P. Hilton, *Homotopy Theory and Duality*, Gordon and Breach Science Publishers, New York, 1965.

[2] C. Joanna Su, *Some examples of nontrivial homotopy groups of modules*, Int. J. Math. Math. Sci. **27** (2001), no. 3, 189–195.

C. Joanna Su: Department of Mathematics and Computer Science, Providence College, Providence, RI 02918, USA

E-mail address: jsu@providence.edu