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FIXED POINTS VIA A GENERALIZED LOCAL COMMUTATIVITY: THE COMPACT CASE

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In an earlier paper, the concept of semigroups of self-maps which are nearly commutative at a function $g : X \to X$ was introduced. We now continue the investigation, but with emphasis on the compact case. Fixed-point theorems for such semigroups are obtained in the setting of semimetric and metric spaces.

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1. Introduction. By a semigroup of maps we mean a family H of self-maps of a set X which is closed with respect to the composition of maps and includes the identity map. In [6] we obtained fixed-point theorems for semigroups of maps by introducing the following concept.

Definition 1.1 [6]. A semigroup H of self-maps of a set X is *nearly commutative (n.c.)* at $g : X \to X$ if and only if $f \in H$ implies that there exists $h \in H$ such that $fg = gh$.

As in [6], we consider this concept in the context of generalized metric spaces, namely, semimetric spaces. A semimetric on a set X is a function $d : X \times X \to [0,\infty)$ such that $d(x,y) = 0$ if and only if $x = y$ and $d(x,y) = d(y,x)$ for $x,y \in X$. For $p \in X$ and $\epsilon > 0$, we let $S(p,\epsilon) = \{x \in X : d(x,p) < \epsilon\}$. A semimetric space is a pair $(X;d)$ in which X is a topological space and d is a semimetric on X. The topology on X is the family $t(d) = \{U \subset X : p \in U \Rightarrow S(p,\epsilon) \subset U \text{ for some } \epsilon > 0\}$. We require that the point p be an interior point of the set $S(p,\epsilon)$; that is, there exists $U \in t(d)$ such that $p \in U \subset S(p,\epsilon)$. Consequently, a sequence $\{x_n\}$ in X converges in $t(d)$ to $p \in X$, denoted as $x_n \to p$ if and only if $d(x_n,p) \to 0$. And a function (map) $g : X \to X$ is continuous if and only if $fx_n \to fx$ whenever $x_n \to x$. To ensure unique limits, all spaces X will be assumed to be Hausdorff (T_2). We will conclude with two examples of semimetrics, one of which is not T_2. For further detail regarding semimetric spaces, see [1, 2, 6].

This paper is a continuation of [6] but with focus on the effects of compactness requirements. We obtain fixed-point theorems for semigroups H of self-maps of X which are n.c. at continuous maps $g : X \to X$. These theorems generalize known results involving, for example, the semigroups $C_g = \{f : X \to X \mid fg = gf\}$. The main result we will need from [6, Proposition 2.6] is the fact that whenever a semigroup H of self-maps of a set X is n.c. at $g : X \to X$, then H is n.c. at the composite g^n for all $n \in \mathbb{Z}^+$. ($\mathbb{Z}^+$ denotes the set of positive integers, $\omega = \mathbb{Z}^+ \cup \{0\}$, and $\mathbb{R}$ denotes the real numbers.) Observe also that if H is a semigroup of self-maps of X and $a, b \in X$, then $H(a) = \{h(a) : h \in H\}$ and $H(a,b) = H(a) \cup H(b)$. Furthermore, if $A \subset X$, $\delta(A) = \sup\{d(x,y) : x, y \in A\}$ and $\mathrm{cl}(A)$ denotes the closure of A.

2. Preliminaries. We now state and prove our first lemma. Note that most results stipulate that the semimetric d in $(X;d)$ is upper semicontinuous (u.s.c.). Thus, $d^{-1}(-\infty, a)$ is open for $a \in \mathbb{R}$. Specifically, if $a_n \to a$, $b_n \to b$ in $(X;d)$ and $d(a_n, b_n) \downarrow \eta$, then $\eta \le d(a,b)$.

Lemma 2.1. *Let $(X;d)$ be a semimetric space with d u.s.c. and let $g, f : X \to X$. Suppose there is a nonempty compact subset A of X such that $g(A) = f(A) = A$ and a semigroup H of self-maps of X such that $h(A) \subset A$ for $h \in H$. If there exist $m, n \in \omega$ such that*

$$(f^m x \ne g^n y) \Longrightarrow d(f^m x, g^n y) < \delta(H(x,y)), \quad \text{for } x, y \in A, \tag{2.1}$$

then there is a unique point $a \in A$ such that $a = fa = ga = ha$ for all $h \in H$. In fact, $A = \{a\}$.

Proof. Since A is compact, $A \times A$ is compact. Therefore, there exists $a, b \in A$ such that $d(a,b) = \delta(A)$, since d is u.s.c. But $f(A) = g(A) = A = f^m(A) = g^n(A)$, so there exists $c, d \in A$ such that $f^m c = a$ and $g^n d = b$. If $a \ne b$, (2.1) implies

$$\delta(A) = d(a,b) = d(f^m c, g^n d) < \delta(H(c,d)). \tag{2.2}$$

But $c, d \in A$ so that $hc, hd \in A$ for all $h \in H$ by hypothesis; that is, $H(c,d) \subset A$. Thus, (2.2) yields the contradiction $\delta(A) < \delta(H(c,d)) \le \delta(A)$. Consequently, $a = b$ so that $A = \{a\}$. Therefore, by hypothesis, $f(\{a\}) = g(\{a\}) = h(\{a\}) = \{a\}$ for all $h \in H$; that is, $a = fa = ga = ha$ for all $h \in H$. Clearly, a is the only point of A which is a common fixed point of f, g, and $h \in H$. $\square$

Remark 2.2. The definition of a semimetric space $(X;d)$ requires that $S(x,\epsilon)$ be a neighborhood of x; that is, the topological interior of $S(x,\epsilon)$ is a set in $t(d)$ which contains x. It is an easy matter to show that this fact assures us that any compact semimetric space is sequentially compact.

Lemma 2.3. *Let $(X;d)$ be a semimetric space and let $f, g : X \to X$. Suppose that H is a semigroup of self-maps of X which is n.c. at f and at g. Then the following hold, where $A = \cap\{(gf)^n(X) : n \in \mathbb{Z}^+\} (= \cap (gf)^n(X))$:*

(1) *H is n.c. at gf;*
(2) *$h(A) \subset A$ for all $h \in H$;*

(3) *if $f,g \in H$ and $gf(A) = A$, then $f(A) = g(A) = A$;*
(4) *$gf(A) = A$ when $(X;d)$ is compact and gf is continuous.*

Proof. Since H is n.c. at g and at f for each $h \in H$, there exists $h_1, h_2 \in H$ such that

$$h(gf) = (hg)f = (gh_1)f = g(h_1f) = g(fh_2) = (gf)h_2. \tag{2.3}$$

Thus (1) holds. But then, since H is n.c. at g^n for $n \in \mathbb{Z}^+$ if H is n.c. at g, (1) implies that for each $n \in \mathbb{Z}^+$ and $h \in H$, there exists $h_n \in H$ such that $h(gf)^n = (gf)^n h_n$. So

$$h(A) \subset \cap h(gf)^n(X) = \cap (gf)^n h_n(X) \subset \cap (gf)^n(X) = A. \tag{2.4}$$

Consequently, (2) is true. Moreover, if $f,g \in H$ and $gf(A) = A$, then $g(A) = A$ since $f \in H$ implies that $f(A) \subset A$. Therefore,

$$(A = gf(A)) \Longrightarrow (A = gf(A) \subset g(A) \subset A) \tag{2.5}$$

since $g \in H$. To see that $f(A) = A$, first note that since $g \in H$ and H is n.c. at f, there exists $h \in H$ such that $gf(A) = fh(A)$. But $h(A) \subset A$ (by (2)) and $f \in H$, so we can write

$$(A = gf(A)) \Longrightarrow (A = gf(A) = fh(A) \subset f(A) \subset A). \tag{2.6}$$

Thus $f(A) = A$, and (3) holds.

To prove (4), first note that $gf(A) \subset A$ by the definition of A. To show that $A \subset gf(A)$, let $a \in A$. Then $a \in (gf)^{n+1}(X) \subset (gf)^n(X)$ for each $n \in \mathbb{Z}^+$, so we can choose $x_n \in (gf)^n(X)$ such that $gf(x_n) = a$. Since $(X;d)$ is compact (sequentially), there exists a subsequence $\{x_{i_n}\}$ of $\{x_n\}$ and $p \in X$ such that $x_{i_n} \to p$. But $i_n \geq n$ by definition of subsequence, so for any $k \in \mathbb{Z}^+$, $x_{i_n} \in (gf)^k(X)$ for all $n \geq k$. Therefore, for a fixed k, since $(gf)^k(X)$ is compact and thus closed (X is T_2), $p \in (gf)^k(X)$. Hence $p \in A$. But gf is continuous and therefore, $a = gf(x_{i_n}) \to gf(p)$. Thus, $gf(p) = a$. □

Remark 2.4. If $(X;d)$ is compact and gf is continuous, then $A = \cap (gf)^n(X) \neq \emptyset$ and is compact by the finite intersection property.

3. Main results

Theorem 3.1. *Let f and g be self-maps of a compact semimetric space $(X;d)$ with d u.s.c. and gf continuous. Let H be a semigroup of self-maps of X containing f and g, and suppose that H is n.c. at f and at g. If there exist $m,n \in \omega$ such that*

$$(f^m x \neq g^n y) \Longrightarrow d(f^m x, g^n y) < \delta(H(x,y)) \quad \text{for } x,y \in X, \tag{3.1}$$

then there is a unique point $a \in X$ such that $a = fa = ga = ha$ for all $h \in H$. Moreover, (i) $\delta((gf)^n(X)) \to 0$ *and* (ii) $(gf)^n(x) \to a$ *uniformly on X.*

Proof. Let $A = \cap\{(gf)^n(X) : n \in \mathbb{Z}^+\}$. By Lemma 2.3(4), $gf(A) = A$ and therefore, $A = f(A) = g(A)$ by Lemma 2.3(3). Since $(X;d)$ is compact and gf is continuous, A is compact and nonempty. Moreover, Lemma 2.3(2) tells us that $h(A) \subset A$ for all $h \in H$. But then Lemma 2.1 implies that $A = \{a\}$ and a is the desired common fixed point. The point a is clearly unique since any common fixed point b of f and g satisfies $(gf)^n(b) = b$ for $n \in \mathbb{Z}^+$ and is therefore in A.

To see that (i) holds, note that gf is continuous and therefore $(gf)^n(X)$ is compact for $n \in \mathbb{Z}^+$. Thus for each $n \in \mathbb{Z}^+$, there exists $x_n, y_n \in (gf)^n(X)$ such that

$$\alpha_n = d(x_n, y_n) = \delta((gf)^n(X)) \tag{3.2}$$

since the semimetric d is u.s.c. Clearly, $0 \le \alpha_{n+1} \le \alpha_n$ and therefore $\alpha_n \downarrow b$ for some $b \ge 0$. Since X is compact, there exist subsequences $\{x_{k_n}\}$ and $\{y_{k_n}\}$ of $\{x_n\}$ and $\{y_n\}$ which converge to x, y, respectively, for some $x, y \in X$. As in the proof of Lemma 2.3(4), $x, y \in A = \{a\}$, so $x = y = a$. Then the upper semicontinuity of the semimetric d yields

$$b = \lim_{n\to\infty} d(x_{k_n}, y_{k_n}) \le d(x, y) = 0. \tag{3.3}$$

To verify (ii), let $\epsilon > 0$. By (i), we can choose $k \in \mathbb{Z}^+$ such that $\delta((gf)^n(X)) < \epsilon$ for $n \ge k$. Therefore, if $x \in X$, $a, (gf)^n(x) \in (gf)^n(X)$, so $d(a, (gf)^n(x)) < \epsilon$ for $n \ge k$. $\square$

In Theorem 3.1, (ii) states that the point a is a uniformly contractive point of gf, or attracts X (see [7]). The following theorem tells us that f and g need not be members of H if $fg = gf$.

Theorem 3.2. *Let f and g be commuting maps of a compact semimetric space $(X;d)$ with d u.s.c. Let H be a semigroup of self-maps of X which is n.c. at gf. If gf is continuous and (3.1) holds, then there is a unique point $a \in X$ such that for all $h \in X$, $a = fa = ha$. Moreover, Theorem 3.1(i) and (ii) hold.*

Proof. As above, let $A = \cap\{(gf)^n(X) : n \in \mathbb{Z}^+\}$. By Remark 2.4, we know that A is nonempty and compact. Moreover, since $(X;d)$ is compact and gf is continuous, the proof of Lemma 2.3(4) tells us that $gf(A) = A$. And the argument in the proof of Lemma 2.3(2) is valid under our hypothesis, so $h(A) \subset A$ for $h \in H$. Finally, since f and g commute, they commute with $(gf)^n$ for $n \in \mathbb{Z}^+$, so $g(A) \subset A$ and $f(A) \subset A$ (substitute $g(f)$ for h and h_n in (2.4)). Consequently, $A = gf(A) \subset g(A) \subset A$ and thus $A = g(A)$. Therefore, since $gf = fg$, $A = gf(A) = fg(A) = f(A)$. Thus, by Lemma 2.1, $a = fa = ga = ha$

for $h \in H$. And the proof of (i) and (ii) given in the proof of Theorem 3.1 holds under the present hypothesis. □

Remark 3.3. Theorem 3.2 generalizes [5, Theorem 4.2] by requiring that the underlying space be a semimetric space $(X;d)$ with u.s.c. semimetric d instead of being a metric space (X,d), in which case d is uniformly continuous (see [4]). Moreover, it substitutes the more general semigroup H for C_{gf}. Note also that [6, Theorem 5.1] is a consequence of Theorem 3.2. In fact, Theorem 3.2 extends [6, Theorem 5.1] since the underlying space in Theorem 5.1 is a metric space, and Theorem 5.1 does not include results (i) and (ii) of Theorem 3.2.

We now lift the requirement that the space $(X;d)$ be compact by demanding that $H(a)$ be relatively compact (i.e., $\mathrm{cl}(H(a))$ is compact) for some $a \in X$. And if there exists $x \in X$ such that $hx = x$ for all $h \in H$, we say that the semigroup H has a *fixed point*.

Theorem 3.4. *Let (X,d) be a semimetric space with d u.s.c. and let H be a semigroup of continuous self-maps of X which is n.c. at $f,g \in H$. Suppose $H(a)$ is relatively compact for some $a \in X$. If (2.1) holds for $x,y \in \mathrm{cl}(H(a))$, then H has a fixed point in $\mathrm{cl}(H(a))$.*

Proof. Now $h(H(a)) \subset H(a)$ for $h \in H$ since H is a semigroup. Moreover, since each $h \in H$ is continuous, $h(\mathrm{cl}(H(a))) \subset \mathrm{cl}(h(H(a))) \subset \mathrm{cl}(H(a))$. Thus, $\mathrm{cl}(H(a))$ is a nonempty compact h-invariant subset of X for $h \in H$. We can therefore apply Theorem 3.1 to the compact semimetric space $(M;d)$ with $M = \mathrm{cl}(H(a))$ to obtain our conclusion. □

Since any closed and bounded subset of $\mathbb{R}^n$ is compact, we have the following result. Recall that a semigroup H is n.c. at g^n if it is n.c. at g.

Corollary 3.5. *Suppose $X \subset \mathbb{R}^n$ and H is a semigroup of continuous self-maps of X n.c. at $g \in H$. If $H(a)$ is bounded for some $a \in X$ and there exists $m,n \in \omega$ such that*

$$g^m(x) \neq g^n(y) \Longrightarrow d(g^m x, g^n y) < \delta(H(x,y)) \quad \text{for } x,y \in \mathrm{cl}(H(a)), \tag{3.4}$$

then H has a fixed point.

The following example demonstrates the generality of Corollary 3.5 in that the semigroup H constructed is n.c. at the function g but H is not n.c. (i.e., n.c. at each $h \in H$, see [3, 6]).

Remark 3.6. Suppose that $g_i : X_i \to X_i$ for i in some indexing set λ and that H_i is a semigroup of maps $h_i : X_i \to X_i$ which is n.c. at g_i for $i \in \lambda$. Let $g = (g_i)_\lambda$, where for $x = (x_i)_\lambda \in X = \prod\{X_i : i \in \lambda\}$, $g(x) = g((x_i)_\lambda) = (g_i(x_i))_\lambda$, and $h = (h_i)$ is defined in a similar manner. If $H = \{(h_i)_\lambda : h_i \in H_i \text{ for } i \in \lambda\}$, then it is immediate that H is a semigroup of self-maps of X which is n.c. at $g : X \to X$.

Example 3.7. Let $X = X_1 \times X_2 \subset \mathbb{R}^2$ with the usual metric d, where $X_1 = X_2 = [0,\infty)$, and let $b \in (0,1)$. Define $g : X \to X$, where $g = (g_1, g_2)$ by $g_1(x_1) = bx_1$ and $g_2(x_2) = x_2^{1/2} = \sqrt{x_2}$. Let H_1 and H_2 be semigroups defined by

$$\begin{aligned} H_1 &= \{h_1 : h_1(x_1) = b^m x_1^n : m \in \omega,\ n \in \mathbb{Z}^+\}, \\ H_2 &= \{h_2 : h_2(x_2) = x_2^{(1/k)},\ k \in \mathbb{Z}^+\}. \end{aligned} \tag{3.5}$$

Then H_1 is not n.c. but is n.c. at g_1 (see [6, Example 2.2]), and H_2 is n.c. at g_2 since any member of H_2 commutes with g_2. Thus, $H = H_1 \times H_2$ is n.c. at g by Remark 3.6. Moreover,

$$H\left(\left(\frac{1}{2},\frac{1}{2}\right)\right) = \left\{\left(b^m \frac{1}{2^n}, \left(\frac{1}{2}\right)^{1/k}\right) : m \in \omega;\ n,k \in \mathbb{Z}^+\right\} \subset \left[0,\frac{1}{2}\right] \times \left[\frac{1}{2},1\right], \tag{3.6}$$

and is thus bounded. Furthermore, if $x_1, y_1 \in [0,1/2]$, $x_2, y_2 \in [1/2,1]$, and $(x_1,x_2) \neq (y_1,y_2)$, then

$$\begin{aligned} d(g((x_1,x_2),g(y_1,y_2))) &= d((g_1(x_1),g_2(x_2)),(g_1(y_1),g_2(y_2))) \\ &= \sqrt{(b(x_1-y_1))^2 + (\sqrt{x_2}-\sqrt{y_2})^2} \\ &< \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}. \end{aligned} \tag{3.7}$$

Note that for $x_1 \neq y_1$, $|\sqrt{x_2} - \sqrt{y_2}| < |x_2 - y_2|$ since $x_2, y_2 \in [1/2,1]$. Now the identity function i_d ($i_d(x_i) = x_i$) is a member of H_i ($i = 1,2$), and if $h_2(x_2) = \sqrt{x_2}$, then $h_2 \in H_2$. Therefore, (3.4) is satisfied with $m = n = 1$ and $a = (1/2,1/2)$, and H has a fixed point by Corollary 3.5. In fact, for any $h = (h_1,h_2) \in H$, the composites $h_1^p(x_1) \to 0$ and $h_2^p(x_2) \to 1$, as $p \to \infty$ for $x_1 \in [0,1/2]$ and $x_2 \in [1/2,1]$. Thus, $(0,1)$ is a fixed point of H.

In the above example, with $M = [0,1/2] \times [1/2,1]$, $g(M) \subset M$ and $d(g(p),g(q)) < d(p,q)$ when $p \neq q$ on M; that is, g was a contraction on M. Thus g has a fixed point by the classic theorem of Edelstein. We now provide an example in which g is not a contraction but satisfies the hypothesis of Corollary 3.5.

Example 3.8. Let $X = [0,\infty)$ and let $g(x) = (15/4)x^3 - (59/4)x^2 + 15x + 1$ for $x \in X$. If $H = O_g = \{g^n : n \in \omega\}$, then g satisfies condition (3.4) with $m = n = 1$, and has $a = 2$ as a fixed point. For $x < y$,

$$d(gx,gy) < \begin{cases} d(i_d(x),\max\{gx,gy\}), & x < 2, \\ d(gx,g^2y), & x \geq 2. \end{cases} \tag{3.8}$$

Clearly, g is not a contraction on X. However, on the small interval $J = [1.87, 2]$, g is a contraction and $g^n x \uparrow 2$ on J. So suppose we let g be the continuous piecewise linear function $g : X \to X$ given by $g(x) = 3x + 1$ on $[0,1]$, $g(x) = -2x + 6$ on $[1,2]$, and $g(x) = 2x - 2$ for $x > 2$. Then g satisfies (3.8), has 2 as a fixed point, and g so defined is a contraction on no subinterval of X.

4. Retrospect. The arguments given in the proofs of Lemma 2.1 and Theorem 3.1 appear to be "standard" in that they are similar in form to those given in the metric case (see, e.g., [5, Theorem 4.2]). However, these proofs demonstrate that much can be done in spaces not having all of the attributes of metric spaces such as the triangle inequality or the continuity of the distance function. Moreover, by using semigroups H of maps n.c. at a function g instead of C_g, the results are appreciably generalized. See [6] for further examples of semigroups n.c. at a function g.

We now give two examples of nonmetric semimetrics. The first produces a topology which is not T_2 and therefore does not yield a semimetric space.

Example 4.1. Let $X = \mathbb{R}$, and define $d(x,y) = 1/|x-y|$ if $x \neq y$ and $d(x,x) = 0$, for $x, y \in \mathbb{R}$. Clearly, d is a semimetric on X. To see that $(X;d)$ is a semimetric space, note that for any $a \in X$ and $\epsilon > 0$, $S(a,\epsilon) = \{a\} \cup (-\infty, a - 1/\epsilon) \cup (a + 1/\epsilon, \infty)$. It is then an easy matter to show that $S(a,\epsilon) \in t(d)$; that is, $S(a,\epsilon)$ is "open." Moreover, it is clear that $S(a,\epsilon) \cap S(b,\delta) \neq \varnothing$ for any $a, b \in X$ and $\epsilon, \delta > 0$ and therefore the definition of $t(d)$ assures us that any two nonempty members of $t(d)$ have a nonempty intersection.

The next semimetric induces a topology which does not produce a semimetric space since p is not always an interior point of $S(p,\epsilon)$.

Example 4.2. Let $X = [0,1]$ and define

$$d(x,y) = |x-y|, \quad \text{if } x, y \in X,\ x, y > 0,$$
$$d(y,0) = d(0,y) = \begin{cases} y, & \text{if } y \in X,\ y \text{ is rational}, \\ 2, & \text{if } y \in X,\ y \text{ is irrational}. \end{cases} \tag{4.1}$$

Let $\epsilon \in (0,2)$. Then 0 is not an interior point of $S(0,\epsilon)$. Suppose there exists $U \in t(d)$ such that

$$0 \in U \subset S(0,\epsilon). \tag{4.2}$$

Since $U \in t(d)$, we can choose a positive rational r such that

$$S(0,r) \subset U \subset S(0,\epsilon). \tag{4.3}$$

Now r is rational, so that $r/2 \in S(0,r)$ and therefore $r/2 \in U$. Since $U \in t(d)$ by definition of $t(d)$, there exists $\alpha \in (0,r/2)$ such that

$$S\left(\frac{r}{2},\alpha\right) \subset U \subset S(0,\epsilon). \tag{4.4}$$

Thus, the definition of the function d implies

$$I = \left(\frac{r}{2}-\alpha,\frac{r}{2}+\alpha\right) \subset U \subset S(0,\epsilon). \tag{4.5}$$

But since $0 < \epsilon < 2$, $S(0,\epsilon)$ contains no irrationals whereas the interval I does. This contradiction assures us that 0 is not an interior point of $S(0,\epsilon)$.

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