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A RELATIVE INTEGRAL BASIS OVER $\mathbb{Q}(\sqrt{-3})$ FOR THE NORMAL CLOSURE OF A PURE CUBIC FIELD

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Let K be a pure cubic field. Let L be the normal closure of K. A relative integral basis (RIB) for L over $\mathbb{Q}(\sqrt{-3})$ is given. This RIB simplifies and completes the one given by Haghighi (1986).

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1. Introduction. Let K be the pure cubic field $\mathbb{Q}(d^{1/3})$, where d is a cube-free integer, and let L be the normal closure of K so that $\mathbb{Q} \subset K \subset L$, $[L:K] = 2$, and $[K:\mathbb{Q}] = 3$. Let k be the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$ so that $\mathbb{Q} \subset k \subset L$, $[L:k] = 3$, and $[k:\mathbb{Q}] = 2$. The ring of all algebraic integers is denoted by Ω. The rings of integers of K, k, L are $O_K = K \cap \Omega$, $O_k = k \cap \Omega$, $O_L = L \cap \Omega$, respectively. As O_k is a principal ideal domain, L/k possesses a relative integral basis (RIB) [3, Corollary 3, page 401]. Haghighi [2, Theorems 5.1, 5.3, 5.6] has given a RIB for L/k. However, Haghighi's RIB for L/k contains two difficulties. The first is that in certain cases the RIB makes use of an element of norm 3 in a pure cubic field, a quantity which is not easy to determine, see [2, Theorem 5.1]. The second problem is that the RIB is not completely general, see [2, Theorem 5.3]. In this note, we give a simple and completely general RIB for L/k.

2. Preliminary remarks. As d is a cube-free integer, we can define integers a and b by

$$d = ab^2, \quad (a,b) = 1, \ a,b \text{ square-free.} \tag{2.1}$$

If $a^2 \not\equiv b^2 \pmod 9$, an integral basis for K is

$$\left\{1, (ab^2)^{1/3}, (a^2b)^{1/3}\right\}, \tag{2.2}$$

and if $a^2 \equiv b^2 \pmod 9$, an integral basis is

$$\left\{1, (ab^2)^{1/3}, \frac{b + ab(ab^2)^{1/3} + (a^2b)^{1/3}}{3}\right\}. \tag{2.3}$$

These integral bases are due to Dedekind [1]. From (2.2) and (2.3), we deduce that the discriminant $d(K)$ of K is given by

$$d(K) = -3f^2, \tag{2.4}$$

where

$$f = \begin{cases} 3ab, & \text{if } a^2 \not\equiv b^2 \pmod 9, \\ ab, & \text{if } a^2 \equiv b^2 \pmod 9. \end{cases} \tag{2.5}$$

The relative discriminant $d(L/k)$ of L/k is given by

$$d(L/k) = f^2 = \begin{cases} 9a^2b^2, & \text{if } a^2 \not\equiv b^2 \pmod 9, \\ a^2b^2, & \text{if } a^2 \equiv b^2 \pmod 9, \end{cases} \tag{2.6}$$

see [1]. We note that if $\alpha, \beta \in O_L$ are such that

$$d_{L/k}(1, \alpha, \beta) = d(L/k), \tag{2.7}$$

then $\{1, \alpha, \beta\}$ is a RIB for L/k.

3. RIB for L/k. We show that $\{1, \alpha, \beta\}$ is a RIB for L/k, where α and β are given in Table 3.1.

TABLE 3.1

Case	Condition	α	β
(i)	$3 \mid a$, $3 \nmid b$	$(ab^2)^{1/3}$	$\dfrac{(a^2b)^{1/3}}{\sqrt{-3}}$
(ii)	$3 \nmid a$, $3 \mid b$	$\dfrac{(ab^2)^{1/3}}{\sqrt{-3}}$	$(a^2b)^{1/3}$
(iii)	$3 \nmid a$, $3 \nmid b$, $9 \nmid a^2 - b^2$	$(ab^2)^{1/3}$	$\dfrac{b + ab(ab^2)^{1/3} + (a^2b)^{1/3}}{\sqrt{-3}}$
(iv)	$3 \nmid a$, $3 \nmid b$, $9 \mid a^2 - b^2$	$\dfrac{(ab^2)^{1/3} - a}{\sqrt{-3}}$	$\dfrac{b + ab(ab^2)^{1/3} + (a^2b)^{1/3}}{3}$

An easy calculation making use of (2.2), (2.3), (2.4), and (2.5) shows that

$$d_{L/k}(1, \alpha, \beta) = \begin{cases} 9a^2b^2, & \text{if } a^2 \not\equiv b^2 \pmod 9, \\ a^2b^2, & \text{if } a^2 \equiv b^2 \pmod 9, \end{cases} \tag{3.1}$$

so that (2.7) holds in view of (2.6). Clearly, $\alpha \in L$ and $\beta \in L$. We now show that $\alpha \in \Omega$ and $\beta \in \Omega$ so that $\alpha \in O_L$ and $\beta \in O_L$, proving that $\{1, \alpha, \beta\}$ is a RIB for L/k. Clearly, $\alpha \in \Omega$ in Cases (i) and (iii), and $\beta \in \Omega$ in Cases (ii) and (iv), see (2.3) for the latter. In the remaining cases, it suffices to give a monic polynomial $f_\alpha(x) \in \mathbb{Z}[x]$ of which α is a root in Cases (ii) and (iv), and a monic polynomial $f_\beta(x) \in \mathbb{Z}[x]$ of which β is a root in Cases (i) and (iii).

Case (i). Here,

$$f_\beta(x) = x^6 + 3a_1^4 b^2, \quad a_1 = \frac{a}{3} \in \mathbb{Z}. \tag{3.2}$$

Case (ii). Here,

$$f_\alpha(x) = x^6 + 3a^2 b_1^4, \quad b_1 = \frac{b}{3} \in \mathbb{Z}. \tag{3.3}$$

Case (iii). We have

$$a^2 \equiv b^2 \equiv 1 (\operatorname{mod} 3), \qquad a^2 - b^2 \equiv 0 (\operatorname{mod} 3), \tag{3.4}$$

so that

$$\begin{aligned} a^4b^4 - 3a^2b^2 + a^2 + b^2 &= (a^2 - b^2)^2 + (a^2 - 1)(b^2 - 1)(a^2b^2 + a^2 + b^2) \\ &\equiv 0 (\operatorname{mod} 9), \end{aligned} \tag{3.5}$$

and we define $m \in \mathbb{Z}$ by

$$m = \frac{(a^4b^4 - 3a^2b^2 + a^2 + b^2)}{9}. \tag{3.6}$$

In this case,

$$f_\beta(x) = x^6 + (2a^2 + 1)b^2x^4 + ((a^2 - 1)^2 b^2 - 6m)b^2x^2 + 3b^2m^2. \tag{3.7}$$

Case (iv). We have

$$a^2 \equiv b^2 \equiv 1 (\operatorname{mod} 3), \qquad a^2 - b^2 \equiv 0 (\operatorname{mod} 9), \qquad a^2 + 2b^2 \equiv 0 (\operatorname{mod} 3) \tag{3.8}$$

so that we can define $r, s \in \mathbb{Z}$ by

$$r = \frac{(a^2 + 2b^2)}{3}, \qquad s = \frac{(a^2 - b^2)}{9}. \tag{3.9}$$

Here,

$$f_\alpha(x) = x^6 + a^2x^4 + a^2rx^2 + 3a^2s^2. \tag{3.10}$$

This completes the proof that $\{1, \alpha, \beta\}$ is a RIB for L/k.

We conclude with four examples.

Example 3.1 (cf. [2, Illustration 5.2]). A RIB for $\mathbb{Q}(\sqrt[3]{213}, \sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (i))

$$\left\{1, 213^{1/3}, \frac{213^{2/3}}{\sqrt{-3}}\right\}. \tag{3.11}$$

Example 3.2. A RIB for $\mathbb{Q}(\sqrt[3]{9},\sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (ii))

$$\left\{1,\frac{9^{1/3}}{\sqrt{-3}},3^{1/3}\right\}. \tag{3.12}$$

Example 3.3 (cf. [2, Illustration 5.5]). A RIB for $\mathbb{Q}(\sqrt[3]{2},\sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (iii))

$$\left\{1,2^{1/3},\frac{1+2\cdot 2^{1/3}+2^{2/3}}{\sqrt{-3}}\right\}. \tag{3.13}$$

Example 3.4 (cf. [2, Illustration 5.7]). A RIB for $\mathbb{Q}(\sqrt[3]{10},\sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (iv))

$$\left\{1,\frac{10^{1/3}-10}{\sqrt{-3}},\frac{1+10\cdot 10^{1/3}+10^{2/3}}{3}\right\}. \tag{3.14}$$

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