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ON THE CONVOLUTION PRODUCT OF THE DISTRIBUTIONAL KERNEL $K_{\alpha,\beta,\gamma,\nu}$

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We introduce a distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ which is related to the operator $\oplus^k$ iterated k times and defined by $\oplus^k = [(\sum_{r=1}^{p}\partial^2/\partial x_r^2)^4 - (\sum_{j=p+1}^{p+q}\partial^2/\partial x_j^2)^4]^k$, where $p+q=n$ is the dimension of the space $\mathbb{R}^n$ of the n-dimensional Euclidean space, $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, k is a nonnegative integer, and α, β, γ, and ν are complex parameters. It is found that the existence of the convolution $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'}$ is depending on the conditions of p and q.

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1. Introduction. The operator $\oplus^k$ can be factorized in the form

$$\begin{aligned}\oplus^k &= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\ &\quad\times\left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k\left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k,\end{aligned} \tag{1.1}$$

where $p+q=n$ is the dimension of the space $\mathbb{R}^n$, $i=\sqrt{-1}$, and k is a nonnegative integer. The operator $(\sum_{r=1}^{p}\partial^2/\partial x_r^2)^2 - (\sum_{j=p+1}^{p+q}\partial^2/\partial x_j^2)^2$ is first introduced by Kananthai [2] and named the Diamond operator denoted by

$$\diamond = \left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2. \tag{1.2}$$

We denote the operators L_1 and L_2 by

$$\begin{aligned}L_1 &= \sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}, \\ L_2 &= \sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}.\end{aligned} \tag{1.3}$$

Thus (1.1) can be written by

$$\oplus^k = \Diamond^k L_1^k L_2^k. \tag{1.4}$$

Now consider the convolution $R_\alpha^H(u) * R_\beta^e(v) * S_\gamma(w) * T_\nu(z)$ where $R_\alpha^H(u)$, $R_\beta^e(v)$, $S_\gamma(w)$, and $T_\nu(z)$ are defined by (2.2), (2.4), (2.6), and (2.7), respectively.

We defined the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ by

$$K_{\alpha,\beta,\gamma,\nu} = R_\alpha^H(u) * R_\beta^e(v) * S_\gamma(w) * T_\nu(z). \tag{1.5}$$

Since the functions $R_\alpha^H(u)$, $R_\beta^e(v)$, $S_\gamma(w)$, and $T_\nu(z)$ are all tempered distributions and the supports of $R_\alpha^H(u)$ and $R_\beta^e(v)$ are compact (see [2, pages 30–31] and [1, pages 152–153]), then the convolution on the right-hand side of (1.5) exists and also is a tempered distributions. Thus $K_{\alpha,\beta,\gamma,\nu}$ is well defined and also is a tempered distribution.

For $\alpha = \beta = \gamma = \nu = 2k$, we obtain $(-1)^k K_{2k,2k,2k,2k}$ as an elementary solution of the operator $\oplus^k$, see [3]. That is $\oplus^k(-1)^k K_{2k,2k,2k,2k}(x) = \delta$ where δ is the Dirac-delta distribution and $\oplus^k$ is defined by (1.4).

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write

$$x = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \quad p + q = n. \tag{2.1}$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the interior of forward cone and $\overline{\Gamma}_+$ denote its closure. For any complex number α, we define the function

$$R_\alpha^H(x) = \begin{cases} \dfrac{u^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+, \\ 0, & \text{if } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma((2+\alpha-n)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p)/2)\Gamma((p-\alpha)/2)}, \tag{2.3}$$

the function R_α^H is first introduced by Nozaki [4, page 72] and is called the ultra-hyperbolic kernel of Marcel Riesz. Hence $R_\alpha^H(x)$ is an ordinary function if $\operatorname{Re}(\alpha) \geq n$ and is a distribution of α if $\operatorname{Re}(\alpha) < n$. Let $\operatorname{supp} R_\alpha^H(u) \subset \overline{\Gamma}_+$ where $\operatorname{supp} R_\alpha^H(u)$ denotes the support of $R_\alpha^H(u)$.

Definition 2.2. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write $v = x_1^2 + x_2^2 + \cdots + x_n^2$.

For any complex number β, define the function

$$R_\beta^e(v) = \frac{v^{(\beta-n)/2}}{W_n(\beta)}, \tag{2.4}$$

where $W_n(\beta) = \pi^{n/2} 2^\beta \Gamma(\beta)/\Gamma((n-\beta)/2)$, the function $R_\beta^e(v)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if $\mathrm{Re}(\beta) \geq n$ and is a distribution of β if $\mathrm{Re}(\beta) < n$.

Definition 2.3. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the space $\mathbb{R}^n$. Write

$$\begin{aligned} w &= x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}), \\ z &= x_1^2 + x_2^2 + \cdots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}), \quad p+q = n,\ i = \sqrt{-1}. \end{aligned} \tag{2.5}$$

For any complex numbers γ and ν, define

$$S_\gamma(\omega) = \frac{\omega^{(\gamma-n)/2}}{W_n(\gamma)}, \tag{2.6}$$

$$T_\nu(z) = \frac{z^{(\nu-n)/2}}{W_n(\nu)}, \tag{2.7}$$

where

$$W_n(\gamma) = \frac{\pi^{n/2} 2^\gamma \Gamma(\gamma/2)}{\Gamma((n-\gamma)/2)}, \qquad W_n(\nu) = \frac{\pi^{n/2} 2^\nu \Gamma(\nu/2)}{\Gamma((n-\nu)/2)}. \tag{2.8}$$

Lemma 2.4 (the convolution product of $R_\beta^e(v)$). *The convolution $R_\beta^e * R_{\beta'}^e = R_{\beta+\beta'}^e$ where R_β^e and $R_{\beta'}$ are given by (2.2).*

Proof. See [5, page 20]. □

Lemma 2.5 (the convolution product of $R_\alpha^H(x)$). *The convolution product is given by*

(i)

$$R_\alpha^H * R_{\alpha'}^H = \frac{\cos(\alpha(\pi/2))\cos(\alpha'(\pi/2))}{\cos((\alpha+\beta)/2)\pi} \cdot R_{\alpha+\alpha'}^H, \tag{2.9}$$

where R_α^H and $R_{\alpha'}^H$ are defined by (2.1) with p even,

(ii) $R^H_\alpha * R^H_\alpha = R^H_{\alpha+\alpha'} + A_{\alpha,\alpha'}$ *for p odd, where*

$$A_{\alpha,\alpha'} = \frac{2\pi i}{4}\frac{C((-\alpha-\alpha')/2)}{C(-\alpha/2)C(-\alpha'/2)}[H^+_{\alpha+\alpha'} - H^-_{\alpha+\alpha'}],$$
$$C(r) = \Gamma(r)\Gamma(1-r),$$
$$H^\pm_r = H_r(x \pm i0, n) = e^{\mp r(\pi/2)i}e^{\pm q(\pi/2)i}a\left(\frac{r}{2}\right)(u \pm i0)^{(r-n)/2}, \tag{2.10}$$
$$a\left(\frac{r}{2}\right) = \Gamma\left(\frac{n-r}{2}\right)\left(2^r\pi^{n/2}\Gamma\left(\frac{r}{2}\right)\right)^{-1},$$
$$(u \pm i0, n)^\lambda = \lim_{\epsilon\to 0}(u \pm i\epsilon, n)^\lambda,$$

$u = u(x)$ is defined by (2.1) and $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ in particular $R^H_\alpha * R^H_{-2k} = R^H_{\alpha-2k}$ and $R^H_\alpha * R^H_{2k} = R^H_{\alpha+2k}$.

The proof of this lemma is given by Téllez [6, pages 121–123].

Lemma 2.6 (the convolutions product of $S_\gamma(w)$ and $T_\nu(z)$)**.** *The convolutions product is given by*

(i) $S_\gamma * S_{\gamma'} = (i)^{q/2} S_{\gamma+\gamma'}$,

(ii) $T_\nu * T_{\nu'} = (-i)^{q/2} T_{\nu+\nu'}$ *where S_γ and T_ν are defined by (2.6) and (2.7), respectively.*

Proof. (i) Now

$$\langle S_\gamma(w), \varphi(x)\rangle = \frac{1}{W_n(\gamma)}\int_{\mathbb{R}^n} \omega^{(\gamma-n)/2}\varphi(x)\,dx, \tag{2.11}$$

where $\varphi \in \mathcal{D}$ the space of infinitely differentiable function with compact supports. We have $\omega = x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2)$, $p + q = n$. By changing the variables $x_1 = y_1$, $x_2 = y_2, \ldots, x_p = y_p$, $x_{p+1} = y_{p+1}/\sqrt{-i}$, $x_{p+2} = y_{p+2}/\sqrt{-i}, \ldots$, and $x_{p+q} = y_{p+q}/\sqrt{-i}$. Thus we obtain $\omega = y_1^2 + y_2^2 + \cdots + y_p^2 + y_{p+1}^2 + y_{p+2}^2 + \cdots + y_{p+q}^2$. Let $r^2 = y_1^2 + y_2^2 + \cdots + y_{p+q}^2$, $p + q = n$. Thus (2.11) can be written in the form

$$\begin{aligned}\langle S_\gamma(w), \varphi(x)\rangle &= \frac{1}{W_n(\gamma)}\int_{\mathbb{R}^n} r^{\gamma-n}\varphi\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)}dy_1 dy_2\cdots dy_n\\ &= \frac{1}{(-i)^{q/2}}\frac{1}{W_n(\gamma)}\int_{\mathbb{R}^n} r^{\gamma-n}\varphi\,dy\\ &= \frac{(i)^{q/2}}{W_n(\gamma)}\langle r^{\gamma-n}, \varphi\rangle.\end{aligned} \tag{2.12}$$

Thus $S_\gamma(w) = ((i)^{q/2}/w_n(\gamma))r^{\gamma-n} = (i)^{q/2}R^e_\gamma(w)$ by (2.4).

Consider the convolution $S_\gamma * S_{\gamma'}$. We have

$$\begin{aligned} S_\gamma * S_{\gamma'} &= (i)^{q/2} R^e_\gamma(w) * (i)^{q/2} R^e_{\gamma'}(w) \\ &= (i)^q R^e_{\gamma+\gamma'}(w) \quad \text{by Lemma 2.4 and [1, pages 157–159]} \\ &= (i)^{q/2}(i)^{q/2} R^e_{\gamma+\gamma'}(w) \\ &= (i)^{q/2} S_{\gamma+\gamma'}. \end{aligned} \tag{2.13}$$

Similarly, for (ii) we also have

$$T_\nu * T_{\nu'} = (-i)^{q/2} T_{\nu+\nu'}. \tag{2.14}$$

□

3. Main results

Theorem 3.1. *Let $K_{\alpha,\beta,\gamma,\nu}$ be the distributional kernel defined by (1.5). Then we obtain the following:*

(i) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}$ *for $\alpha,\beta,\gamma,\nu,\alpha',\beta',\gamma'$, and ν' positive even numbers with $\alpha=\beta=\gamma=\nu$, $\alpha'=\beta'=\gamma'=\nu'$;*

(ii) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = B_{\alpha,\alpha'} \cdot K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}$ *for p even, $\alpha,\beta,\gamma,\nu,\alpha'$, β',γ', and ν' any complex numbers, and* $B_{\alpha,\alpha'} = \cos(\alpha\pi/2)\cos(\alpha'\pi/2)/\cos((\alpha+\alpha')/2)\pi$*;*

(iii) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'} + A_{\alpha,\alpha'} * R^e_{\beta+\beta'} * S_{\gamma+\gamma'} * T_{\nu+\nu'}$ *for p odd, $\alpha,\beta,\gamma,\nu,\alpha',\beta',\gamma'$, and ν' any complex numbers R^e_β, S_γ, and T_ν defined by (2.4), (2.6), and (2.7), respectively. And*

$$\begin{gathered} A_{\alpha,\alpha'} = \frac{C((-\alpha-\alpha')/2)}{C(-\alpha/2)} C\left(-\frac{\alpha'}{2}\right) \cdot \frac{2\pi i}{4}\left[H^+_{\alpha+\alpha'} - H^-_{\alpha+\alpha'}\right], \\ C(r) = \Gamma(r)\Gamma(1-r), \\ H^\pm_r = H_r(u \pm i0, n) = e^{\mp r(\pi/2)i} e^{\pm q(\pi/2)i} a\left(\frac{r}{2}\right)(u \pm i0)^{r-n/2}, \\ a\left(\frac{r}{2}\right) = \Gamma\left(\frac{n-r}{2}\right)\left[2^r \pi^{n/2}\Gamma\left(\frac{r}{2}\right)\right]^{-1}, \\ (u \pm i0)^\lambda = \lim_{\epsilon \to 0}(u + i \in |x|^2)^\lambda, \end{gathered} \tag{3.1}$$

where $u = u(x)$ is defined by (2.1) and

$$|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}. \tag{3.2}$$

Proof. The proof of (i) follows from [3, Theorem 3.1, page 66]. The proof of (ii) and (iii) is obtained by Lemmas 2.4, 2.5, and 2.6. □

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