

GROUP METHOD ANALYSIS OF TWO-DIMENSIONAL PLATE IN HEAT FLUX

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Received 24 June 2002

The group transformation theoretic approach is applied to present an analytic study of the temperature distribution in a triangular plate, Ω , placed in the field of heat flux, along one boundary, in a form of polynomial functions of any degree “ n .” The Laplace’s equation has been reduced to second-order linear ordinary differential equation with an appropriate boundary conditions. Exact solution has been obtained for general shape of Ω and different boundary conditions.

2000 Mathematics Subject Classification: 58J35, 54H15.

1. Introduction. The Laplace’s equation arises in many branches of physics, from which it attracts a wide band of researchers. Electrostatic potential, temperature in the case of steady state heat conduction, velocity potential in the case of steady irrotational flow of an ideal fluid, concentration of a substance that is diffusing through a solid, and the displacements of a two-dimensional membrane in equilibrium state are counter examples in which the Laplace’s equation is satisfied.

The mathematical technique used in the present analysis is the parameter-group transformation. The group methods, as a class of methods lead to the reduction of the number of independent variables, were first introduced by Birkhoff [6] in 1948, where he made use of one-parameter transformation groups. In 1952, Morgan [9] presented a theory which has led to improvements over earlier similarity methods. The method has been applied intensively by Abd-el-Malek et al. [1, 2, 3, 4, 5, 7].

In this paper, we present a general procedure for applying one-parameter group transformation to the Laplace’s equation in a triangular domain. Under the transformation, the partial differential equation with boundary conditions in polynomial form, of any degree, is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is then solved analytically for the general form of the triangular domain and boundary conditions.

2. Mathematical formulation. The governing equation for the distribution of temperature $T(x, y)$ is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad (x, y) \in \Omega \quad (2.1)$$

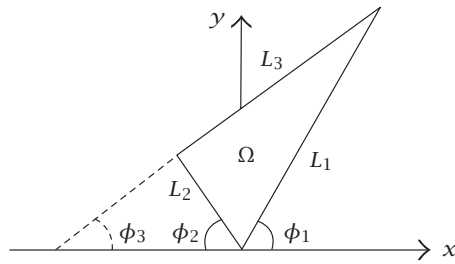


FIGURE 2.1. Geometrical configuration of the problem.

with the following boundary conditions:

$$\begin{aligned} T(x, y) &= \alpha q(x), \quad (x, y) \in L_1, \\ \frac{\partial T}{\partial y} &= \beta x^n, \quad (x, y) \in L_2. \end{aligned} \quad (2.2)$$

It is required to find the distribution of the temperature $T(x, y)$ inside the domain Ω , defined in Figure 2.1, and the heat flux across L_3 , where

$$\begin{aligned} L_1 : y &= x \tan \Phi_1, \\ L_2 : y &= -x \tan \Phi_2, \\ L_3 : y &= x \tan \Phi_3 + b, \quad b \neq 0, n \in \{0, 1, 2, 3, \dots\}, \alpha \text{ and } \beta \text{ are constants.} \end{aligned} \quad (2.3)$$

Write

$$T(x, y) = w(x, y)q(x), \quad q(x) \neq 0 \text{ in } \Omega, \quad (2.4)$$

by which differential equation (2.1) takes the form

$$q(x) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 2 \frac{\partial w}{\partial x} \frac{dq}{dx} + w \frac{d^2 q}{dx^2} = 0, \quad (2.5)$$

and the boundary conditions (2.2) take the form

$$\begin{aligned} w(x, y) &= \alpha, \quad (x, y) \in L_1, \\ \frac{\partial w}{\partial y} &= \frac{\beta x^n}{q(x)}, \quad (x, y) \in L_2. \end{aligned} \quad (2.6)$$

3. Solution of the problem. The method of solution depends on the application of a one-parameter group transformation to the partial differential equation (2.1). Under this transformation, the two independent variables will be reduced by one and the differential equation (2.1) transforms into an ordinary differential equation in only one independent variable, which is the similarity variable.

3.1. The group systematic formulation. The procedure is initiated with the group G , a class of transformation of one-parameter “ a ” of the form

$$G : \bar{S} = C^S(a)S + K^S(a), \tag{3.1}$$

where S stands for x, y, w , and q and the C 's and K 's are real valued and at least differentiable in the real argument “ a .”

3.2. The invariance analysis. To transform the differential equation, transformations of the derivatives of w and q are obtained from G via chain-rule operations

$$\bar{S}_i = \left(\frac{C^S}{C^i}\right)S_i, \quad \bar{S}_{ij} = \left(\frac{C^S}{C^i C^j}\right)S_{ij}, \quad i = x, y; j = x, y, \tag{3.2}$$

where S stands for w and q .

Equation (2.5) is said to be invariantly transformed, for some function $H_1(a)$, whenever

$$\bar{q}(\bar{w}_{\bar{x}\bar{x}} + \bar{w}_{\bar{y}\bar{y}}) + 2\bar{w}_{\bar{x}}\bar{q}_{\bar{x}} + \bar{w}\bar{q}_{\bar{x}\bar{x}} = H_1(a)[q(w_{xx} + w_{yy}) + 2w_x q_x + w q_{xx}]. \tag{3.3}$$

Substitution from (3.1) into (3.3) yields

$$\begin{aligned} & q\left(\left[\frac{C^q C^w}{(C^x)^2}\right]w_{xx} + \left[\frac{C^q C^w}{(C^y)^2}\right]w_{yy}\right) \\ & + 2\left[\frac{C^q C^w}{(C^x)^2}\right]w_x q_x + \left[\frac{C^q C^w}{(C^x)^2}\right]w q_{xx} + \zeta_1(a) \\ & = H_1(a)[q(w_{xx} + w_{yy}) + 2w_x q_x + w q_{xx}], \end{aligned} \tag{3.4}$$

where

$$\zeta_1(a) = (K^q C^w)\left(\frac{w_{xx}}{(C^x)^2} + \frac{w_{yy}}{(C^y)^2}\right) + \left[\frac{K^w C^q}{(C^x)^2}\right]q_{xx}. \tag{3.5}$$

The invariance of (3.4) implies that $\zeta_1(a) \equiv 0$. This is satisfied by putting

$$K^q = K^w = 0, \quad \left[\frac{C^q C^w}{(C^x)^2}\right] = \left[\frac{C^q C^w}{(C^y)^2}\right] = H_1(a), \tag{3.6}$$

which yields

$$C^x = C^y. \tag{3.7}$$

Moreover, the boundary conditions (2.6) are also invariant in form, implying that

$$K^x = K^q = K^w = 0, \quad C^q C^w = (C^x)^{n+1}, \quad C^w = 1. \tag{3.8}$$

Finally, we get the one-parameter group G which transforms invariantly the differential equation (2.5) and the boundary conditions (2.6). The group G is of the form

$$G: \begin{cases} \bar{x} = C^x x, \\ \bar{y} = C^x y + K^y, \\ \bar{w} = w, \\ \bar{q} = [(C^x)^{n+1}]q. \end{cases} \tag{3.9}$$

3.3. The complete set of absolute invariants. Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation. Then, we have to proceed in our analysis to obtain a complete set of absolute invariants.

If $\eta \equiv \eta(x, y)$ is the absolute invariant of the independent variables, then

$$g_j(x, y; w, q) = F_j[\eta(x, y)]; \quad j = 1, 2 \tag{3.10}$$

are the two absolute invariants corresponding to w and q . The application of a basic theorem in group theory, see [8], states that *a function $g(x, y; w, q)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation:*

$$\sum_{i=1}^4 (\alpha_i S_i + \beta_i) \frac{\partial g}{\partial S_i} = 0, \tag{3.11}$$

where S_i stands for x, y, w , and q , respectively, and

$$\alpha_i = \frac{\partial C^{S_i}}{\partial a}(a^0), \quad \beta_i = \frac{\partial K^{S_i}}{\partial a}(a^0), \quad i = 1, 2, 3, 4 \tag{3.12}$$

and a^0 denotes the value of “ a ” which yields the identity element of the group.

From (3.6), (3.7), (3.8), and (3.12), we get

$$\alpha_1 = \alpha_2, \quad \alpha_3 = 0, \quad \beta_1 = \beta_3 = \beta_4 = 0. \tag{3.13}$$

We take $\beta_2 = 0$.

Owing to (3.11), $\eta(x, y)$ is an absolute invariant if it satisfies

$$x \frac{\partial \eta}{\partial x} + y \frac{\partial \eta}{\partial y} = 0, \tag{3.14}$$

which has a solution in the form

$$\eta(x, y) = \frac{y}{x}. \tag{3.15}$$

To obtain the absolute invariant of the dependent variables w and q , apply (3.11), we get

$$q(x) = R(x)\theta(\eta). \tag{3.16}$$

Since $q(x)$ and $R(x)$ are independent of y , while η is a function of x and y , then $\theta(\eta)$ must be a constant, say $\theta(\eta) = 1$, hence

$$q(x) = R(x), \tag{3.17}$$

and the second absolute invariant is

$$w(x, y) = F(\eta). \tag{3.18}$$

4. The reduction to an ordinary differential equation. As the general analysis proceeds, the established forms of the dependent and the independent absolute invariant are used to obtain an ordinary differential equation. Generally, the absolute invariant $\eta(x, y)$ has the form given in (3.15).

Substituting from (3.15) and (3.18) into (2.5) yields

$$(\eta^2 + 1) \frac{d^2F}{d\eta^2} - 2\eta \left[\frac{x}{R} \frac{dR}{dx} - 1 \right] \frac{dF}{d\eta} + \left[\frac{x^2}{R} \frac{d^2R}{dx^2} \right] F = 0. \tag{4.1}$$

For (4.1) to be reduced to an expression in the single independent invariant η , the coefficients should be constants or functions of η alone. Thus,

$$\frac{x}{R} \frac{dR}{dx} = C_1, \tag{4.2}$$

$$\frac{x^2}{R} \frac{d^2R}{dx^2} = C_2. \tag{4.3}$$

It follows then that, from (4.2),

$$R(x) = C_3 x^{C_1}. \tag{4.4}$$

Also, from (4.2) and (4.3), we can show that

$$C_2 = C_1(C_1 - 1). \tag{4.5}$$

Take $C_3 = 1$ and $C_1 = n + 1$, we get $C_2 = n(n + 1)$, hence we have

$$(\eta^2 + 1) \frac{d^2F}{d\eta^2} - 2\eta n \frac{dF}{d\eta} + n(n + 1)F = 0. \tag{4.6}$$

Under the similarity variable η , the boundary conditions are

$$F(\tan \phi_1) = \alpha, \quad \frac{dF}{d\eta}(-\tan \phi_2) = \beta, \quad (4.7)$$

such that the boundary L_1 or L_2 does not coincide with the vertical axis.

5. Analytical solution. The solution corresponds to $n \geq 0$ is

$$\begin{aligned} F(\eta) = & b_0 \sum_{k=0}^{[(n+1)/2]} (-1)^k \binom{n+1}{2k} \eta^{2k} \\ & + \left(\frac{b_1}{n+1} \right) \sum_{k=0}^{[n/2]} (-1)^k \binom{n+1}{2k+1} \eta^{2k+1} \end{aligned} \quad (5.1)$$

hence we get

$$\begin{aligned} T(x, y) = & b_0 \sum_{k=0}^{[(n+1)/2]} (-1)^k \binom{n+1}{2k} y^{2k} x^{n-2k+1} \\ & + \left(\frac{b_1}{n+1} \right) \sum_{k=0}^{[n/2]} (-1)^k \binom{n+1}{2k+1} y^{2k+1} x^{n-2k}. \end{aligned} \quad (5.2)$$

The heat flux across L_3 is

$$\frac{\partial T}{\partial n}(x, y)|_{L_3} = -\frac{\partial T}{\partial x} \sin \phi_3 + \frac{\partial T}{\partial y} \cos \phi_3, \quad (5.3)$$

hence we get

$$\begin{aligned} \frac{\partial T}{\partial n}(x, y)|_{L_3} = & b_0 [-\sin \phi_3 M_{0,1} + \cos \phi_3 M_{0,2}] \\ & + \left(\frac{b_1}{n+1} \right) [-\sin \phi_3 M_{1,1} + \cos \phi_3 M_{1,2}]. \end{aligned} \quad (5.4)$$

Applying the boundary conditions (4.7), we get

$$\begin{aligned} \alpha = & b_0 z_{0,1} + \left(\frac{b_1}{n+1} \right) z_{1,1}, \\ \beta = & b_0 z_{0,2} + \left(\frac{b_1}{n+1} \right) z_{1,2}, \end{aligned} \quad (5.5)$$

where

$$\begin{aligned}
 M_{0,1} &= \sum_{k=0}^{[n/2]} (-1)^k (n-2k+1) \binom{n+1}{2k} y^{2k} x^{n-2k}, \\
 M_{1,1} &= \sum_{k=0}^{[(n-1)/2]} (-1)^k (n-2k+1) \binom{n+1}{2k+1} y^{2k+1} x^{n-2k-1}, \\
 M_{0,2} &= \sum_{k=1}^{[(n+1)/2]} (-1)^k (2k) \binom{n+1}{2k} y^{2k-1} x^{n-2k+1}, \\
 M_{1,2} &= \sum_{k=0}^{[n/2]} (-1)^k (2k+1) \binom{n+1}{2k+1} y^{2k} x^{n-2k}, \\
 z_{0,1} &= \sum_{k=0}^{[(n+1)/2]} (-1)^k \binom{n+1}{2k} \tan^{2k} \phi_1, \\
 z_{1,1} &= \sum_{k=0}^{[(n+1)/2]} (-1)^k \binom{n+1}{2k+1} \tan^{2k+1} \phi_1, \\
 z_{0,2} &= \sum_{k=0}^{[(n+1)/2]} (-1)^k (2k) \binom{n+1}{2k} \tan^{2k-1} \phi_2, \\
 z_{1,2} &= \sum_{k=0}^{[n/2]} (-1)^k (2k+1) \binom{n+1}{2k+1} \tan^{2k} \phi_2.
 \end{aligned} \tag{5.6}$$

Solving (5.5) for a given value of “ n ,” we get b_0 and b_1 .

THEOREM 5.1. *The system of (5.5) possesses a solution, if the difference between “ Φ_1/n ” and the vertex angle between L_1 and L_2 is not an odd integral multiple of “ $\pi/(2n)$.”*

PROOF. Let $z = e^{i\phi_1}$. Hence,

$$\begin{aligned}
 z^{n+1} + z^{-(n+1)} &= (\cos \phi_1 + i \sin \phi_1)^{n+1} + (\cos \phi_1 - i \sin \phi_1)^{n+1} \\
 &= \cos^{n+1} \left([1 + i \tan \phi_1]^{n+1} + [1 - i \tan \phi_1]^{n+1} \right).
 \end{aligned} \tag{5.7}$$

Applying the binomial theorem, we obtain

$$z^{n+1} + z^{-(n+1)} = 2[\cos^{n+1} \phi_1] z_{0,1}. \tag{5.8}$$

On the other hand,

$$z^{n+1} + z^{-(n+1)} = 2[\cos(n+1)\phi_1]. \tag{5.9}$$

Equating the right-hand side of (5.8) and (5.9) and solving for $z_{0,1}$, we get

$$z_{0,1} = \frac{\cos(n+1)\phi_1}{\cos^{n+1} \phi_1}. \tag{5.10}$$

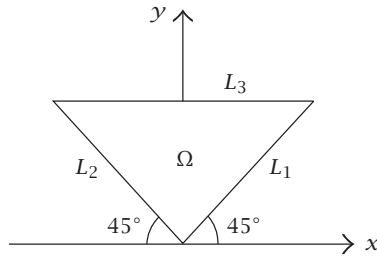


FIGURE 6.1. Geometrical configuration of the problem of Case 6.1.

Similarly, we get

$$\begin{aligned}
 z_{1,1} &= \frac{\sin(n+1)\phi_1}{\cos^{n+1}\phi_1}, \\
 z_{0,2} &= \frac{(n+1)\sin n\phi_2}{\cos^n\phi_2}, \\
 z_{1,2} &= \frac{(n+1)\cos n\phi_2}{\cos^n\phi_2}.
 \end{aligned}
 \tag{5.11}$$

The system of (5.5) has a unique solution if the determinant

$$D = \begin{vmatrix} z_{0,1} & z_{0,2} \\ z_{1,1} & z_{1,2} \end{vmatrix} \neq 0.
 \tag{5.12}$$

That is,

$$D = z_{0,1}z_{1,2} - z_{0,2}z_{1,1} = \frac{\cos[n(\phi_1 + \phi_2) + \phi_1]}{\cos^{n+1}\phi_1 \cos^n\phi_2} (n+1) \neq 0
 \tag{5.13}$$

which is satisfied if the difference between “ Φ_1/n ” and the vertex angle between L_1 and L_2 is not an odd multiple of “ $\pi/(2n)$.” □

6. Special cases

CASE 6.1. Boundary conditions are combinations of two different degrees of polynomials.

The governing equation for the distribution of temperature $T(x, y)$ is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad (x, y) \in \Omega
 \tag{6.1}$$

with the following boundary conditions:

$$\begin{aligned} T(x, y) &= q(x), \quad (x, y) \in L_1, \\ \frac{\partial T}{\partial y} &= \beta_1 x + \beta_2 x^5, \quad (x, y) \in L_2, \\ \phi_1 &= 45^\circ, \quad \phi_2 = 45^\circ, \quad \phi_3 = 0^\circ. \end{aligned} \quad (6.2)$$

From the principle of superposition, write

$$T(x, y) = T_1(x, y) + T_2(x, y), \quad (6.3)$$

where the boundary conditions for $T_1(x, y)$ are

$$\begin{aligned} T_1(x, y) &= \alpha_1 q_1(x), \quad (x, y) \in L_1, \\ \frac{\partial T_1}{\partial y} &= \beta_1 x, \quad (x, y) \in L_2, \end{aligned} \quad (6.4)$$

and the boundary conditions for $T_2(x, y)$ are

$$\begin{aligned} T_2(x, y) &= \alpha_2 q_2(x), \quad (x, y) \in L_1, \\ \frac{\partial T_2}{\partial y} &= \beta_2 x^5, \quad (x, y) \in L_2, \end{aligned} \quad (6.5)$$

where

$$q(x) = \alpha_1 q_1(x) + \alpha_2 q_2(x). \quad (6.6)$$

Setting $n = 1$ in the general solution (5.2) and (5.4), we get

$$\begin{aligned} T_1(x, y) &= \frac{\beta_1 - \alpha_1}{2} (x^2 - y^2) + \alpha_1 y x, \\ \frac{\partial T_1}{\partial n}(x, y)|_{L_3} &= b(\alpha_1 - \beta_1) + \alpha_1 x. \end{aligned} \quad (6.7)$$

Setting $n = 5$ in the general solution (5.2) and (5.4), we get

$$\begin{aligned} T_2(x, y) &= \frac{3\alpha_2 - \beta_2}{24} (x^6 - 15y^2x^4 - 15y^4x^2 - y^6) \\ &\quad - \frac{\alpha_2}{8} (6yx^5 - 20y^3x^3 + 6y^5x), \\ \frac{\partial T_2}{\partial n}(x, y)|_{L_3} &= -\frac{3\alpha_2 - \beta_2}{4} b(5x^4 - 10b^2x^2 + b^4) \\ &\quad - \frac{3\alpha_2}{4} (x^5 - 10b^2x^3 + 5b^4x). \end{aligned} \quad (6.8)$$

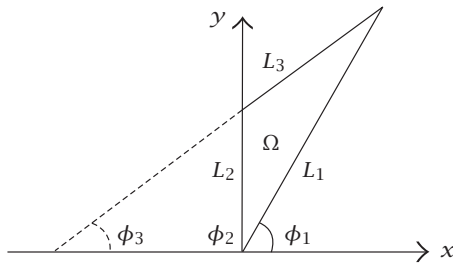


FIGURE 6.2. Geometrical configuration of the problem of Case 6.2.

Hence, the analytic solutions have the form

$$\begin{aligned}
 T(x, y) &= \frac{\beta_1 - \alpha_1}{2}(x^2 - y^2) + \frac{3\alpha_2 - \beta_2}{24}(x^6 - 15y^2x^4 - 15y^4x^2 - y^6) \\
 &\quad - \frac{\alpha_2}{8}(6yx^5 - 20y^3x^3 + 6y^5x), \\
 \frac{\partial T}{\partial n}(x, y)|_{L_3} &= b(\alpha_1 - \beta_1) + \alpha_1x - \frac{3\alpha_2 - \beta_2}{4}b(5x^4 - 10b^2x^2 + b^4) \\
 &\quad - \frac{3\alpha_2}{4}(x^5 - 10b^2x^3 + 5b^4x).
 \end{aligned}
 \tag{6.9}$$

CASE 6.2. One of the boundaries is vertical.

The governing equation for the distribution of temperature $T(x, y)$ is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad (x, y) \in \Omega
 \tag{6.10}$$

with the following boundary conditions:

$$\begin{aligned}
 T(x, y) &= \alpha q(y), \quad (x, y) \in L_1, \\
 \frac{\partial T}{\partial x} &= \beta y^n, \quad (x, y) \in L_2, \quad \phi_2 = 90^\circ.
 \end{aligned}
 \tag{6.11}$$

Write

$$T(x, y) = w(x, y)q(y), \quad q(y) \neq 0 \text{ in } \Omega,
 \tag{6.12}$$

by which differential equation (6.10) takes the form

$$q(y) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 2 \frac{\partial w}{\partial y} \frac{dq}{dy} + w \frac{d^2 q}{dy^2} = 0,
 \tag{6.13}$$

and the boundary conditions (6.11) take the form

$$\begin{aligned} w(x, y) &= \alpha, \quad (x, y) \in L_1, \\ \frac{\partial w}{\partial x}(x, y) &= \frac{\beta y^n}{q(y)}, \quad (x, y) \in L_2. \end{aligned} \tag{6.14}$$

Applying the invariant analysis, we get

$$G: \begin{cases} \tilde{x} = C^x x, \\ \tilde{y} = C^x y, \\ \tilde{w} = w, \\ \tilde{q} = [(C^x)^{n+1}]q, \end{cases} \tag{6.15}$$

and the absolute invariant $\eta(x, y)$ is

$$\eta(x, y) = \frac{x}{y}. \tag{6.16}$$

The complete set of absolute invariants corresponding to “ w ” and “ q ” are

$$q(y) = R(y), \quad w(x, y) = F(\eta). \tag{6.17}$$

Substituting (6.16) and (6.17) in (6.13), with $R(y) = y^{n+1}$, we get

$$(\eta^2 + 1) \frac{d^2 F}{d\eta^2} - 2n\eta \frac{dF}{d\eta} + n(n+1)F = 0. \tag{6.18}$$

Under the similarity variable η , the boundary conditions take the form

$$F(\tan \phi_1) = \alpha, \quad \frac{dF}{d\eta}(0) = \beta. \tag{6.19}$$

Solution of (6.18) with the boundary conditions (6.19) is

$$\begin{aligned} T(x, y) &= b_0 \sum_{k=0}^{[(n+1)/2]} (-1)^k \binom{n+1}{2k} \left(\frac{x}{y}\right)^{2k} y^{n+1} \\ &+ \left(\frac{b_1}{n+1}\right) \sum_{k=0}^{[n/2]} (-1)^k \binom{n+1}{2k+1} \left(\frac{x}{y}\right)^{2k+1} y^{n+1}, \end{aligned} \tag{6.20}$$

where

$$\begin{aligned} \alpha &= b_0 \sum_{k=0}^{[(n+1)/2]} (-1)^k \binom{n+1}{2k} \cot^{2k+1} \phi_1 \\ &+ \left(\frac{b_1}{n+1}\right) \sum_{k=0}^{[n/2]} (-1)^k \binom{n+1}{2k+1} \cot^{2k+1} \phi_1, \\ \beta &= b_1. \end{aligned} \tag{6.21}$$

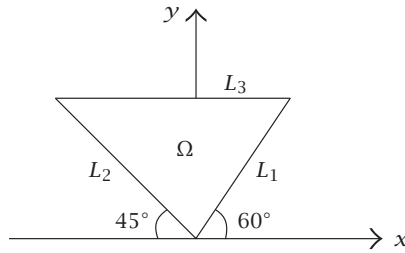


FIGURE 6.3. Geometrical configuration of the problem of Case 6.3.

Solving (6.21) for the given value of “ n ,” we get both b_0 and b_1 .

The heat flux across L_3 is

$$\begin{aligned} \frac{\partial T}{\partial n}(x, y)|_{L_3} = & b_0[-\sin \phi_3 N_{0,1} + \cos \phi_3 N_{0,2}] \\ & + \left(\frac{b_1}{n}\right)[- \sin \phi_3 N_{1,1} + \cos \phi_3 N_{1,2}], \end{aligned} \tag{6.22}$$

where

$$\begin{aligned} N_{0,1} &= \sum_{k=0}^{[(n+1)/2]} (-1)^k (2k) \binom{n+1}{2k} x^{2k-1} y^{n-2k}, \\ N_{1,1} &= \sum_{k=0}^{[n/2]} (-1)^k (2k+1) \binom{n+1}{2k+1} x^{2k} y^{n-2k-1}, \\ N_{0,2} &= \sum_{k=1}^{[n/2]} (-1)^k (n-2k+1) \binom{n+1}{2k} x^{2k} y^{n-2k}, \\ N_{1,2} &= \sum_{k=0}^{[(n-1)/2]} (-1)^k (n-2k) \binom{n+1}{2k+1} x^{2k+1} y^{n-2k+1}. \end{aligned} \tag{6.23}$$

CASE 6.3. The two boundary conditions are identical.

The governing equation for the distribution of temperature $T(x, y)$ is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad (x, y) \in \Omega \tag{6.24}$$

with the following boundary conditions:

$$\begin{aligned} T(x, y) &= \alpha q(x), \quad (x, y) \in L_1, \\ \frac{\partial T}{\partial y} &= -\frac{3}{4} \alpha x^2, \quad (x, y) \in L_2, \\ \phi_1 &= 60^\circ, \quad \phi_2 = 45^\circ, \quad \phi_3 = 0^\circ. \end{aligned} \tag{6.25}$$

Write

$$T(x, y) = w(x, y)q(x), \quad q(x) \neq 0 \text{ in } \Omega, \tag{6.26}$$

by which differential equation (6.24) takes the form

$$q(x) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 2 \frac{\partial w}{\partial x} \frac{dq}{dx} + w \frac{d^2 q}{dx^2} = 0, \tag{6.27}$$

and the boundary conditions (6.11) take the form

$$\begin{aligned} w(x, y) &= \alpha, \quad (x, y) \in L_1, \\ \frac{\partial w}{\partial y}(x, y) &= -\frac{3\alpha x^2}{4q(x)}, \quad (x, y) \in L_2. \end{aligned} \tag{6.28}$$

Applying the invariant analysis, we get

$$G: \begin{cases} \bar{x} = C^x x, \\ \bar{y} = C^x y, \\ \bar{w} = w, \\ \bar{q} = [(C^x)^3] q, \end{cases} \tag{6.29}$$

and the absolute invariant $\eta(x, y)$ is

$$\eta(x, y) = \frac{y}{x}. \tag{6.30}$$

The complete set of absolute invariants corresponding to “ w ” and “ q ” are

$$q(x) = R(x), \quad w(x, y) = F(\eta). \tag{6.31}$$

Substituting (6.30) and (6.31) in (6.27), with $R(x) = x^3$, we get

$$(\eta^2 + 1) \frac{d^2 F}{d\eta^2} - 4\eta \frac{dF}{d\eta} + 6F = 0. \tag{6.32}$$

Under the similarity variable η , the boundary conditions take the form

$$F(\sqrt{3}) = \alpha, \quad \frac{dF}{d\eta}(-1) = -\frac{3}{4}\alpha. \tag{6.33}$$

It can be easily shown that the two conditions (6.33) are identical.

Hence, to find the second condition, assume that

$$\frac{\partial T}{\partial y}(x, y)|_{L_3} = \frac{3}{4} \alpha b x + \gamma(x^2 - b^2), \tag{6.34}$$

where “ γ ” is a constant.

The solution of (6.32) with the aid of the boundary conditions (6.33) and (6.34) is

$$T(x, y) = -\frac{\alpha}{8}(x^3 - 3xy^2) + \frac{y}{3}(3x^2y - y^3). \quad (6.35)$$

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