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ON HOPF DEMEYER-KANZAKI GALOIS EXTENSIONS

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Let H be a finite-dimensional Hopf algebra over a field k, B a left H-module algebra, and H^* the dual Hopf algebra of H. For an H^*-Azumaya Galois extension B with center C, it is shown that B is an H^*-DeMeyer-Kanzaki Galois extension if and only if C is a maximal commutative separable subalgebra of the smash product $B\#H$. Moreover, the characterization of a commutative Galois algebra as given by S. Ikehata (1981) is generalized.

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1. Introduction. Let H be a finite-dimensional Hopf algebra over a field k, B a left H-module algebra, and H^* the dual Hopf algebra of H. In [7], the class of Azumaya Galois extensions of a ring as studied in [1, 2] was generalized to H^*-Azumaya Galois extensions. An H^*-Azumaya Galois extension B was characterized in terms of the smash product $B\#H$ see [7, Theorem 3.4]. Observing that the commutator $V_B(B^H)$ of B^H in B is also an H^*-Azumaya Galois extension (see [7, Lemma 4.1]), in the present paper, we will give a characterization of an H^*-Azumaya Galois extension B in terms of $V_B(B^H)$. Moreover, we will investigate the class of H^*-Azumaya Galois extensions B such that $V_B(B^H) = C$, where C is the center of B. We note that when $H = kG$, where G is a finite automorphism group of B, such a B is precisely a DeMeyer-Kanzaki Galois extension with Galois group G [3, 6, 8, 9]. Several equivalent conditions are then given for an H^*-Azumaya Galois extension being an H^*-DeMeyer-Kanzaki Galois extension, and the characterization of a commutative Galois algebra as given by Ikehata [5, Theorem 2] is generalized to an H^*-DeMeyer-Kanzaki Galois extension.

2. Basic definitions and notation. Throughout, H denotes a finite-dimensional Hopf algebra over a field k with comultiplication Δ and counit ε, H^* the dual Hopf algebra of H, B a left H-module algebra, C the center of B, $B^H = \{b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H\}$, and $B\#H$ the smash product of B with H, where $B\#H = B \otimes_k H$ such that, for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1b')\#h_2h'$, where $\Delta(h) = \sum h_1 \otimes h_2$.

For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, \ldots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$ and $\sum ba_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. A ring B is called an H^*-Galois extension of B^H if B is a right H^*-comodule algebra with structure map $\rho : B \to B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \to B \otimes_k H^*$ is a bijection where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$. An H^*-Galois extension B is called an H^*-Azumaya Galois extension if B is separable over B^G which is an Azumaya algebra over C^G, and an H^*-DeMeyer-Kanzaki Galois extension if B is an H^*-Azumaya Galois extension and $V_B(B^H) = C$.

Let P be a finitely generated and projective module over a commutative ring R. Then for a prime ideal p of R, P_p $(= P \otimes_R R_p)$ is a free module over R_p $(=$ the local ring of R at $p)$, and the rank of P_p over R_p is the number of copies of R_p in P_p, that is, $\text{rank}_{R_p}(P_p) = m$ for some integer m. It is known that the $\text{rank}_R(P)$ is a continuous function $(\text{rank}_R(P)(p) = \text{rank}_{R_p}(P_p) = m)$ from $\text{Spec}(R)$ to the set of nonnegative integers with the discrete topology (see [4, Corollary 4.11, page 31]). We will use the $\text{rank}_R(P)$-function for a finitely generated and projective module P over a commutative ring R.

3. H^*-Azumaya Galois extensions.

In this section, keeping all notations as given in Section 2, we will characterize an H^*-Azumaya Galois extension B in terms of the commutator $V_B(B^H)$ of B^H in B.

Theorem 3.1. *If $B = B^H \cdot V_B(B^H)$, then $(V_B(B^H))^H = C^H$.*

Proof. Since $C \subset V_B(B^H)$, $C^H \subset (V_B(B^H))^H$. Conversely, since $V_B(B^H) \subset B$, $(V_B(B^H))^H \subset B^H$. Hence $(V_B(B^H))^H \subset B^H \cap V_B(B^H) \subset$ the center of $V_B(B^H)$. But $B = B^H \cdot V_B(B^H)$, so the center of $V_B(B^H)$ is C. Thus, $(V_B(B^H))^H \subset C^H$. □

Theorem 3.2. *A ring B is an H^*-Azumaya Galois extension of B^H if and only if $B = B^H \cdot V_B(B^H)$ such that $V_B(B^H)$ is an H^*-Azumaya Galois extension of C^H and B^H is an Azumaya C^H-algebra.*

Proof. $(\Rightarrow)$ Since B is an H^*-Azumaya Galois extension of B^H, then $V_B(B^H)$ is an H^*-Azumaya Galois extension of $(V_B(B^H))^H$ (see [7, Lemma 4.1]) and B^H is an Azumaya C^H-algebra (see [7, Theorem 3.4]). Moreover, by the proof of [7, Lemma 4.1], $B\#H$ is an Azumaya C^H-algebra such that $B\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H) \cong B^H(V_B(B^H)\#H)$, where B^H and $V_B(B^H)\#H$ are Azumaya C^H-algebras. But H is a finite-dimensional Hopf algebra over a field k, so $B \cong B^H \otimes_{C^H} V_B(B^H)$ from the isomorphism $B\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H)$, and so $B = B^H \cdot V_B(B^H)$. Hence $(V_B(B^H))^H = C^H$ by Theorem 3.1. Thus $V_B(B^H)$ is an H^*-Azumaya Galois C^H-algebra.

$(\Leftarrow)$ Since $V_B(B^H)$ is an H^*-Azumaya Galois algebra over C^H, $V_B(B^H)\#H$ is an Azumaya C^H-algebra [7, Theorem 3.4]. By hypothesis, B^H is an Azumaya C^H-algebra, so $B^H \otimes_{C^H} (V_B(B^H)\#H) \cong B^H V_B(B^H)\#H = B\#H$ which is an Azumaya

C^H-algebra. Thus $B\#H$ is a Hirata separable extension of B (see [5, Theorem 1]). Moreover, $V_B(B^H)$ is a separable C^H-algebra (see [7, Theorem 3.4]) and B^H is an Azumaya C^H-algebra by hypothesis, so $B^H \cdot V_B(B^H)$ $(= B)$ is also a separable C^H-algebra. Thus B is an H^*-Azumaya Galois extension of B^H [7, Theorem 3.4]. □

Next we generalize the characterization of a commutative Galois algebra as given by Ikehata (see [5, Theorem 2]) to a commutative H^*-Galois algebra.

Lemma 3.3. *If C is a commutative H^*-Galois algebra over C^H, then C is a maximal commutative subalgebra of $C\#H$.*

Proof. Since C is a commutative H^*-Galois algebra over C^H, $C\#H \cong \mathrm{Hom}_{C^H}(C,C)$ [6, Theorem 1.7]. Hence it suffices to show that $V_{\mathrm{Hom}_{C^H}(C,C)}(C_L) = C_L$ where $C_L = \{c_L$, the left multiplication map induced by $c \in C\}$. In fact, $C_L \subset V_{\mathrm{Hom}_{C^H}(C,C)}(C_L)$ is clear. Conversely, let $f \in V_{\mathrm{Hom}_{C^H}(C,C)}(C_L)$. Then, for each $c \in C$, $(cf)(x) = (fc)(x)$ for all $x \in C$. Hence $cf(x) = f(cx)$, and so $cf(1) = f(c)$ for all $c \in C$. Thus $f(c) = d_f(c)$ for all $c \in C$, where $d_f = f(1) \in C$, that is, $f = (d_f)_L \in C_L$. □

Theorem 3.4. *Let C be a commutative separable C^H-algebra containing C^H as a direct summand as a C^H-module. Then, C is a commutative H^*-Galois algebra over C^H if and only if $C \otimes_{C^H} (C\#H) \cong M_n(C)$, the matrix algebra over C of order n where n is the dimension of H over k.*

Proof. (⇒) Since C is an H^*-Galois algebra over C^H, $C\#H \cong \mathrm{Hom}_{C^H}(C,C)$ such that C is finitely generated and projective over C^H [6, Theorem 1.7]. Hence $C\#H$ is an Azumaya C^H-algebra and C is a maximal commutative subalgebra of the Azumaya C^H-algebra $C\#H$ by Lemma 3.3. By hypothesis, C is also a separable C^H-algebra, so C is a splitting ring for the Azumaya C^H-algebra $C\#H$ such that $C \otimes_{C^H} (C\#H) \cong \mathrm{Hom}_C(C\#H, C\#H)$ (see the proof of [4, Theorem 5.5, page 64]). Noting that $C\#H = C \otimes_k H$ which is a free C-module of rank n where $n = \dim_k(H)$, we have that $C \otimes_{C^H} (C\#H) \cong M_n(C)$.

(⇐) Since $C \otimes_{C^H} (C\#H) \cong M_n(C)$, $C \otimes_{C^H} (C\#H)$ is an Azumaya C-algebra. By hypothesis, C^H is a direct summand of C as a C^H-module, so $C\#H$ is an Azumaya C^H-algebra [4, Corollary 1.10, page 45]. Hence $C\#H$ is a Hirata separable extension of C. But C is a separable C^H-algebra by hypothesis, so C is an H^*-Galois algebra over C^H [7, Theorem 3.4].

We remark that the necessity does not need the hypothesis that C^H is a direct summand of C. □

4. H^*-DeMeyer-Kanzaki Galois extensions. We recall that B is an H^*-DeMeyer-Kanzaki Galois extension of B^H if B is an H^*-Azumaya Galois extension of B^H and $V_B(B^H) = C$. In this section, we characterize an H^*-DeMeyer-Kanzaki Galois extension in terms of the smash product $V_B(B^H)\#H$ and prove that C is a splitting ring for the Azumaya C^H-algebras $V_B(B^H)\#H$ and $B\#H$.

Theorem 4.1. *Let B be an H^*-Azumaya Galois extension of B^H. Then the following statements are equivalent:*

(1) *B is an H^*-DeMeyer-Kanzaki Galois extension of B^H;*

(2) $\operatorname{rank}_{C^H}(V_B(B^H)) = \operatorname{rank}_{C^H}(C)$*;*

(3) *C is a maximal commutative separable subalgebra of $V_B(B^H)\#H$.*

Proof. (1)⇒(2). It is clear.

(2)⇒(1). Since B is an H^*-Azumaya Galois extension of B^H, $V_B(B^H)$ is an H^*-Azumaya Galois algebra over C^H by Theorem 3.2 such that $V_B(B^H)$ is a separable and finitely generated projective module over C^H (see [7, Theorem 3.4]). Hence the rank function $\operatorname{rank}_{C^H}(V_B(B^H))$ is defined and $V_B(B^H)$ is an Azumaya algebra over its center [4, Theorem 3.8, page 55]. But $B = B^H \cdot V_B(B^H)$ by Theorem 3.2, so the center of $V_B(B^H)$ is C. Thus $V_B(B^H)$ is an Azumaya C-algebra; and so C is a direct summand $V_B(B^H)$ as a C-module. This implies that C is a direct summand $V_B(B^H)$ as a C^H-module. Therefore the rank function $\operatorname{rank}_{C^H}(C)$ is also defined. Now by hypothesis, $\operatorname{rank}_{C^H}(V_B(B^H)) = \operatorname{rank}_{C^H}(C)$, so $V_B(B^H) = C$, that is, B is an H^*-DeMeyer-Kanzaki Galois extension of B^H.

(1)⇒(3). Since B is an H^*-DeMeyer-Kanzaki Galois extension of B^H, B is an H^*-Azumaya Galois extension such that $V_B(B^H) = C$. Hence $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{C^H} C$ such that C is an H^*-Galois algebra over C^H by Theorem 3.2, and so C is a separable C^H-algebra containing C^H as a direct summand as a C^H-module [7, Theorem 3.4]. Hence C is a maximal commutative separable subalgebra of $C\#H$ where $C = V_B(B^H)$ by Lemma 3.3.

(3)⇒(2). Since B is an H^*-Azumaya Galois extension of B^H, $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{C^H} V_B(B^H)$ such that $V_B(B^H)$ is an H^*-Azumaya Galois algebra over C^H by Theorem 3.2. Hence $V_B(B^H)\#H$ is an Azumaya C^H-algebra and $V_B(B^H)$ is an Azumaya C-algebra [7, Theorem 3.4]. By hypothesis, C is a maximal commutative separable subalgebra of $V_B(B^H)\#H$, so

$$C \otimes_{C^H} (V_B(B^H)\#H) \cong \operatorname{Hom}_C(V_B(B^H)\#H, V_B(B^H)\#H) \tag{4.1}$$

(see [4, Theorem 5.5, page 64]). On the other hand, $V_B(B^H)\#H \cong \operatorname{Hom}_{C^H}(V_B(B^H), V_B(B^H))$ (see [7, Theorem 3.4]). Thus

$$\begin{aligned} C \otimes_{C^H} (V_B(B^H)\#H) &\cong C \otimes_{C^H} \operatorname{Hom}_{C^H}(V_B(B^H), V_B(B^H)) \\ &\cong \operatorname{Hom}_C(C \otimes_{C^H} V_B(B^H), C \otimes_{C^H} V_B(B^H)); \end{aligned} \tag{4.2}$$

and so $\operatorname{Hom}_C(V_B(B^H)\#H, V_B(B^H)\#H) \cong \operatorname{Hom}_C(C \otimes_{C^H} V_B(B^H), C \otimes_{C^H} V_B(B^H))$. This implies that $V_B(B^H)\#H \cong P \otimes_C (C \otimes_{C^H} V_B(B^H))$ for some finitely generated projective C-module P of rank 1, that is, $V_B(B^H)\#H \cong P \otimes_{C^H} V_B(B^H)$. Taking $\operatorname{rank}_{C^H}(\ )$ both sides, we have that $n \cdot \operatorname{rank}_{C^H}(V_B(B^H)) = (\operatorname{rank}_{C^H}(P)) \cdot (\operatorname{rank}_{C^H}(V_B(B^H)))$ where $n = \dim_k(H)$. But $\operatorname{rank}_{C^H}(V_B(B^H))$ is also n, so $\operatorname{rank}_{C^H}(C) = \operatorname{rank}_{C^H}(P) = n = \operatorname{rank}_{C^H}(V_B(B^H))$. □

Theorem 4.1 implies that the Azumaya C^H-algebras $V_B(B^H)\#H$ and $B\#H$ have a nice splitting ring C which is an H^*-Galois algebra over C^H and separable over C^H such that $C\otimes_{C^H}(V_B(B^H)\#H)$ and $C\otimes_{C^H}(B\#H)$ are matrix algebras.

Corollary 4.2. *If B is an H^*-DeMeyer-Kanzaki Galois extension of B^H, then $C\otimes_{C^H}(V_B(B^H)\#H)\cong M_n(C)$, the matrix algebra over C of order n where $n=\dim_k(H)$.*

Proof. By hypothesis, B is an H^*-DeMeyer-Kanzaki Galois extension of B^H, so C $(=V_B(B^H))$ is an H^*-Galois algebra over C^H by Theorem 3.2. Hence C is a separable C^H-algebra and $C\#H$ is an Azumaya C^H-algebra [7, Theorem 3.4]. Thus C^H is a direct summand of C as a C^H-module. Therefore, $C\otimes_{C^H}(C\#H)\cong M_n(C)$ by Theorem 3.4. □

Corollary 4.3. *If B is an H^*-DeMeyer-Kanzaki Galois extension of B^H, then $C\otimes_{C^H}(B\#H)\cong M_n(B)$, the matrix algebra over B of order n where $n=\dim_k(H)$.*

Proof. By Corollary 4.2, $C\otimes_{C^H}(C\#H)\cong M_n(C)$, so

$$B^H\otimes_{C^H}C\otimes_{C^H}(C\#H)\cong B^H\otimes_{C^H}M_n(C). \tag{4.3}$$

Since $B=B^H\cdot V_B(B^H)\cong B^H\otimes_{C^H}V_B(B^H)=B^H\otimes_{C^H}C$, we have that

$$\begin{aligned} C\otimes_{C^H}(B\#H) &\cong C\otimes_{C^H}((B^H\otimes_{C^H}C)\#H)\\ &\cong C\otimes_{C^H}B^H\otimes_{C^H}(C\#H)\\ &\cong B^H\otimes_{C^H}C\otimes_{C^H}(C\#H)\\ &\cong B^H\otimes_{C^H}M_n(C)\cong M_n(B^H\otimes_{C^H}C)\\ &\cong M_n(B). \end{aligned} \tag{4.4}$$

□

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