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FLUID QUEUES DRIVEN BY A DISCOURAGED ARRIVALS QUEUE

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We consider a fluid queue driven by a discouraged arrivals queue and obtain explicit expressions for the stationary distribution function of the buffer content in terms of confluent hypergeometric functions. We compare it with a fluid queue driven by an infinite server queue. Numerical results are presented to compare the behaviour of the buffer content distributions for both these models.

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1. Introduction. Stochastic fluid flow models are increasingly used in the performance analysis of communication and manufacturing models. Recent measurements have revealed that in high-speed telecommunication networks, like the ATM-based broadband ISDN, traffic conditions exhibit long-range dependence and burstiness over a wide range of time scales. Fluid models characterize such traffic as a continuous stream with a parameterized flow rate.

Fluid queue models, where the fluid rates are controlled by state-dependent rates, have been studied in the literature. van Doorn and Scheinhardt [3] analyse the content of the buffer which receives and releases fluid flows at rates which are determined by the state of an infinite birth-death process evolving in the background. Lam and Lee [7] investigate a fluid flow model with linear adaptive service rates. Lenin and Parthasarathy [9] provide closed form expressions for the eigenvalues and eigenvectors for fluid queues driven by an $M/M/1/N$ queue. Resnick and Samorodnitsky [12] have obtained the steady-state distribution of the buffer content for $M/G/\infty$ input fluid queues using large deviation approach.

In this paper, we obtain explicit expressions for the stationary distribution function of the buffer content for fluid processes driven by two distinct queueing models, namely, discouraged arrivals queue and infinite server queue, respectively. Both these models have the same steady-state probabilities. We show that the buffer content distributions of fluid queues modulated by the two models vary considerably as depicted in the graph. The discouraged arrivals single-server queueing system is useful to model a computing facility that is solely dedicated to batch-job processing (see [11]). The well-known infinite server queue is often used to analyze open loop statistical multiplexing of data connections on an ATM network (see [6]).

2. Confluent hypergeometric function. We obtain explicit expressions for the buffer content distributions of the fluid queues driven by discouraged arrivals queue and infinite server queue by employing well-known identities of confluent hypergeometric function. Some of the identities are presented in this section.

The *confluent hypergeometric function*, also referred to as Kummer function, is denoted by ${}_1F_1(a;c;z)$ and is defined by

$$\begin{aligned}{}_1F_1(a;c;z) &= 1+\frac{a}{c}\frac{z}{1!}+\frac{a(a+1)}{c(c+1)}\frac{z^2}{2!}+\cdots \\ &= \sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!}\end{aligned} \tag{2.1}$$

for $z\in\mathbb{C}$, parameters $a,c\in\mathbb{C}$ (c a nonnegative integer), with $(\alpha)_n$, known as Pochhammer symbol, defined as

$$(\alpha)_n=\begin{cases}1, & n=0,\\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), & n\ge 1.\end{cases} \tag{2.2}$$

Observe that ${}_1F_1(0;c;z)=1$ and ${}_1F_1(a;a;z)=e^z$. The confluent hypergeometric function satisfies the relations (see [1])

$$\begin{aligned}c(c-1)\,{}_1F_1(a-1;c-1;z)-az\,{}_1F_1(a+1;c+1;z)\\ =c(c-1-z)\,{}_1F_1(a;c;z),\end{aligned} \tag{2.3}$$

$${}_1F_1(a;c;z)=e^z\,{}_1F_1(c-a;c;-z). \tag{2.4}$$

The following identities are from [2]:

$$(c-a)\,{}_1F_1(a;c+1;z)+a\,{}_1F_1(a+1;c+1;z)=c\,{}_1F_1(a;c;z), \tag{2.5}$$

$$c\,{}_1F_1(a+1;c;z)-c\,{}_1F_1(a;c;z)=z\,{}_1F_1(a+1;c+1;z), \tag{2.6}$$

$$(c-a)\,{}_1F_1(a-1;c;z)+(2a-c+z)\,{}_1F_1(a;c;z)=a\,{}_1F_1(a+1;c;z). \tag{2.7}$$

3. Model description. Consider a fluid model driven by a single server queueing process with state-dependent arrival and service rates. It consists of an infinitely large buffer in which the fluid flow is regulated by the state of the background queueing process. Denote the background queueing process by $\mathscr{X}:=\{X(t),\ t\ge 0\}$ taking values in the state space $\mathscr{S}$ of nonnegative integers, where $X(t)$ denotes the state of the process at time t. Let λ_n and μ_n denote the mean arrival and service rates, respectively, when there are n units in the system.

The background queueing process modulates the fluid model in such a way that during the busy periods of the server, the fluid accumulates in an infinite capacity buffer at a constant rate $r>0$. The buffer depletes the fluid during the idle periods of the server at a constant rate $r_0<0$ as long as the buffer is nonempty. We denote by $C(t)$ the content of the buffer at time t. Clearly, the 2-dimensional process $\{(X(t), C(t)),\ t \geq 0\}$ constitutes a Markov process, and it possesses a unique stationary distribution under a suitable stability condition.

The stationary state probabilities p_i, $i \in \mathcal{S}$, of the background process are given by

$$p_i = \frac{\pi_i}{\sum_{j \in \mathcal{S}} \pi_j}, \quad i \in \mathcal{S}, \tag{3.1}$$

where $\pi_i = (\lambda_0 \lambda_1 \cdots \lambda_{i-1})/(\mu_1 \mu_2 \cdots \mu_i)$, $i = 1, 2, 3, \ldots$, and $\pi_0 = 1$ are called the *potential coefficients*. To ensure the stability of the process $\{(X(t), C(t)),\ t \geq 0\}$, we assume the mean aggregate input rate to be negative, that is,

$$r_0 + r \sum_{i=1}^{\infty} \pi_i < 0. \tag{3.2}$$

Letting

$$F_n(t, x) \equiv P(X(t) = n,\ C(t) \leq x), \quad n \in \mathcal{S},\ t, x \geq 0, \tag{3.3}$$

the Kolmogorov forward equations for the Markov process $\{X(t), C(t)\}$ are given by

$$\begin{aligned} \frac{\partial F_0(t,x)}{\partial t} + r_0 \frac{\partial F_0(t,x)}{\partial x} &= -\lambda_0 F_0(t,x) + \mu_1 F_1(t,x), \\ \frac{\partial F_n(t,x)}{\partial t} + r \frac{\partial F_n(t,x)}{\partial x} &= \lambda_{n-1} F_{n-1}(t,x) - (\lambda_n + \mu_n) F_n(t,x) \\ &\quad + \mu_{n+1} F_{n+1}(t,x), \quad n \in \mathcal{S} \setminus \{0\},\ t, x \geq 0 \end{aligned} \tag{3.4}$$

(see [3]). Assume that the process is in equilibrium so that $\partial F_n(t,x)/\partial t \equiv 0$ and in that case $\lim_{t \to \infty} F_n(t,x) \equiv F_n(x)$. Hence, the above system reduces to a system of ordinary differential equations

$$\begin{aligned} r_0 F_0'(x) &= -\lambda_0 F_0(x) + \mu_1 F_1(x), \\ r F_n'(x) &= \lambda_{n-1} F_{n-1}(x) - (\lambda_n + \mu_n) F_n(x) \\ &\quad + \mu_{n+1} F_{n+1}(x), \quad x \geq 0,\ n = 1, 2, 3, \ldots. \end{aligned} \tag{3.5}$$

When the net input rate of fluid flow into the buffer is positive, the buffer content increases and the buffer cannot stay empty. It follows that the solution to (3.5) must satisfy the boundary conditions

$$F_n(0) = 0, \quad n = 1, 2, 3, \ldots. \tag{3.6}$$

But $F_0(0)$ is nonzero and is determined later. Also,

$$\lim_{x \to \infty} F_n(x) = p_n. \tag{3.7}$$

We study two fluid models driven by state-dependent queues with arrival and service rates given by

$$\lambda_n = \frac{\lambda}{n+1}, \quad n = 0,1,2,\dots, \qquad \mu_n = \mu, \qquad n = 1,2,3,\dots, \tag{3.8}$$

$$\lambda_n = \lambda, \qquad n = 0,1,2,\dots, \qquad \mu_n = n\mu, \quad n = 1,2,3,\dots. \tag{3.9}$$

For the process to be stable, from (3.2), $(r_0 - r) + re^{\rho} < 0$ where ρ denotes the ratio λ/μ.

Both the queueing models under consideration have the same steady-state probabilities given by $p_n = (\rho^n/n!)e^{-\rho}$. From [13], the stationary probability for the fluid queue to be empty is given by

$$F_0(0) = \frac{(r_0 - r)e^{-\rho} + r}{r_0} \tag{3.10}$$

for both these models.

Our task is to solve the system of (3.5) with rates suggested by (3.8) and (3.9) subject to conditions (3.6) and (3.7). The stationary buffer content distribution can then be obtained.

In this sequel, let $\hat{F}_n(s)$ denote the Laplace transform of the function $F_n(x)$.

4. Discouraged arrivals queue. In this section, we consider a fluid queue driven by a state-dependent queueing model with rates given by (3.8) and obtain an explicit expression for the quantity $F_n(x)$ using well-known identities of confluent hypergeometric functions. As suggested by the birth and death rates, it is seen that the arrivals decrease as the queue length increases and hence the name *discouraged arrivals queue*. The governing system of forward Kolmogrov equations for this model is

$$\begin{aligned} r_0 F_0'(x) &= -\lambda F_0(x) + \mu F_1(x), \\ rF_n'(x) &= \frac{\lambda}{n} F_{n-1}(x) - \left(\frac{\lambda}{n+1} + \mu\right) F_n(x) + \mu F_{n+1}(x), \quad n = 1,2,3,\dots. \end{aligned} \tag{4.1}$$

Laplace transform yields

$$(r_0 s + \lambda)\hat{F}_0(s) - r_0 F_0(0) = \mu \hat{F}_1(s), \tag{4.2}$$

$$\left(rs + \frac{\lambda}{n+1} + \mu\right)\hat{F}_n(s) = \frac{\lambda}{n}\hat{F}_{n-1}(s) + \mu \hat{F}_{n+1}(s), \quad n = 1,2,3,\dots. \tag{4.3}$$

We obtain the solution of the above system of equations in terms of confluent hypergeometric function. Defining

$$\hat{g}_n(s) = \frac{(\lambda rs/(rs+\mu)^2 + n + 1)\cdots(\lambda rs/(rs+\mu)^2 + 1)}{((n+1)/s)(\lambda/(rs+\mu))^n} \times \hat{F}_n(s), \tag{4.4}$$

it is observed that (4.3) reduces to

$$\left(\frac{\lambda rs}{(rs+\mu)^2}+n+2\right)\left(\frac{\lambda rs}{(rs+\mu)^2}+n+1\right)\hat{g}_{n-1}(s)-(n+2)\left(-\frac{\lambda\mu}{(rs+\mu)^2}\right)\hat{g}_{n+1}(s)$$
$$=\left(\frac{\lambda rs}{(rs+\mu)^2}+n+2\right)\left(\frac{\lambda}{rs+\mu}+n+1\right)\hat{g}_n(s). \tag{4.5}$$

We identify that the term $\hat{g}_n(s)$ satisfies (2.3) with $a=n+2$, $c=\lambda rs/(rs+\mu)^2+n+2$, and $z=-\lambda\mu/(rs+\mu)^2$. Thus, we can deduce from (4.5) and (2.3) that

$$\hat{g}_n(s)={}_1F_1\left(n+2;\frac{\lambda rs}{(rs+\mu)^2}+n+2;-\frac{\lambda\mu}{(rs+\mu)^2}\right),\quad n=1,2,3,\ldots \tag{4.6}$$

and hence

$$\hat{F}_n(s)$$
$$=\frac{((n+1)/s)(\lambda/(rs+\mu))^n\,{}_1F_1(n+2;\lambda rs/(rs+\mu)^2+n+2;-\lambda\mu/(rs+\mu)^2)}{(\lambda rs/(rs+\mu)^2+n+1)\cdots(\lambda rs/(rs+\mu)^2+1)},$$
$$n=1,2,3,\ldots. \tag{4.7}$$

In order that $\hat{F}_n(s)$ satisfies (4.2), we *redefine*

$$\hat{F}_n(s)$$
$$=\frac{(r_0F_0(0)/r)((n+1)/s)(\lambda/(rs+\mu))^n\,{}_1F_1(n+2;\lambda rs/(rs+\mu)^2+n+2;-\lambda\mu/(rs+\mu)^2)}{(\lambda rs/(rs+\mu)^2+n+1)\cdots(\lambda rs/(rs+\mu)^2+1)}$$
$$\times\frac{1}{(1-(1-r_0/r)({}_1F_1(2;\lambda rs/(rs+\mu)^2+2;-\lambda\mu/(rs+\mu)^2)/(\lambda rs/(rs+\mu)^2+1)))},$$
$$n=0,1,\ldots \tag{4.8}$$

so that both (4.2) and (4.3) are satisfied. The fact that $\hat{F}_0(s)$ and $\hat{F}_1(s)$ satisfy (4.2) can be verified by using identities (2.5) and (2.7) (see Appendix A). Since $\hat{F}_n(s)$ represents the Laplace transform of a probability distribution function, in view of (2.4) we can express

$$\hat{F}_n(s)$$
$$=\frac{(r_0F_0(0)/r)((n+1)/s)(\lambda/(rs+\mu))^n\,{}_1F_1(n+2;\lambda rs/(rs+\mu)^2+n+2;-\lambda\mu/(rs+\mu)^2)}{(\lambda rs/(rs+\mu)^2+n+1)\cdots(\lambda rs/(rs+\mu)^2+1)}$$
$$\times\sum_{j=0}^{\infty}\left(1-\frac{r_0}{r}\right)^j(\hat{\phi}(s))^j, \tag{4.9}$$

where

$$\hat{\phi}(s) = \frac{{}_1F_1(2;\lambda rs/(rs+\mu)^2+2;-\lambda\mu/(rs+\mu)^2)}{(\lambda rs/(rs+\mu)^2+1)}. \tag{4.10}$$

Now, we invert (4.9) by expanding the function as

$$\begin{aligned}\hat{F}_n(s) &= r_0F_0(0)\left(\frac{(n+1)\lambda^{n+1}}{(rs+\mu)^n}\right)\frac{(rs+\mu)^{2(n+1)}}{\lambda rs[\lambda rs+(rs+\mu)^2]\cdots[\lambda rs+(n+1)(rs+\mu)^2]}\\ &\quad\times\sum_{k=0}^{\infty}\frac{(n+2)_k}{(\lambda rs/(rs+\mu)^2+n+2)_k}\frac{(-\lambda\mu/(rs+\mu)^2)^k}{k!}\sum_{j=0}^{\infty}\left(1-\frac{r_0}{r}\right)^j(\hat{\phi}(s))^j\\ &= r_0F_0(0)\sum_{k=0}^{\infty}\frac{\lambda^{n+k+1}(-\mu)^k(n+k+1)!}{n!k!(rs+\mu)^{2k+n}}\frac{(rs+\mu)^{2(n+k+1)}}{\prod_{i=0}^{n+k+1}[\lambda rs+i(rs+\mu)^2]}\\ &\quad\times\sum_{j=0}^{\infty}\left(1-\frac{r_0}{r}\right)^j(\hat{\phi}(s))^j.\end{aligned} \tag{4.11}$$

Laplace inversion yields

$$\begin{aligned}F_n(x) = r_0F_0(0)&\sum_{k=0}^{\infty}\frac{\lambda^{n+k+1}(-\mu)^k}{n!k!}\sum_{m=0}^{n+k+1}\binom{n+k+1}{m}(-1)^m g_{n+2k,m}(x)\\ &\times\sum_{j=0}^{\infty}\left(1-\frac{r_0}{r}\right)^j\phi^{*(j)}(x),\quad n\ge 0,\end{aligned} \tag{4.12}$$

where $\phi^{*(j)}(x)$ denotes the j-fold convolution of $\phi(x)$,

$$\begin{aligned}g_{0,0}(x) &= \frac{1}{r\lambda},\\ g_{0,1}(x) &= \frac{1}{2r\sqrt{\lambda^2/4+\lambda\mu}}\left\{e^{-(\lambda/2+\mu-\sqrt{\lambda^2/4+\lambda\mu})(x/r)}-e^{-(\lambda/2+\mu+\sqrt{\lambda^2/4+\lambda\mu})(x/r)}\right\},\\ g_{\ell,0}(x) &= \frac{1}{\lambda r^{\ell+1}(\ell-1)!}\int_0^x e^{-(\mu/r)y}y^{\ell-1}\,dy,\\ g_{\ell,m}(x) &= \frac{1}{(\ell-1)!2mr^{\ell+1}\sqrt{\lambda^2/4m^2+\lambda\mu/m}}\\ &\quad\times\left\{e^{-(\lambda/2m+\mu-\sqrt{\lambda^2/4m^2+\lambda\mu/m})(x/r)}\int_0^x e^{(\lambda/2m-\sqrt{\lambda^2/4m^2+\lambda\mu/m})y}y^{\ell-1}dy\right.\\ &\quad\left.-e^{-(\lambda/2m+\mu+\sqrt{\lambda^2/4m^2+\lambda\mu/m})(x/r)}\int_0^x e^{(\lambda/2m+\sqrt{\lambda^2/4m^2+\lambda\mu/m})y}y^{\ell-1}dy\right\},\\ \phi(x) &= \delta(x)-\lambda r g_{0,1}'(x)+\sum_{k=1}^{\infty}\frac{(k+1)(-\lambda\mu)^k}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^i g_{2(k-1),m+1}(x),\end{aligned} \tag{4.13}$$

where $\delta(x)$ is the Dirac delta function. We now verify the boundary condition (3.7). Using the fact that ${}_1F_1(a,a,z) = e^z$, observe that ${}_1F_1(2,\lambda rs/(rs+\mu)^2 + 2, -\lambda\mu/(rs+\mu)^2)$ and hence $\hat{\phi}(s)$ tends to $e^{-\rho}$ as $s \to 0$. Hence we have

$$\begin{aligned}\lim_{x\to\infty} F_n(x) &= \lim_{s\to 0} s\hat{F}_n(s) \\ &= \left(\frac{r_0 F_0(0)}{r}\right)\frac{\rho^n e^{-\rho}}{n!} \times \sum_{j=0}^{\infty}\left(1-\frac{r_0}{r}\right)^j e^{-j\rho} \\ &= \left(\frac{r_0 F_0(0)}{r}\right)\frac{\rho^n e^{-\rho}}{n!}\frac{1}{1-(1-r_0/r)e^{-\rho}} \\ &= \left(\frac{r_0 F_0(0)}{(r_0-r)e^{-\rho}+r}\right)\frac{\rho^n e^{-\rho}}{n!} \\ &= \frac{\rho^n e^{-\rho}}{n!} \\ &= p_n\end{aligned} \tag{4.14}$$

(from (3.10)).

5. Infinite server queue. In this section, we consider a fluid queue driven by an infinite server queue and obtain an explicit expression for $G_n(x)$, thereby highlighting the variation in their expressions, although both the underlying queueing models have the same steady-state probabilities. For the sake of clarity in notation, we use $G_n(x)$ in place of $F_n(x)$. The forward Kolmogrov equations for this model are

$$\begin{aligned} r_0 G_0'(x) &= -\lambda G_0(x) + \mu G_1(x), \\ r G_n'(x) &= \lambda G_{n-1}(x) - (\lambda + n\mu) G_n(x) + (n+1)\mu G_{n+1}(x). \end{aligned} \tag{5.1}$$

Laplace transform yields

$$(r_0 s + \lambda)\hat{G}_0(s) - \mu\hat{G}_1(s) = r_0 G_0(0), \tag{5.2}$$

$$(rs + \lambda + n\mu)\hat{G}_n(s) = \lambda\hat{G}_{n-1}(s) + (n+1)\mu\hat{G}_{n+1}(s). \tag{5.3}$$

Here, $G_0(0) = F_0(0)$. Analysing as before, if

$$\hat{k}_n(s) = \mu\left(\frac{\mu}{\lambda}\right)^n\left(\frac{rs}{\mu}\right)\left(\frac{rs}{\mu}+1\right)\cdots\left(\frac{rs}{\mu}+n\right)\hat{G}_n(s), \tag{5.4}$$

then (5.3) reduces to

$$\begin{aligned}\left(\frac{rs}{\mu}+n+1\right)\left(\frac{rs}{\mu}+n\right)\hat{k}_{n-1}(s) - (n+1)\left(-\frac{\lambda}{\mu}\right)\hat{k}_{n+1}(s) \\ = \left(\frac{rs+\lambda}{\mu}+n\right)\left(\frac{rs}{\mu}+n+1\right)\hat{k}_n(s).\end{aligned} \tag{5.5}$$

We observe that $\hat{k}_n(s)$ satisfies the recurrence relation (2.3) with $a = n+1$, $c = rs/\mu + n + 1$, and $z = -\lambda/\mu$. Thus we have

$$\hat{k}_n(s) = {}_1F_1\left(n+1, \frac{rs}{\mu} + n + 1, -\frac{\lambda}{\mu}\right), \quad n = 1,2,3,\ldots \tag{5.6}$$

and hence

$$\hat{G}_n(s) = \left(\frac{\lambda}{\mu}\right)^n \frac{(1/\mu)\,{}_1F_1(n+1, rs/\mu + n + 1, -\lambda/\mu)}{(rs/\mu)(rs/\mu+1)\cdots(rs/\mu+n)}, \quad n = 1,2,3,\ldots. \tag{5.7}$$

By a similar argument as in the previous section in order to satisfy (5.2), we *redefine*

$$\begin{aligned}
\hat{G}_n(s) &= r_0G_0(0)\left(\frac{\lambda}{\mu}\right)^n \frac{(1/\mu)\,{}_1F_1(n+1, rs/\mu+n+1, -\lambda/\mu)}{(rs/\mu)(rs/\mu+1)\cdots(rs/\mu+n)} \\
&\quad\times \frac{1}{1-(1-r_0/r)\,{}_1F_1(1, rs/\mu+1, -\lambda/\mu)} \\
&= \frac{r_0G_0(0)}{rs}\left(\frac{\lambda}{\mu}\right)^n \frac{{}_1F_1(n+1, rs/\mu+n+1, -\lambda/\mu)}{(rs/\mu+1)\cdots(rs/\mu+n)} \\
&\quad\times \sum_{j=0}^{\infty}(1-r_0/r)^j\hat{\psi}^j(s), \quad n = 0,1,2,\ldots,
\end{aligned} \tag{5.8}$$

where

$$\hat{\psi}(s) = {}_1F_1\left(1, \frac{rs}{\mu}+1, -\frac{\lambda}{\mu}\right). \tag{5.9}$$

Subject to the above definition, the verification of (5.2), being satisfied by $\hat{G}_0(s)$ and $\hat{G}_1(s)$, is done through certain algebra involving the application of identities (2.5) and (2.6) (see Appendix B).

To facilitate the Laplace inversion, we write

$$\hat{G}_n(s) = (r_0G_0(0))\hat{h}_n(s)\sum_{j=0}^{\infty}\left(1-\frac{r_0}{r}\right)^j\hat{\psi}^j(s), \tag{5.10}$$

where

$$\begin{aligned}
\hat{h}_n(s) &= \left(\frac{\lambda}{\mu}\right)^n \frac{(1/\mu)\,{}_1F_1(n+1, rs/\mu+n+1, -\lambda/\mu)}{(rs/\mu)(rs/\mu+1)\cdots(rs/\mu+n)} \\
&= \left(\frac{\lambda}{\mu}\right)^n \frac{(1/\mu)\sum_{k=0}^{\infty}\left((n+1)_k/(rs/\mu+n+1)_k\right)\left((-\lambda/\mu)^k/k!\right)}{(rs/\mu)(rs/\mu+1)\cdots(rs/\mu+n)} \\
&= \sum_{k=0}^{\infty}\frac{((-1)^k/\mu)(\lambda/\mu)^{n+k}((n+1)_k/k!)}{(rs/\mu)(rs/\mu+1)\cdots(rs/\mu+n)\cdots(rs/\mu+n+k)} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(n+k)!}{n!k!r}\left(\frac{\lambda}{\mu}\right)^{n+k}\sum_{m=0}^{n+k}\frac{(-1)^m}{m!(n+k-m)!(s+m\mu/r)}.
\end{aligned} \tag{5.11}$$

On inversion, we get

$$\begin{aligned} h_n(x) &= \frac{1}{r}\sum_{k=0}^{\infty}\frac{(-1)^k}{n!k!}\left(\frac{\lambda}{\mu}\right)^{n+k}\sum_{m=0}^{n+k}\binom{n+k}{m}(-1)^m e^{-(m\mu/r)x} \\ &= \frac{1}{r}\sum_{k=0}^{\infty}\frac{(-1)^k}{n!k!}\left(\frac{\lambda}{\mu}\right)^{n+k}\left(1-e^{-(\mu/r)x}\right)^{n+k} \\ &= \frac{1}{r}\left(\frac{\lambda}{\mu}\right)^{n}\frac{\left(1-e^{-(\mu/r)x}\right)^{n}}{n!}\exp\left\{-\frac{\lambda}{\mu}\left(1-e^{-(\mu/r)x}\right)\right\}. \end{aligned} \tag{5.12}$$

Hence, we have

$$\begin{aligned} G_n(x) = {} & \frac{r_0 G_0(0)}{r}\left(\frac{\lambda}{\mu}\right)^{n}\frac{\left(1-e^{-(\mu/r)x}\right)^{n}}{n!}\exp\left\{-\frac{\lambda}{\mu}\left(1-e^{-(\mu/r)x}\right)\right\} \\ & \times\sum_{j=0}^{\infty}\left(1-\frac{r_0}{r}\right)^{j}\psi^{*(j)}(x), \end{aligned} \tag{5.13}$$

where

$$\psi(x) = \delta(x) - \frac{\lambda}{r}e^{-(\mu/r)x}\exp\left\{-\frac{\lambda}{\mu}\left(1-e^{-(\mu/r)x}\right)\right\}. \tag{5.14}$$

Using ${}_1F_1(a,a,z) = e^z$, we verify below the boundary condition (3.7) for the buffer content distribution as

$$\begin{aligned} &\lim_{x\to\infty} G_n(x) \\ &\quad = \lim_{s\to 0} s\hat{G}_n(s) \\ &\quad = \left(\frac{r_0 G_0(0)}{r}\right)\left(\frac{\rho^n}{n!}\right){}_1F_1\left(n+1,n+1,-\frac{\lambda}{\mu}\right)\sum_{j=0}^{\infty}\left(1-\frac{r_0}{r}\right)^{j}\left({}_1F_1\left(1,1,-\frac{\lambda}{\mu}\right)\right)^{j} \\ &\quad = \left(\frac{r_0 G_0(0)}{r}\right)\left(\frac{\rho^n}{n!}e^{-\rho}\right)\sum_{j=0}^{\infty}\left(1-\frac{r_0}{r}\right)^{j}e^{-j\rho} \\ &\quad = \left(\frac{r_0 G_0(0)}{(r_0-r)e^{-\rho}+r}\right)\left(\frac{\rho^n}{n!}\right)e^{-\rho} \\ &\quad = p_n \end{aligned} \tag{5.15}$$

(from (3.10)).

In this way, we analytically obtain closed form expressions for $F_n(x)$ and $G_n(x)$ for both the models as given by (4.12) and (5.13), respectively. Hence, we obtain the stationary distribution of the buffer content given by

$$\lim_{t\to\infty}\Pr(C(t) > x) = 1 - \sum_{n=0}^{\infty}F_n(x) = 1 - \left(1-\frac{r_0}{r}\right)F_0(x) - \frac{r_0}{r}F_0(0), \tag{5.16}$$

where $F_0(0)$ is given by (3.10).

6. Asymptotic analysis. In this section, we discuss the large deviations calculation that gives the asymptotic straight line fit to the two models under consideration. Large buffers are obtained by having the birth-death process avoid zero more often than average. Suppose that for a time t the average occupancy of the state zero in the birth-death process is x, then the drift of the fluid buffer is on average $r_0x+r(1-x)=r-x(r-r_0)$, which is positive. The probability that the occupancy of state zero is near x is obtained by Sanov's theorem. Let $m(t)$ represent the fraction of time that the birth-death process is zero in $[0,t]$. Then

$$P(m(t)\approx x)=\exp[-tI(x)], \tag{6.1}$$

where

$$I(x)=\inf_{\nu\in H(x)}\sum_i \nu_i\log\frac{\nu_i}{p_i} \tag{6.2}$$

and $H(x)=\{\nu_i:\sum_i\nu_i=1,\ \nu_i\geq 0,\ \nu_0=x\}$. Following standard arguments as sketched in Schwartz and Weiss [14, Section 2.4], we use a Lagrange multiplier to find the minimum in $I(x)$ as follows. We write $I(x)=\inf_{\nu\in H(x)}\sum_i p_i\alpha_i\log\alpha_i$, where $\alpha_i=\nu_i/p_i$ for $i>0$ and $\alpha_0=x/p_0$. We then look for extreme points of the function

$$\sum_{i=0}^{\infty}p_i\alpha_i\log\alpha_i+K(\alpha_ip_i-1), \tag{6.3}$$

where the Lagrange multiplier K is chosen so that the condition $\sum_i\alpha_ip_i=1$ is satisfied. Setting the partial derivatives of the function with respect to α_i equal to zero, for $i>0$, we obtain that all α_i, $i\geq 1$, are equal, say α. Therefore

$$\alpha=\frac{1-x}{1-p_0}. \tag{6.4}$$

Hence, we find that

$$I(x)=x\log\frac{x}{p_0}+(1-p_0)\alpha\log\alpha, \tag{6.5}$$

where x is the parameter to be determined. Recall that $p_0=e^{-\rho}$.

Now to estimate the probability that the fluid buffer is above some level B, we estimate the probability that $m(t)$ is near x for sufficient time t. Note that the fluid buffer fills at rate $r-x(r-r_0)$ so that the time required is $t=B/(r-x(r-r_0))$. Therefore, the probability that the buffer fills to B is approximately

$$\exp(-tI(x))=\exp(-BI(x)/(r-x(r-r_0))). \tag{6.6}$$

We can find x which minimise the quotient $I(x)/(r-x(r-r_0))$. We numerically determine the value of x and this minimum is unique. For example,

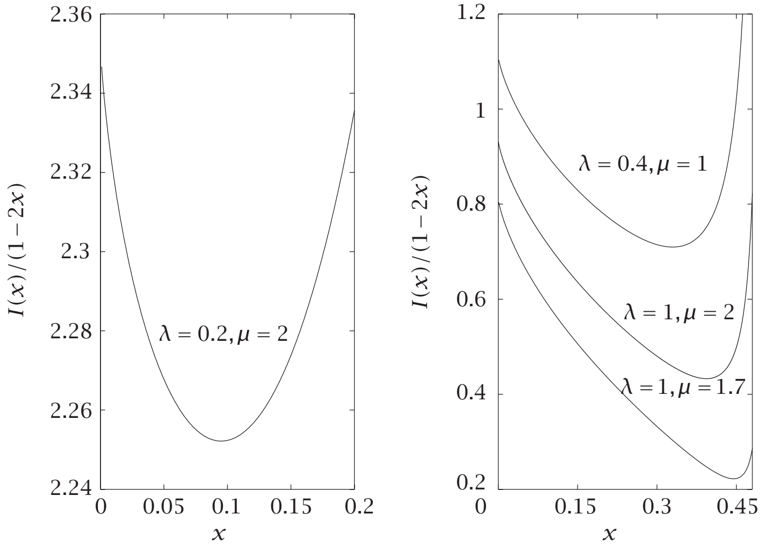

FIGURE 6.1. Behaviour of $I(x)/(r - x(r - r_0))$ against x for $r_0 = -1$, $r = 1$, and varying values of λ and μ.

TABLE 6.1. Value of x at which $I(x)/(1-2x)$ attains the minimum and the corresponding minimum for different values of λ and μ.

λ	μ	x	$I(x)/(1-2x)$
1.0	1.7	0.44469	0.22213441870537
0.4	1.0	0.32968	0.70963293158894
1.0	2.0	0.39347	0.43275212957147
0.8	2.0	0.32968	0.70963293158894
0.2	2.0	0.09516	2.25216846109190

when $r = 1 = -r_0$, with $\lambda = 1$ and $\mu = 1.7$, we find $x = 0.44469$, quotient $= 0.22213441870537$, that is, $P(\text{fluid} > t) \approx \exp(-0.22213441870537t)$. Observe that for sufficiently small values of t, $\exp(-0.22213441870537t)$ turns out to be a straight line.

Figure 6.1 depicts the behaviour of this function $I(x)/(r - x(r - r_0))$ against x for $r = 1 = -r_0$ and the varying values of λ and μ where $I(x)$ is given by (6.5). Table 6.1 gives the value of x at which the quotient attains minimum for the various parameter values.

7. Numerical illustrations. In this section, we briefly discuss the method of numerically evaluating the stationary buffer content distribution for the two models under consideration.

The governing system of differential-difference equations given by (3.5) can be written in the matrix notation as

$$\frac{d\mathbf{F}(x)}{dx} = \mathbf{R}^{-1}\mathbf{Q}^{T}\mathbf{F}(x), \tag{7.1}$$

where $\mathbf{F}(x) = [F_0(x), F_1(x), F_2(x), \ldots]^T$, $\mathbf{R} = \operatorname{diag}\{r_0, r, r, \ldots\}$, and $\mathbf{Q}$ denotes the infinitesimal generator of the background birth and death process given by

$$\mathbf{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{7.2}$$

The capacity of the background birth and death process is unrestricted in our theoretical study. However, for the purpose of numerical investigations, we truncate the size of the process by a finite quantity, say N. Hence $\mathbf{R}^{-1}\mathbf{Q}^T$ takes the form

$$\mathbf{R}^{-1}\mathbf{Q}^{T} = \begin{pmatrix} -\dfrac{\lambda_0}{r_0} & \dfrac{\mu_1}{r_0} & & & \\ \dfrac{\lambda_0}{r} & -\dfrac{\lambda_1+\mu_1}{r} & \dfrac{\mu_2}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \dfrac{\mu_N}{r} \\ & & & \dfrac{\lambda_{N-1}}{r} & -\dfrac{\mu_N}{r} \end{pmatrix}_{N+1}. \tag{7.3}$$

Mitra [10] have shown that $\mathbf{R}^{-1}\mathbf{Q}^T$ has exactly N_+ negative eigenvalues, $N_- - 1$ positive eigenvalues, and one zero eigenvalue, where N_+ is the cardinality of the set $S^+ \equiv \{j \in \mathcal{S} : r_j > 0\}$ and N_- is that of $S^- \equiv \{j \in \mathcal{S} : r_j < 0\}$.

Suppose that ξ_j, $j = 0,1,2,\ldots,N$, are the eigenvalues of the matrix $\mathbf{R}^{-1}\mathbf{Q}^T$ such that

$$\xi_j < 0, \quad j = 0,1,\ldots,N-1, \qquad \xi_N = 0 \tag{7.4}$$

and $\mathbf{y}^\ell = [y_0^\ell, y_1^\ell, \ldots, y_N^\ell]^T$, $\mathbf{z}^\ell = [z_0^\ell, z_1^\ell, \ldots, z_N^\ell]$ are the right and the left eigenvectors of the matrices $\mathbf{R}^{-1}\mathbf{Q}^T$ and $\mathbf{Q}^T\mathbf{R}^{-1}$, respectively, corresponding to the eigenvalue ξ_ℓ.

Then,

$$\begin{aligned} &y_0^\ell = 1 = z_0^\ell \quad \text{for } \ell \in \mathcal{S}, \\ &y_j^\ell = \frac{B_j(\xi_\ell)}{c_{j0}}, \quad z_j^\ell = \frac{r_0 B_j(\xi_\ell)}{r c_{j0}} \quad \text{for } j,\ell \in \mathcal{S}\setminus\{0\}, \end{aligned} \tag{7.5}$$

where the polynomials $B_j(s)$ are recursively defined as follows:

$$\begin{aligned}
B_0(s) &= 1,\\
B_1(s) &= \left(s+\frac{\lambda_0}{r_0}\right)B_0(s),\\
B_2(s) &= \left(s+\frac{\lambda_1+\mu_1}{r}\right)B_1(s)-\frac{\lambda_0\mu_1}{r_0 r}B_0(s),\\
B_j(s) &= \left(s+\frac{\lambda_{j-1}+\mu_{j-1}}{r}\right)B_{j-1}(s)-\frac{\lambda_{j-2}\mu_{j-1}}{r^2}B_{j-2}(s),\quad j=3,4,\dots,N,\\
B_{N+1}(s) &= \left(s+\frac{\mu_N}{r}\right)B_N(s)-\frac{\lambda_{N-1}\mu_N}{r^2}B_{N-1}(s)
\end{aligned}\tag{7.6}$$

and

$$c_{j0}=\frac{\mu_1\mu_2\cdots\mu_j}{r_0 r^{j-1}}.\tag{7.7}$$

From the knowledge of the eigenvalues, left and right eigenvectors, the equilibrium distribution of the buffer occupancy is given by (see [8])

$$F_j(x)=p_j+\sum_{\ell=0}^{N-1}\beta_j^\ell\exp\left(\xi_\ell x\right)\quad\text{for } j\in\mathcal{S},\ x\ge 0,\tag{7.8}$$

where

$$\beta_j^\ell=y_j^\ell\frac{r_0F_0(0)}{r\sum_{k=0}^N y_k^\ell z_k^\ell}.\tag{7.9}$$

The unknown $F_0(0)$ representing the distribution of the buffer occupancy when the buffer is empty and the background process is in state zero is obtained as

$$F_0(0)=\frac{p_0\left[\pi_0r_0+r\sum_{j=1}^N\pi_j\right]}{r_0}.\tag{7.10}$$

Determination of eigenvalues. We determine the eigenvalues of $\mathbf{R}^{-1}\mathbf{Q}^T$ from its associated characteristic polynomial denoted by $\mathbf{P}(s)$:

$$\mathbf{P}(s)=\begin{vmatrix}
s+\dfrac{\lambda_0}{r_0} & -\dfrac{\mu_1}{r_0} & & & \\
-\dfrac{\lambda_0}{r} & s+\dfrac{\lambda_1+\mu_1}{r} & -\dfrac{\mu_2}{r} & & \\
 & \ddots & \ddots & \ddots & \\
 & & & & -\dfrac{\mu_N}{r}\\
 & & & -\dfrac{\lambda_{N-1}}{r} & s+\dfrac{\mu_N}{r}
\end{vmatrix}_{N+1}.\tag{7.11}$$

It can be written as

$$\mathbf{P}(s)=\begin{vmatrix} s+\dfrac{\lambda_0}{r_0} & \dfrac{\lambda_0}{r_0} & & & \\ \dfrac{\mu_1}{r} & s+\dfrac{\lambda_1+\mu_1}{r} & \dfrac{\lambda_1}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \dfrac{\lambda_{N-1}}{r} \\ & & & \dfrac{\mu_N}{r} & s+\dfrac{\mu_N}{r} \end{vmatrix}_{N+1}. \tag{7.12}$$

Doing the operations: (1) row i = (row i) + (row $i+1$) for $i=1,2,\ldots,N$,

(2) diminishing the second column by the first, the third column by the new second column, and so on in the above determinant, we get

$$\mathbf{P}(s)=s\times\begin{vmatrix} s+\dfrac{\lambda_0}{r_0}+\dfrac{\mu_1}{r} & \dfrac{\mu_1}{r} & & & \\ \dfrac{\lambda_1}{r} & s+\dfrac{\lambda_1}{r}+\dfrac{\mu_2}{r} & \dfrac{\mu_2}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \dfrac{\mu_{N-1}}{r} \\ & & & \dfrac{\lambda_{N-1}}{r} & s+\dfrac{\lambda_{N-1}}{r}+\dfrac{\mu_N}{r} \end{vmatrix}_{N}. \tag{7.13}$$

Thus zero is an eigenvalue of $\mathbf{R}^{-1}\mathbf{Q}^T$. The above determinant is sign-symmetric and hence can be written as

$$\mathbf{P}(s)=s\times\begin{vmatrix} s+\dfrac{\lambda_0}{r_0}+\dfrac{\mu_1}{r} & \dfrac{\sqrt{\lambda_1\mu_1}}{r} & & & \\ \dfrac{\sqrt{\lambda_1\mu_1}}{r} & s+\dfrac{\lambda_1}{r}+\dfrac{\mu_2}{r} & \dfrac{\sqrt{\lambda_2\mu_2}}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \dfrac{\sqrt{\lambda_{N-1}\mu_{N-1}}}{r} \\ & & & \dfrac{\sqrt{\lambda_{N-1}\mu_{N-1}}}{r} & s+\dfrac{\lambda_{N-1}}{r}+\dfrac{\mu_N}{r} \end{vmatrix}_{N}. \tag{7.14}$$

The other eigenvalues of $\mathbf{R}^{-1}\mathbf{Q}^T$ are determined from the associated real symmetric matrix of this reduced determinant $\mathbf{P}(s)$ by using the method of bisection suggested by Evans et al. [4] with suitable modifications.

Determination of eigenvectors. Let $\mathbf{M}(s)=((a_{ij}))$ denote the matrix $(s\mathbf{I}-\mathbf{R}^{-1}\mathbf{Q}^T)$ where $\mathbf{I}$ is the identity matrix of order $N+1$. In expression (7.5) of the eigenvector of the underlying matrices, the polynomials $B_j(s)$ play a major role. Since the system of equations given by (7.6) is an underdetermined system, at least one of the equations is redundant. If the kth equation is redundant, we may assume $B_k(s) = 1$ and solve the rest of the equations. Fernando [5] provides a method to overcome the instability that arises because of this particular normalization. This is achieved by computing the diagonal entries of the matrix $\mathcal{F}$, which is obtained by elementwise reciprocation of the inverse of $\mathbf{M}(s)^T$ based on LDU and UDL factorization of the tridiagonal matrix $\mathbf{M}(s)$.

We consider the LDU factorization of $\mathbf{M}(s)$. The diagonal elements $d_i(s)$ of D are given recursively as

$$\begin{aligned} d_0(s) &= a_{0,0}, \\ d_i(s) &= a_{i,i} - \frac{a_{i-1,i}a_{i,i-1}}{d_{i-1}(s)}, \quad \text{for } i = 1,2,\ldots,N, \end{aligned} \tag{7.15}$$

where s is the eigenvalue of the matrix $\mathbf{R}^{-1}\mathbf{Q}^T$. Now, we consider the UDL factorization of $\mathbf{M}(s)$. The diagonal elements $\delta_i(s)$ are given recursively as

$$\begin{aligned} \delta_N(s) &= a_{N,N}, \\ \delta_i(s) &= a_{i,i} - \frac{a_{i+1,i}a_{i,i+1}}{\delta_{i+1}(s)}, \quad \text{for } i = N-1,N-2,\ldots,0. \end{aligned} \tag{7.16}$$

Then the diagonal elements $\eta_i(s)$ of the matrix $\mathcal{F}$ are given by

$$\begin{aligned} \eta_1(s) &= \delta_1(s), \\ \eta_i(s) &= \delta_i - \frac{a_{i-1,i}a_{i,i-1}}{d_{i-1}(s)}, \quad \text{for } i = 2,3,\ldots,N+1. \end{aligned} \tag{7.17}$$

The following algorithm may be used for computing $B_j(s)$ with suitable normalization suggested by the algorithm.

Algorithm 7.1. (1) Compute $\eta_k = \min_{0\le i\le N}\{\eta_i\}$.

(2) Set $B_k(s) = 1$, where k is corresponding to the suffix k of η_k in step (1).

(3) Compute other $B_j(s)$ using

$$\begin{aligned} B_j(s) &= -\frac{a_{j,j+1}}{d_j(s)}B_{j+1}(s), \quad j = k-1,k-2,\ldots,0, \\ B_j(s) &= -\frac{a_{j,j-1}}{\delta_j(s)}B_{j-1}(s), \quad j = k+1,k+2,\ldots,N. \end{aligned} \tag{7.18}$$

To visualize the foregoing discussion, we plot the graphs of buffer content distribution for the two models by assuming certain values for the parameter λ and μ. The variation in the buffer content distribution is well brought out by evaluating them numerically. Figure 7.1 depicts the behaviour of the stationary buffer content distribution against the content of the buffer x for both the models with $r_0 = -1$, $r = 1$ and N truncated at 30. It is observed from the graph that the buffer content distribution decreases with the increase in λ and decrease in μ.

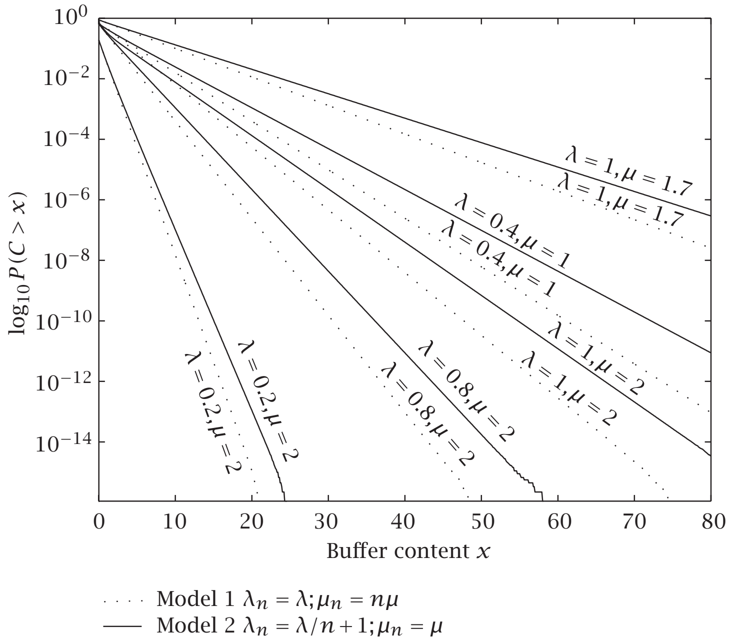

FIGURE 7.1. Buffer content distribution with $r_0 = -1$, $r = 1$, and $N = 30$.

Appendices

A. We verify below that $\hat{F}_0(s)$ and $\hat{F}_1(s)$ satisfy (4.2). Consider

$$(r_0 s+\lambda)\hat{F}_0(s) - r_0 F_0(0) = \mu \hat{F}_1(s). \tag{A.1}$$

Substituting for $\hat{F}_0(s)$ and $\hat{F}_1(s)$ from (4.8), we need to verify

$$\begin{aligned}(r_0 s+\lambda)\,{}_1F_1\left(2;\frac{\lambda rs}{(rs+\mu)^2}+2;-\frac{\lambda\mu}{(rs+\mu)^2}\right)&\\ -\frac{\lambda\mu}{(rs+\mu)}\,\frac{2\,{}_1F_1(3;\lambda rs/(rs+\mu)^2+3;-\lambda\mu/(rs+\mu)^2)}{\lambda rs/(rs+\mu)^2+2}&\\ = s\left[r\left(\frac{\lambda rs}{(rs+\mu)^2}+1\right)+(r_0-r)\,{}_1F_1\left(2;\frac{\lambda rs}{(rs+\mu)^2}+2;-\frac{\lambda\mu}{(rs+\mu)^2}\right)\right]&.\end{aligned} \tag{A.2}$$

Let $a = 2$, $c = \lambda rs/(rs+\mu)^2+2$, and $z = -\lambda\mu/(rs+\mu)^2$. Then we need to verify

$$\begin{aligned}(r_0 s+\lambda)\,{}_1F_1(a;c;z) - \frac{\lambda\mu}{(rs+\mu)}\,\frac{a\,{}_1F_1(a+1;c+1;z)}{c}&\\ = s\left[r(c-1)+(r_0-r)\,{}_1F_1(a;c;z)\right]&.\end{aligned} \tag{A.3}$$

Observe that

$$
\begin{aligned}
&(r_0 s+\lambda)\,{}_1F_1(a;c;z)-\frac{\lambda\mu}{(rs+\mu)}\frac{a\,{}_1F_1(a+1;c+1;z)}{c} \\
&\quad=\frac{1}{c(rs+\mu)}\big[c(r_0 s+\lambda)(rs+\mu)\,{}_1F_1(a;c;z)-a\lambda\mu\,{}_1F_1(a+1;c+1;z)\big] \\
&\quad=\frac{1}{c(rs+\mu)}\big[cr_0 s(rs+\mu)\,{}_1F_1(a;c;z)+c\lambda rs\,{}_1F_1(a;c;z) \\
&\qquad\qquad+\lambda\mu\big(c\,{}_1F_1(a;c;z)-a\,{}_1F_1(a+1;c+1;z)\big)\big].
\end{aligned}
\tag{A.4}
$$

Using identity (2.5), we obtain

$$
\begin{aligned}
&(r_0 s+\lambda)\,{}_1F_1(a;c;z)-\frac{\lambda\mu}{(rs+\mu)}\frac{a\,{}_1F_1(a+1;c+1;z)}{c} \\
&\quad=\frac{1}{c(rs+\mu)}\big[cr_0 s(rs+\mu)\,{}_1F_1(a;c;z)+c\lambda rs\,{}_1F_1(a;c;z) \\
&\qquad\qquad+\lambda\mu(c-a)\,{}_1F_1(a;c+1;z)\big] \\
&\quad=\frac{1}{c(rs+\mu)}\big[cr_0 s(rs+\mu)\,{}_1F_1(a;c;z) \\
&\qquad\qquad+c\lambda rs\,{}_1F_1(a;c;z)-\lambda rsz\,{}_1F_1(a;c+1;z)\big] \\
&\quad=\frac{1}{c(rs+\mu)}\big[cr_0 s(rs+\mu)\,{}_1F_1(a;c;z) \\
&\qquad\qquad+\lambda rs\big(c\,{}_1F_1(a;c;z)-z\,{}_1F_1(a;c+1;z)\big)\big].
\end{aligned}
\tag{A.5}
$$

Using (2.6), we get

$$
\begin{aligned}
&(r_0 s+\lambda)\,{}_1F_1(a;c;z)-\frac{\lambda\mu}{(rs+\mu)}\frac{a\,{}_1F_1(a+1;c+1;z)}{c} \\
&\quad=\frac{1}{c(rs+\mu)}\big[cr_0 s(rs+\mu)\,{}_1F_1(a;c;z)+c\lambda rs\,{}_1F_1(a-1;c;z)\big] \\
&\quad=s\left[r_0\,{}_1F_1(a;c;z)+\frac{\lambda r}{rs+\mu}\,{}_1F_1(a-1;c;z)\right].
\end{aligned}
\tag{A.6}
$$

Adding and subtracting $r\,{}_1F_1(a;c;z)$ yields

$$
\begin{aligned}
&(r_0 s+\lambda)\,{}_1F_1(a;c;z)-\frac{\lambda\mu}{(rs+\mu)}\frac{a\,{}_1F_1(a+1;c+1;z)}{c} \\
&\quad=s\left[(r_0-r)\,{}_1F_1(a;c;z)+r\left\{{}_1F_1(a;c;z)+\frac{\lambda}{(rs+\mu)}\,{}_1F_1(a-1;c;z)\right\}\right].
\end{aligned}
\tag{A.7}
$$

Using (2.7), we obtain

$$\begin{aligned}(r_0 s+\lambda)\,{}_1F_1(a;c;z)-\frac{\lambda\mu}{(rs+\mu)}\frac{a\,{}_1F_1(a+1;c+1;z)}{c}&\\ =s\big[(r_0-r)\,{}_1F_1(a;c;z)+r(c-a+1)\big]&\\ =s\big[(r_0-r)\,{}_1F_1(a;c;z)+r(c-1)\big].&\end{aligned}\tag{A.8}$$

Hence the verification.

B. We verify below that $\hat{G}_0(s)$ and $\hat{G}_1(s)$ satisfy (5.2). Consider

$$(r_0 s+\lambda)\hat{G}_0(s)-\mu\hat{G}_1(s)=r_0G_0(0).\tag{B.1}$$

Substituting for $\hat{G}_0(s)$ and $\hat{G}_1(s)$ from (5.8), we need to verify

$$\begin{aligned}(r_0 s+\lambda)\,{}_1F_1\Big(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\Big)-\frac{\lambda}{rs/\mu+1}\,{}_1F_1\Big(2;\frac{rs}{\mu}+2;-\frac{\lambda}{\mu}\Big)&\\ =s\Big[r+(r_0-r)\,{}_1F_1\Big(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\Big)\Big].&\end{aligned}\tag{B.2}$$

Observe that

$$\begin{aligned}&(r_0 s+\lambda)\,{}_1F_1\Big(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\Big)-\frac{\lambda}{rs/\mu+1}\,{}_1F_1\Big(2;\frac{rs}{\mu}+2;-\frac{\lambda}{\mu}\Big)\\ &\quad=\frac{1}{rs/\mu+1}\Big[(r_0 s+\lambda)\Big(\frac{rs}{\mu}+1\Big)\,{}_1F_1\Big(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\Big)-\lambda\,{}_1F_1\Big(2;\frac{rs}{\mu}+2;-\frac{\lambda}{\mu}\Big)\Big]\\ &\quad=\frac{1}{rs/\mu+1}\Big[r_0 s\Big(\frac{rs}{\mu}+1\Big)\,{}_1F_1\Big(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\Big)\\ &\qquad\qquad+\lambda\Big\{\Big(\frac{rs}{\mu}+1\Big)\,{}_1F_1\Big(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\Big)-{}_1F_1\Big(2;\frac{rs}{\mu}+2;-\frac{\lambda}{\mu}\Big)\Big\}\Big].\end{aligned}\tag{B.3}$$

Using identity (2.5) with $a=1$, $c=rs/\mu+1$, and $z=-\lambda/\mu$, we obtain

$$\begin{aligned}&(r_0 s+\lambda)\,{}_1F_1\Big(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\Big)-\frac{\lambda}{rs/\mu+1}\,{}_1F_1\Big(2;\frac{rs}{\mu}+2;-\frac{\lambda}{\mu}\Big)\\ &\quad=\frac{1}{rs/\mu+1}\Big[r_0 s\Big(\frac{rs}{\mu}+1\Big)\,{}_1F_1\Big(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\Big)+\lambda\frac{rs}{\mu}\,{}_1F_1\Big(1;\frac{rs}{\mu}+2;-\frac{\lambda}{\mu}\Big)\Big].\end{aligned}\tag{B.4}$$

Adding and subtracting the term $r(rs/\mu+1)\,{}_1F_1(1;rs/\mu+1;-\lambda/\mu)$ and using identity (2.6), we obtain

$$\begin{aligned}
&(r_0 s+\lambda)\,{}_1F_1\left(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\right)-\frac{\lambda}{rs/\mu+1}\,{}_1F_1\left(2;\frac{rs}{\mu}+2;-\frac{\lambda}{\mu}\right)\\
&\quad=\frac{s}{rs/\mu+1}\left[(r_0-r)\left(\frac{rs}{\mu}+1\right){}_1F_1\left(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\right)+r\left(\frac{rs}{\mu}+1\right)\right] \qquad \text{(B.5)}\\
&\quad=s\left[r+(r_0-r)\,{}_1F_1\left(1;\frac{rs}{\mu}+1;-\frac{\lambda}{\mu}\right)\right].
\end{aligned}$$

Hence the verification.

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