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RICCI AND BIANCHI IDENTITIES FOR h-NORMAL Γ-LINEAR CONNECTIONS ON $J^1(T,M)$

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The aim of this paper is to describe the local Ricci and Bianchi identities of an h-normal Γ-linear connection on the first-order jet fibre bundle $J^1(T,M)$. We present the physical and geometrical motives that determined our study and introduce the h-normal Γ-linear connections on $J^1(T,M)$, emphasizing their particular local features. We describe the expressions of the local components of torsion and curvature d-tensors produced by an h-normal Γ-linear connection $\nabla\Gamma$, and analyze the local Ricci identities induced by $\nabla\Gamma$, together with their derived local deflection d-tensors identities. Finally, we expose the local expressions of Bianchi identities which geometrically connect the local torsion and curvature d-tensors of connection $\nabla\Gamma$.

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1. Introduction. From a physical point of view, it is well known that the jet fibre bundle of order one $J^1(T,M)$ appears as a basic object in the study of continuum mechanics [3], quantum field theories [11], or generalized multi-time field theory [7]. At the same time, the geometrical studies of first-order Lagrangians that govern several important branches of theoretical physics (bosonic strings theory [6], electrodynamics [4, 6], elasticity [12], or magneto-hydrodynamics [3]) required a profound analysis of the differential geometry of 1-jet spaces, in the sense of connections, torsions, and curvatures. In this direction, [13] develops a *contravariant geometry* of jet fibre bundles of arbitrary orders, whose main feature is the global approach of geometrical objects involved. In the same way, but using as a pattern Riemannian geometrical instruments from theory of Lagrange spaces, [4] studies the geometry of particular 1-jet bundle $J^1(\mathbb{R},M) \equiv \mathbb{R}\times\mathbb{T}M$ over the base M, in the sense of d-connections, d-torsions, and d-curvatures. Some interesting geometrical aspects of the 1-jet bundle $J^1(\mathbb{R},M) \equiv \mathbb{R}\times\mathbb{T}M$, regarded over the base space $\mathbb{R}\times M$, are exposed in [14]. In contrast, using the Hamiltonian formalism (i.e., a *covariant geometry* on dual 1-jet spaces) in its polysymplectic or multisymplectic versions, [2, 3] construct various geometrical objects on 1-jet fibre bundles. In this geometrical context, extending by Riemannian methods the geometrical results from [4, 14], our paper analyzes the particular local features of geometrical objects produced on $J^1(T,M)$ by a nonlinear connection Γ. From our point of view, this geometrical study represents a very fruitful domain of mathematics, because not only this differential provides many new ideas, suitable for a geometrical theory of *PDE* systems [8], but also it offers the geometrical background for a multitime field theory [7], whose construction on $J^1(T,M)$ was imposed of certain famous relativistic invariant equations involving many time variables (chiral fields, sine-Gordon, etc.) and of *KP*-hierarchy of integrable systems in which the arbitrary variables t^α and t^β are quite equal in rights and there is no reason to prefer one to another by choosing it as time [1]. At the same time, we believe that our geometrical results may have interesting connections, via the Legendre transformations, with results obtained in covariant Hamiltonian geometry of 1-jet spaces [2, 3]. Finally, it is very important to note that, in the context of generalized multitime field theory described in [7], the expressions of local Ricci and Bianchi identities on 1-jet spaces are decisive for description of local generalized Einstein and Maxwell equations that govern the multitime gravitational electromagnetic fields from [7]. This is because [7] follows the same geometrical ideas as in [4].

2. Components of h-normal Γ-linear connections on first jet bundle $J^1(T,M)$. Let T (resp., M) be a *temporal* (resp., *spatial*) real, smooth manifold of dimension p (resp., n), whose coordinates are $(t^\alpha)_{\alpha=\overline{1,p}}$ (resp., $(x^i)_{i=\overline{1,n}}$). Note that, throughout this paper, the indices $\alpha,\beta,\gamma,\dots$ run from 1 to p while the indices $i,j,k,\dots$ run from 1 to n. We consider the 1-jet fibre bundle $J^1(T,M) \to T\times M$, whose coordinates $(t^\alpha,x^i,x^i_\alpha)$ are produced by T and M. We point out that the coordinate transformations on the product manifold $T\times M$ induce on $J^1(T,M)$ the following geometrical invariance group:

$$\tilde{t}^\alpha = \tilde{t}^\alpha(t^\beta), \qquad \tilde{x}^i = \tilde{x}^i(x^j), \qquad \tilde{x}^i_\alpha = \frac{\partial \tilde{x}^i}{\partial x^j}\frac{\partial t^\beta}{\partial \tilde{t}^\alpha}x^j_\beta. \tag{2.1}$$

Definition 2.1. A pair $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ of local functions on $E = J^1(T,M)$, whose transformation rules are given by

$$\begin{aligned} \tilde{M}^{(j)}_{(\beta)\mu}\frac{\partial \tilde{t}^\mu}{\partial t^\alpha} &= M^{(k)}_{(\gamma)\alpha}\frac{\partial \tilde{x}^j}{\partial x^k}\frac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \frac{\partial \tilde{x}^j_\beta}{\partial t^\alpha}, \\ \tilde{N}^{(j)}_{(\beta)k}\frac{\partial \tilde{x}^k}{\partial x^i} &= N^{(k)}_{(\gamma)i}\frac{\partial \tilde{x}^j}{\partial x^k}\frac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \frac{\partial \tilde{x}^j_\beta}{\partial x^i}, \end{aligned} \tag{2.2}$$

is called a *nonlinear connection* on E. The components $M^{(i)}_{(\alpha)\beta}$ (resp., $N^{(i)}_{(\alpha)j}$) are called the *temporal* (resp., *spatial*) *components* of the nonlinear connection Γ.

Example 2.2. Let $h_{\alpha\beta}(t^\mu)$ (resp., $\varphi_{ij}(x^m)$) be a semi-Riemannian metric on the temporal manifold T (resp., the spatial manifold M). Taking into account the local transformation rules of the Christoffel symbols $H^\gamma_{\alpha\beta}(t^\mu)$ (resp.,

$\gamma^k_{ij}(x^m)$) of these semi-Riemannian metrics, we deduce that the pair $\Gamma_0 = (\overset{0}{M}{}^{(j)}_{(\beta)\alpha}, \overset{0}{N}{}^{(j)}_{(\beta)i})$, where

$$\overset{0}{M}{}^{(j)}_{(\beta)\alpha} = -H^{\gamma}_{\alpha\beta}x^j_{\gamma}, \qquad \overset{0}{N}{}^{(j)}_{(\beta)i} = \gamma^j_{ik}x^k_{\beta}, \tag{2.3}$$

represents a nonlinear connection on E. This is called the *canonical nonlinear connection attached to the semi-Riemannian metrics* $h_{\alpha\beta}$ *and* φ_{ij}. In what follows, we fix $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ a nonlinear connection on E, and we consider

$$\left\{\frac{\delta}{\delta t^{\alpha}}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^i_{\alpha}}\right\} \subset \mathcal{X}(E), \qquad \{dt^{\alpha}, dx^i, \delta x^i_{\alpha}\} \subset \mathcal{X}^*(E), \tag{2.4}$$

the adapted bases of the nonlinear connection Γ, where

$$\begin{gathered}\frac{\delta}{\delta t^{\alpha}} = \frac{\partial}{\partial t^{\alpha}} - M^{(j)}_{(\beta)\alpha}\frac{\partial}{\partial x^j_{\beta}}, \qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{(j)}_{(\beta)i}\frac{\partial}{\partial x^j_{\beta}}, \\ \delta x^i_{\alpha} = dx^i_{\alpha} + M^{(i)}_{(\alpha)\beta}dt^{\beta} + N^{(i)}_{(\alpha)j}dx^j.\end{gathered} \tag{2.5}$$

Proposition 2.3. *The transformation rules of the elements of the adapted bases (2.4) are tensorial ones:*

$$\begin{gathered}\frac{\delta}{\delta t^{\alpha}} = \frac{\partial \tilde{t}^{\beta}}{\partial t^{\alpha}}\frac{\delta}{\delta \tilde{t}^{\beta}}, \qquad \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j}, \qquad \frac{\partial}{\partial x^i_{\alpha}} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial t^{\alpha}}{\partial \tilde{t}^{\beta}}\frac{\partial}{\partial \tilde{x}^j_{\beta}}, \\ dt^{\alpha} = \frac{\partial t^{\alpha}}{\partial \tilde{t}^{\beta}}d\tilde{t}^{\beta}, \qquad dx^i = \frac{\partial x^i}{\partial \tilde{x}^j}d\tilde{x}^j, \qquad \delta x^i_{\alpha} = \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{t}^{\beta}}{\partial t^{\alpha}}\delta \tilde{x}^j_{\beta}.\end{gathered} \tag{2.6}$$

Proof. After local computations, the geometrical invariance transformations (2.1), together with the local transformations rules (2.2), imply relations (2.6). □

Remark 2.4. The simple tensorial transformations rules (2.6) of adapted bases (2.4) confirm our choice to describe the geometrical objects of $J^1(T,M)$ in local adapted components. In order to develop the theory of Γ-linear connections on the 1-jet space E, we need the following proposition.

Proposition 2.5. (i) *The Lie algebra* $\mathcal{X}(E)$ *of vector fields decomposes as*

$$\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_T) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}), \tag{2.7}$$

where

$$\mathcal{X}(\mathcal{H}_T) = \operatorname{Span}\left\{\frac{\delta}{\delta t^{\alpha}}\right\}, \qquad \mathcal{X}(\mathcal{H}_M) = \operatorname{Span}\left\{\frac{\delta}{\delta x^i}\right\}, \qquad \mathcal{X}(\mathcal{V}) = \operatorname{Span}\left\{\frac{\partial}{\partial x^i_{\alpha}}\right\}. \tag{2.8}$$

(ii) *The Lie algebra* $\mathcal{X}^*(E)$ *of covector fields decomposes as*

$$\mathcal{X}^*(E) = \mathcal{X}^*(\mathcal{H}_T) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}), \tag{2.9}$$

where

$$\mathcal{X}^*(\mathcal{H}_T) = \operatorname{Span}\{dt^\alpha\}, \qquad \mathcal{X}^*(\mathcal{H}_M) = \operatorname{Span}\{dx^i\}, \qquad \mathcal{X}^*(\mathcal{V}) = \operatorname{Span}\{\delta x^i_\alpha\}. \tag{2.10}$$

Consider h_T, h_M (horizontal), and v (vertical) as the canonical projections of the above decompositions. In this context, we introduce Definition 2.6.

Definition 2.6. A linear connection $\nabla : \mathcal{X}(E)\times\mathcal{X}(E) \to \mathcal{X}(E)$ is called *a* Γ-*linear connection on* E if and only if $\nabla h_T = 0$, $\nabla h_M = 0$, and $\nabla v = 0$. Obviously, the local description of a Γ-linear connection ∇ on E is given by nine unique adapted components

$$\nabla\Gamma = \left(\bar{G}^\alpha_{\beta\gamma}, G^k_{i\gamma}, G^{(i)(\beta)}_{(\alpha)(j)\gamma}, \bar{L}^\alpha_{\beta j}, L^k_{ij}, L^{(i)(\beta)}_{(\alpha)(j)k}, \bar{C}^{\alpha(\gamma)}_{\beta(k)}, C^{j(\gamma)}_{i(k)}, C^{(i)(\beta)(\gamma)}_{(\alpha)(j)(k)}\right), \tag{2.11}$$

which are locally defined by the relations:

$$\begin{aligned}
&\nabla_{\delta/\delta t^\gamma}\frac{\delta}{\delta t^\beta} = \bar{G}^\alpha_{\beta\gamma}\frac{\delta}{\delta t^\alpha}, && \nabla_{\delta/\delta t^\gamma}\frac{\delta}{\delta x^i} = G^k_{i\gamma}\frac{\delta}{\delta x^k},\\
&\nabla_{\delta/\delta t^\gamma}\frac{\partial}{\partial x^i_\beta} = G^{(k)(\beta)}_{(\alpha)(i)\gamma}\frac{\partial}{\partial x^k_\alpha}, && \nabla_{\delta/\delta x^j}\frac{\delta}{\delta t^\beta} = \bar{L}^\alpha_{\beta j}\frac{\delta}{\delta t^\alpha},\\
&\nabla_{\delta/\delta x^j}\frac{\delta}{\delta x^i} = L^k_{ij}\frac{\delta}{\delta x^k}, && \nabla_{\delta/\delta x^j}\frac{\partial}{\partial x^i_\beta} = L^{(k)(\beta)}_{(\alpha)(i)j}\frac{\partial}{\partial x^k_\alpha},\\
&\nabla_{\partial/\partial x^j_\gamma}\frac{\delta}{\delta t^\beta} = \bar{C}^{\alpha(\gamma)}_{\beta(j)}\frac{\delta}{\delta t^\alpha}, && \nabla_{\partial/\partial x^j_\gamma}\frac{\delta}{\delta x^i} = C^{k(\gamma)}_{i(j)}\frac{\delta}{\delta x^k},\\
&\nabla_{\partial/\partial x^j_\gamma}\frac{\partial}{\partial x^i_\beta} = C^{(k)(\beta)(\gamma)}_{(\alpha)(i)(j)}\frac{\partial}{\partial x^k_\alpha}.
\end{aligned} \tag{2.12}$$

Remark 2.7. The transformation rules of the preceding Γ-linear connection components are completely described in [10].

Example 2.8. Let $\Gamma_0 = (\overset{0}{M}{}^{(i)}_{(\alpha)\beta}, \overset{0}{N}{}^{(i)}_{(\alpha)j})$ be the canonical nonlinear connection of semi-Riemannian metrics pair $(h_{\alpha\beta}, \varphi_{ij})$. Taking into account the transformation rules of Christoffel symbols $H^\gamma_{\alpha\beta}$ and γ^i_{jk}, by local computations, we can show that the local components

$$B\Gamma_0 = \left(\bar{G}^\alpha_{\beta\gamma}, 0, G^{(k)(\beta)}_{(\alpha)(i)\gamma}, 0, L^k_{ij}, L^{(k)(\beta)}_{(\alpha)(i)j}, 0, 0, 0\right), \tag{2.13}$$

where $\bar{G}^\gamma_{\alpha\beta} = H^\gamma_{\alpha\beta}$, $G^{(k)(\beta)}_{(\gamma)(i)\alpha} = -\delta^k_i H^\beta_{\alpha\gamma}$, $L^k_{ij} = \gamma^k_{ij}$, and $L^{(k)(\beta)}_{(\gamma)(i)j} = \delta^\beta_\gamma \gamma^k_{ij}$, verify the transformation rules of components of a Γ_0-linear connection [10]. Consequently, $B\Gamma_0$ is a Γ_0-linear connection on E, which is called the *Berwald connection of the metrics pair* $(h_{\alpha\beta}, \varphi_{ij})$. Now, let $\nabla\Gamma$ be a Γ-linear connection on

E, locally defined by (2.11). The linear connection $\nabla\Gamma$ induces a linear connection on the d-tensors set of the jet fibre bundle $E = J^1(T,M)$, in a natural way. Thus, starting with a d-vector field X and a d-tensor field D, locally expressed by

$$X = X^\alpha \frac{\delta}{\delta t^\alpha} + X^m \frac{\delta}{\delta x^m} + X^{(m)}_{(\alpha)} \frac{\partial}{\partial x^m_\alpha},$$
$$D = D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots} \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x^j_\beta} \otimes dt^\gamma \otimes dx^k \otimes \delta x^l_\delta \cdots, \tag{2.14}$$

we can define the covariant derivative

$$\begin{aligned}\nabla_X D &= X^\varepsilon \nabla_{\delta/\delta t^\varepsilon} D + X^p \nabla_{\delta/\delta x^p} D + X^{(p)}_{(\varepsilon)} \nabla_{\partial/\partial x^p_\varepsilon} D \\ &= \left\{ X^\varepsilon D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots/\varepsilon} + X^p D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots|p} + X^{(p)}_{(\varepsilon)} D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots}|^{(\varepsilon)}_{(p)} \right\} \\ &\quad \times \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x^j_\beta} \otimes dt^\gamma \otimes dx^k \otimes \delta x^l_\delta \cdots,\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots/\varepsilon} &= \frac{\delta D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots}}{\delta t^\varepsilon} + D^{\mu i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots} \bar{G}^\alpha_{\mu\varepsilon} \\
&\quad + D^{\alpha m(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots} G^i_{m\varepsilon} + D^{\alpha i(m)(\delta)\cdots}_{\gamma k(\mu)(l)\cdots} G^{(j)(\mu)}_{(\beta)(m)\varepsilon} + \cdots \\
&\quad - D^{\alpha i(j)(\delta)\cdots}_{\mu k(\beta)(l)\cdots} \bar{G}^\mu_{\gamma\varepsilon} - D^{\alpha i(j)(\delta)\cdots}_{\gamma m(\beta)(l)\cdots} G^m_{k\varepsilon} - D^{\alpha i(j)(\mu)\cdots}_{\gamma k(\beta)(m)\cdots} G^{(m)(\delta)}_{(\mu)(l)\varepsilon} - \cdots, \\
D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots|p} &= \frac{\delta D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots}}{\delta x^p} + D^{\mu i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots} \bar{L}^\alpha_{\mu p} \\
&\quad + D^{\alpha m(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots} L^i_{mp} + D^{\alpha i(m)(\delta)\cdots}_{\gamma k(\mu)(l)\cdots} L^{(j)(\mu)}_{(\beta)(m)p} + \cdots \\
&\quad - D^{\alpha i(j)(\delta)\cdots}_{\mu k(\beta)(l)\cdots} \bar{L}^\mu_{\gamma p} - D^{\alpha i(j)(\delta)\cdots}_{\gamma m(\beta)(l)\cdots} L^m_{kp} - D^{\alpha i(j)(\mu)\cdots}_{\gamma k(\beta)(m)\cdots} L^{(m)(\delta)}_{(\mu)(l)p} - \cdots, \\
D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots}|^{(\varepsilon)}_{(p)} &= \frac{\partial D^{\alpha i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots}}{\partial x^p_\varepsilon} + D^{\mu i(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots} \bar{C}^{\alpha(\varepsilon)}_{\mu(p)} \\
&\quad + D^{\alpha m(j)(\delta)\cdots}_{\gamma k(\beta)(l)\cdots} C^{i(\varepsilon)}_{m(p)} + D^{\alpha i(m)(\delta)\cdots}_{\gamma k(\mu)(l)\cdots} C^{(j)(\mu)(\varepsilon)}_{(\beta)(m)(p)} + \cdots \\
&\quad - D^{\alpha i(j)(\delta)\cdots}_{\mu k(\beta)(l)\cdots} \bar{C}^{\mu(\varepsilon)}_{\gamma(p)} - D^{\alpha i(j)(\delta)\cdots}_{\gamma m(\beta)(l)\cdots} C^{m(\varepsilon)}_{k(p)} - D^{\alpha i(j)(\mu)\cdots}_{\gamma k(\beta)(m)\cdots} C^{(m)(\delta)(\varepsilon)}_{(\mu)(l)(p)} - \cdots.
\end{aligned} \tag{2.16}$$

Definition 2.9. The local derivative operators "$_{/\varepsilon}$", "$_{|p}$", and "$|^{(\varepsilon)}_{(p)}$" are called the *T-horizontal covariant derivative*, *M-horizontal covariant derivative*, and *vertical covariant derivative* of the Γ-linear connection $\nabla\Gamma$.

Remark 2.10. (i) In the particular case when the d-tensor D is a function $f(t^\gamma,x^k,x^k_\gamma)$ on $E=J^1(T,M)$, the preceding covariant derivatives reduce to

$$\begin{aligned} f_{/\varepsilon} &= \frac{\delta f}{\delta t^\varepsilon} = \frac{\partial f}{\partial t^\varepsilon} - M^{(k)}_{(\gamma)\varepsilon}\frac{\partial f}{\partial x^k_\gamma}, \\ f_{|p} &= \frac{\delta f}{\delta x^p} = \frac{\partial f}{\partial x^p} - N^{(k)}_{(\gamma)p}\frac{\partial f}{\partial x^k_\gamma}, \\ f|^{(\varepsilon)}_{(p)} &= \frac{\partial f}{\partial x^p_\varepsilon}. \end{aligned} \tag{2.17}$$

(ii) Considering the d-tensor $D=Y$ like a d-tensor on E, locally expressed by

$$Y = Y^\alpha \frac{\delta}{\delta t^\alpha} + Y^i \frac{\delta}{\delta x^i} + Y^{(i)}_{(\alpha)}\frac{\partial}{\partial x^i_\alpha}, \tag{2.18}$$

the following expressions of local covariant derivatives of $\nabla\Gamma$ hold good:

$$\begin{gathered} Y^\alpha_{/\varepsilon} = \frac{\delta Y^\alpha}{\delta t^\varepsilon} + Y^\mu \bar{G}^\alpha_{\mu\varepsilon}, \qquad Y^i_{/\varepsilon} = \frac{\delta Y^i}{\delta t^\varepsilon} + Y^m G^i_{m\varepsilon}, \\ Y^{(i)}_{(\alpha)/\varepsilon} = \frac{\delta Y^{(i)}_{(\alpha)}}{\delta t^\varepsilon} + Y^{(m)}_{(\mu)} G^{(i)(\mu)}_{(\alpha)(m)\varepsilon}, \\ Y^\alpha_{|p} = \frac{\delta Y^\alpha}{\delta x^p} + Y^\mu \bar{L}^\alpha_{\mu p}, \qquad Y^i_{|p} = \frac{\delta Y^i}{\delta x^p} + Y^m L^i_{mp}, \\ Y^{(i)}_{(\alpha)|p} = \frac{\delta Y^{(i)}_{(\alpha)}}{\delta x^p} + Y^{(m)}_{(\mu)} L^{(i)(\mu)}_{(\alpha)(m)p}, \\ Y^\alpha|^{(\varepsilon)}_{(p)} = \frac{\partial Y^\alpha}{\partial x^p_\varepsilon} + Y^\mu \bar{C}^{\alpha(\varepsilon)}_{\mu(p)}, \qquad Y^i|^{(\varepsilon)}_{(p)} = \frac{\partial Y^i}{\partial x^p_\varepsilon} + Y^m C^{i(\varepsilon)}_{m(p)}, \\ Y^{(i)}_{(\alpha)}|^{(\varepsilon)}_{(p)} = \frac{\partial Y^{(i)}_{(\alpha)}}{\partial x^p_\varepsilon} + Y^{(m)}_{(\mu)} C^{(i)(\mu)(\varepsilon)}_{(\alpha)(m)(p)}. \end{gathered} \tag{2.19}$$

Because the number of components which characterize a Γ-linear connection on E is big (nine local components), we are constrained to study only a particular class of Γ-linear connections on E, which have to be characterized by a reduced number of components. In this direction, fix $h_{\alpha\beta}$ a semi-Riemannian metric on the temporal manifold T, together with its Christoffel symbols $H^\gamma_{\alpha\beta}$. Consider the d-tensor field J on E, locally expressed by

$$J = J^{(i)}_{(\alpha)\beta j}\frac{\partial}{\partial x^i_\alpha} \otimes dt^\beta \otimes dx^j, \tag{2.20}$$

where $J^{(i)}_{(\alpha)\beta j} = h_{\alpha\beta}\delta^i_j$, which is called the *$h$-normalization d-tensor* [9]. In this context, we introduce the following definition.

Definition 2.11. A Γ-linear connection $\nabla\Gamma$ on E, whose local components (2.11) verify the relations

$$\bar{G}^{\alpha}_{\beta\gamma} = H^{\alpha}_{\beta\gamma}, \quad \bar{L}^{\alpha}_{\beta j} = 0, \quad \bar{C}^{\alpha(\gamma)}_{\beta(j)} = 0, \quad \nabla J = 0, \tag{2.21}$$

is called an *h-normal Γ-linear connection* on the 1-jet fibre bundle E.

Theorem 2.12. *The adapted components of an h-normal Γ-linear connection $\nabla\Gamma$ verify the following identities:*

$$\begin{gathered} \bar{G}^{\gamma}_{\alpha\beta} = H^{\gamma}_{\alpha\beta}, \qquad \bar{L}^{\alpha}_{\beta j} = 0, \qquad \bar{C}^{\alpha(\gamma)}_{\beta(j)} = 0, \\ G^{(k)(\beta)}_{(\alpha)(i)\gamma} = \delta^{\beta}_{\alpha} G^{k}_{i\gamma} - \delta^{k}_{i} H^{\beta}_{\alpha\gamma}, \qquad L^{(k)(\beta)}_{(\alpha)(i)j} = \delta^{\beta}_{\alpha} L^{k}_{ij}, \qquad C^{(k)(\beta)(\gamma)}_{(\alpha)(i)(j)} = \delta^{\beta}_{\alpha} C^{k(\gamma)}_{i(j)}. \end{gathered} \tag{2.22}$$

Proof. It is obvious that the first three relations come immediately from the definition of an h-normal Γ-linear connection. To prove the other three ones, we emphasize that, taking into account the local T-horizontal "$_{/\gamma}$", M-horizontal "$_{|k}$", and vertical "$|^{(\gamma)}_{(k)}$" covariant derivatives produced by $\nabla\Gamma$, the condition $\nabla J = 0$ is equivalent to

$$J^{(i)}_{(\alpha)\beta j/\gamma} = 0, \qquad J^{(i)}_{(\alpha)\beta j|k} = 0, \qquad J^{(i)}_{(\alpha)\beta j}|^{(\gamma)}_{(k)} = 0. \tag{2.23}$$

Consequently, the condition $\nabla J = 0$ provides the local identities

$$\begin{gathered} h_{\beta\mu} G^{(i)(\mu)}_{(\alpha)(j)\gamma} = h_{\alpha\beta} G^{i}_{j\gamma} + \delta^{i}_{j}\left[-\frac{\partial h_{\alpha\beta}}{\partial t^{\gamma}} + H_{\beta\gamma\alpha}\right], \\ h_{\beta\mu} L^{(i)(\mu)}_{(\alpha)(j)} = h_{\alpha\beta} L^{i}_{jk}, \qquad h_{\beta\mu} C^{(i)(\mu)(\gamma)}_{(\alpha)(j)(k)} = h_{\alpha\beta} C^{i(\gamma)}_{j(k)}, \end{gathered} \tag{2.24}$$

where $H_{\beta\gamma\alpha} = H^{\mu}_{\beta\gamma} h_{\mu\alpha}$ represent the Christoffel symbols of first kind attached to the semi-Riemannian metric $h_{\alpha\beta}$. Contracting now the above relations by $h^{\beta\varepsilon}$, we obtain the last required identities. □

Remark 2.13. (i) The preceding theorem implies that an h-normal Γ-linear on E is a Γ-linear connection determined by four effective components (instead of nine in the general case), namely,

$$\nabla\Gamma = \left(H^{\gamma}_{\alpha\beta}, G^{k}_{i\gamma}, L^{k}_{ij}, C^{k(\gamma)}_{i(j)}\right). \tag{2.25}$$

The other five components either vanish or are provided by relations (2.22). As a consequence, we can assert that the Berwald Γ_0-linear connection associated to the pair of metrics $(h_{\alpha\beta}, \varphi_{ij})$ is an h-normal Γ_0-linear connection on E, whose four effective components are

$$B\Gamma_0 = (H^{\gamma}_{\alpha\beta}, 0, \gamma^{k}_{ij}, 0). \tag{2.26}$$

(ii) Considering the particular case $(T,h) = (\mathbb{R},\delta)$, we emphasize that the *δ-normal Γ-linear connections* on $J^1(\mathbb{R},M) \equiv \mathbb{R}\times\mathbb{T}M$ represent natural generalizations for the *normal N-linear connections* on $\mathbb{T}M$, used in Lagrangian geometry [4].

3. Components of torsion and curvature d-tensors.

The study of adapted components of the torsion $\mathbf{T}$ and curvature $\mathbf{R}$ d-tensors of an arbitrary Γ-linear connection $\nabla\Gamma$ on $E = J^1(T,M)$ was done in [10]. In that context, we proved that the torsion d-tensor is determined by *twelve* effective local d-tensors, while the curvature d-tensor of $\nabla\Gamma$ is determined by *eighteen* local d-tensors. In what follows, we study the components of torsion and curvature d-tensors for an h-normal Γ-linear connection $\nabla\Gamma$.

Theorem 3.1. *The torsion d-tensor T of an h-normal Γ-linear connection $\nabla\Gamma$ is determined by nine effective adapted local d-tensors (instead of twelve in the general case):*

Table 3.1

	h_T	h_M	v
h_Th_T	0	0	$R^{(m)}_{(\mu)\alpha\beta}$
h_Mh_T	0	$T^m_{\alpha j}$	$R^{(m)}_{(\mu)\alpha j}$
h_Mh_M	0	T^m_{ij}	$R^{(m)}_{(\mu)ij}$
vh_T	0	0	$P^{(m)(\beta)}_{(\mu)\alpha(j)}$
vh_M	0	$P^{m(\beta)}_{i(j)}$	$P^{(m)(\beta)}_{(\mu)i(j)}$
vv	0	0	$S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)}$

where

$$
\begin{gathered}
T^m_{\alpha j} = -G^m_{j\alpha}, \qquad T^m_{ij} = L^m_{ij} - L^m_{ji}, \qquad P^{m(\beta)}_{i(j)} = C^{m(\beta)}_{i(j)}, \\
\begin{aligned}
P^{(m)(\beta)}_{(\mu)\alpha(j)} &= \frac{\partial M^{(m)}_{(\mu)\alpha}}{\partial x^j_\beta} - \delta^\beta_\mu G^m_{j\alpha} + \delta^m_j H^\beta_{\mu\alpha}, \\
P^{(m)(\beta)}_{(\mu)i(j)} &= \frac{\partial N^{(m)}_{(\mu)i}}{\partial x^j_\beta} - \delta^\beta_\mu L^m_{ji}, \\
R^{(m)}_{(\mu)\alpha\beta} &= \frac{\delta M^{(m)}_{(\mu)\alpha}}{\delta t^\beta} - \frac{\delta M^{(m)}_{(\mu)\beta}}{\delta t^\alpha}, \\
R^{(m)}_{(\mu)\alpha j} &= \frac{\delta M^{(m)}_{(\mu)\alpha}}{\delta x^j} - \frac{\delta N^{(m)}_{(\mu)j}}{\delta t^\alpha}, \\
R^{(m)}_{(\mu)ij} &= \frac{\delta N^{(m)}_{(\mu)i}}{\delta x^j} - \frac{\delta N^{(m)}_{(\mu)j}}{\delta x^i}, \\
S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)} &= \delta^\alpha_\mu C^{m(\beta)}_{i(j)} - \delta^\beta_\mu C^{m(\alpha)}_{j(i)}.
\end{aligned}
\end{gathered}
\tag{3.1}
$$

Proof. Particularizing the general local expressions from [10], which give those twelve components of torsion d-tensor of a Γ-linear connection, in the large, for an h-normal Γ-linear connection $\nabla\Gamma$, we deduce that the adapted components $\bar{T}^{\mu}_{\alpha\beta}$, $\bar{T}^{\mu}_{\alpha j}$, and $\bar{P}^{\mu(\beta)}_{\alpha(j)}$ vanish, while the other nine ones are expressed exactly by formulas (3.1). □

Remark 3.2. All torsion d-tensors of the Berwald Γ_0-linear connection $B\Gamma_0$ associated to the metrics $h_{\alpha\beta}$ and φ_{ij} vanish, except

$$R^{(m)}_{(\mu)\alpha\beta} = -H^{\gamma}_{\mu\alpha\beta}x^m_{\gamma}, \qquad R^{(m)}_{(\mu)ij} = r^m_{ijl}x^l_{\mu}, \tag{3.2}$$

where $H^{\gamma}_{\mu\alpha\beta}$ (resp., r^m_{ijl}) are the curvature tensors of the metric $h_{\alpha\beta}$ (resp., φ_{ij}).

Theorem 3.3. *The curvature d-tensor R of an h-normal Γ-linear connection $\nabla\Gamma$ is characterized by seven effective adapted local d-tensors (instead of eighteen in the general case):*

Table 3.2

	h_T	h_M	v
h_Th_T	$H^{\alpha}_{\eta\beta\gamma}$	$R^l_{i\beta\gamma}$	$R^{(l)(\alpha)}_{(\eta)(i)\beta\gamma} = \delta^{\alpha}_{\eta}R^l_{i\beta\gamma} + \delta^l_i H^{\alpha}_{\eta\beta\gamma}$
h_Mh_T	0	$R^l_{i\beta k}$	$R^{(l)(\alpha)}_{(\eta)(i)\beta k} = \delta^{\alpha}_{\eta}R^l_{i\beta k}$
h_Mh_M	0	R^l_{ijk}	$R^{(l)(\alpha)}_{(\eta)(i)jk} = \delta^{\alpha}_{\eta}R^l_{ijk}$
vh_T	0	$P^{l(\gamma)}_{i\beta(k)}$	$P^{(l)(\alpha)(\gamma)}_{(\eta)(i)\beta(k)} = \delta^{\alpha}_{\eta}P^{l(\gamma)}_{i\beta(k)}$
vh_M	0	$P^{l(\gamma)}_{ij(k)}$	$P^{(l)(\alpha)(\gamma)}_{(\eta)(i)j(k)} = \delta^{\alpha}_{\eta}P^{l(\gamma)}_{ij(k)}$
vv	0	$S^{l(\beta)(\gamma)}_{i(j)(k)}$	$S^{(l)(\alpha)(\beta)(\gamma)}_{(\eta)(i)(j)(k)} = \delta^{\alpha}_{\eta}S^{l(\beta)(\gamma)}_{i(j)(k)}$

where

$$\begin{aligned}
H^{\alpha}_{\eta\beta\gamma} &= \frac{\partial H^{\alpha}_{\eta\beta}}{\partial t^{\gamma}} - \frac{\partial H^{\alpha}_{\eta\gamma}}{\partial t^{\beta}} + H^{\mu}_{\eta\beta}H^{\alpha}_{\mu\gamma} - H^{\mu}_{\eta\gamma}H^{\alpha}_{\mu\beta},\\
R^l_{i\beta\gamma} &= \frac{\delta G^l_{i\beta}}{\delta t^{\gamma}} - \frac{\delta G^l_{i\gamma}}{\delta t^{\beta}} + G^m_{i\beta}G^l_{m\gamma} - G^m_{i\gamma}G^l_{m\beta} + C^{l(\mu)}_{i(m)}R^{(m)}_{(\mu)\beta\gamma},\\
R^l_{i\beta k} &= \frac{\delta G^l_{i\beta}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta t^{\beta}} + G^m_{i\beta}L^l_{mk} - L^m_{ik}G^l_{m\beta} + C^{l(\mu)}_{i(m)}R^{(m)}_{(\mu)\beta k},\\
R^l_{ijk} &= \frac{\delta L^l_{ij}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta x^j} + L^m_{ij}L^l_{mk} - L^m_{ik}L^l_{mj} + C^{l(\mu)}_{i(m)}R^{(m)}_{(\mu)jk},\\
P^{l(\gamma)}_{i\beta(k)} &= \frac{\partial G^l_{i\beta}}{\partial x^k_{\gamma}} - C^{l(\gamma)}_{i(k)/\beta} + C^{l(\mu)}_{i(m)}P^{(m)(\gamma)}_{(\mu)\beta(k)},\\
P^{l(\gamma)}_{ij(k)} &= \frac{\partial L^l_{ij}}{\partial x^k_{\gamma}} - C^{l(\gamma)}_{i(k)|j} + C^{l(\mu)}_{i(m)}P^{(m)(\gamma)}_{(\mu)j(k)},\\
S^{l(\beta)(\gamma)}_{i(j)(k)} &= \frac{\partial C^{l(\beta)}_{i(j)}}{\partial x^k_{\gamma}} - \frac{\partial C^{l(\gamma)}_{i(k)}}{\partial x^j_{\beta}} + C^{m(\beta)}_{i(j)}C^{l(\gamma)}_{m(k)} - C^{m(\gamma)}_{i(k)}C^{l(\beta)}_{m(j)}.
\end{aligned} \tag{3.3}$$

Proof. The general formulas that express the local curvature d-tensors of an arbitrary Γ-linear connection [10], applied to the particular case of an h-normal Γ-linear connection $\nabla\Gamma$, imply formulas (3.3) and the relations in Table 3.2. □

Remark 3.4. In the case of the Berwald Γ_0-linear connection $B\Gamma_0$ associated to the pair of metrics $(h_{\alpha\beta},\varphi_{ij})$, all curvature d-tensors vanish, except $H^{\delta}_{\alpha\beta\gamma}$ and $R^{l}_{ijk}=r^{l}_{ijk}$, where r^{l}_{ijk} are the curvature tensors of the metric φ_{ij}.

4. Local Ricci identities. Nonmetrical deflection d-tensors identities. The local Ricci identities for a general Γ-linear connection on $E=J^1(T,M)$ are completely described in [10]. In the particular case of an h-normal Γ-linear connection, these simplify because the number of torsion and curvature d-tensors reduced and their local expressions simplified. A meaningful reduction of the local Ricci identities can be obtained, considering the following particular geometrical concept.

Definition 4.1. An h-normal Γ-linear connection, whose local components,

$$C\nabla\Gamma=\left(H^{\gamma}_{\alpha\beta},G^{k}_{i\gamma},L^{k}_{ij},C^{k(\gamma)}_{i(j)}\right),\tag{4.1}$$

verify the relations $L^{i}_{jk}=L^{i}_{kj}$ and $C^{i(\gamma)}_{j(k)}=C^{i(\gamma)}_{k(j)}$, is called an *$h$-normal Γ-linear connection of Cartan type.*

Remark 4.2. (i) Because the Christoffel symbols γ^{i}_{jk} of the metric φ_{ij} are symmetric, it follows that the Berwald h-normal Γ_0-linear connection $B\Gamma_0$ is of Cartan type.

(ii) The torsion d-tensor $\mathbf{T}$ of an h-normal Γ-linear connection of Cartan type $C\nabla\Gamma$ is characterized only by *eight* adapted local d-tensors because the torsion components $T^{i}_{jk}=L^{i}_{jk}-L^{i}_{kj}$ vanish from Table 3.1.

Theorem 4.3. *The following local Ricci identities for an h-normal Γ-linear connection of Cartan type $C\nabla\Gamma$ are true:*

$$\begin{aligned}
X^{\alpha}_{/\beta/\gamma}-X^{\alpha}_{/\gamma/\beta}&=X^{\mu}H^{\alpha}_{\mu\beta\gamma}-X^{\alpha}|^{(\mu)}_{(m)}R^{(m)}_{(\mu)\beta\gamma},\\
X^{\alpha}_{/\beta|k}-X^{\alpha}_{|k/\beta}&=-X^{\alpha}_{|m}T^{m}_{\beta k}-X^{\alpha}|^{(\mu)}_{(m)}R^{(m)}_{(\mu)\beta k},\\
X^{\alpha}_{|j|k}-X^{\alpha}_{|k|j}&=-X^{\alpha}|^{(\mu)}_{(m)}R^{(m)}_{(\mu)jk},\\
X^{\alpha}_{/\beta}|^{(\gamma)}_{(k)}-X^{\alpha}|^{(\gamma)}_{(k)/\beta}&=-X^{\alpha}|^{(\mu)}_{(m)}P^{(m)(\gamma)}_{(\mu)\beta(k)},\\
X^{\alpha}_{|j}|^{(\gamma)}_{(k)}-X^{\alpha}|^{(\gamma)}_{(k)|j}&=-X^{\alpha}_{|m}C^{m(\gamma)}_{j(k)}-X^{\alpha}|^{(\mu)}_{(m)}P^{(m)(\gamma)}_{(\mu)j(k)},\\
X^{\alpha}|^{(\beta)}_{(j)}|^{(\gamma)}_{(k)}-X^{\alpha}|^{(\gamma)}_{(k)}|^{(\beta)}_{(j)}&=-X^{\alpha}|^{(\mu)}_{(m)}S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)},\\
X^{i}_{/\beta/\gamma}-X^{i}_{/\gamma/\beta}&=X^{m}R^{i}_{m\beta\gamma}-X^{i}|^{(\mu)}_{(m)}R^{(m)}_{(\mu)\beta\gamma},\\
X^{i}_{/\beta|k}-X^{i}_{|k/\beta}&=X^{m}R^{i}_{m\beta k}-X^{i}_{|m}T^{m}_{\beta k}-X^{i}|^{(\mu)}_{(m)}R^{(m)}_{(\mu)\beta k},
\end{aligned}$$

$$
\begin{aligned}
X^i_{|j|k} - X^i_{|k|j} &= X^m R^i_{mjk} - X^i|^{(\mu)}_{(m)} R^{(m)}_{(\mu)jk},\\
X^i_{/\beta}|^{(\gamma)}_{(k)} - X^i|^{(\gamma)}_{(k)/\beta} &= X^m P^{i(\gamma)}_{m\beta(k)} - X^i|^{(\mu)}_{(m)} P^{(m)(\gamma)}_{(\mu)\beta(k)},\\
X^i_{|j}|^{(\gamma)}_{(k)} - X^i|^{(\gamma)}_{(k)|j} &= X^m P^{i(\gamma)}_{mj(k)} - X^i_{|m} C^{m(\gamma)}_{j(k)} - X^i|^{(\mu)}_{(m)} P^{(m)(\gamma)}_{(\mu)j(k)},\\
X^i|^{(\beta)}_{(j)}|^{(\gamma)}_{(k)} - X^i|^{(\gamma)}_{(k)}|^{(\beta)}_{(j)} &= X^m S^{i(\beta)(\gamma)}_{m(j)(k)} - X^i|^{(\mu)}_{(m)} S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)},\\
X^{(i)}_{(\alpha)/\beta/\gamma} - X^{(i)}_{(\alpha)/\gamma/\beta} &= X^{(m)}_{(\alpha)} R^i_{m\beta\gamma} - X^{(i)}_{(\mu)} H^{\mu}_{\alpha\beta\gamma} - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} R^{(m)}_{(\mu)\beta\gamma},\\
X^{(i)}_{(\alpha)/\beta|k} - X^{(i)}_{(\alpha)|k/\beta} &= X^{(m)}_{(\alpha)} R^i_{m\beta k} - X^{(i)}_{(\alpha)|m} T^m_{\beta k} - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} R^{(m)}_{(\mu)\beta k},\\
X^{(i)}_{(\alpha)|j|k} - X^{(i)}_{(\alpha)|k|j} &= X^{(m)}_{(\alpha)} R^i_{mjk} - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} R^{(m)}_{(\mu)jk},\\
X^{(i)}_{(\alpha)/\beta}|^{(\gamma)}_{(k)} - X^{(i)}_{(\alpha)}|^{(\gamma)}_{(k)/\beta} &= X^{(m)}_{(\alpha)} P^{i(\gamma)}_{m\beta(k)} - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} P^{(m)(\gamma)}_{(\mu)\beta(k)},\\
X^{(i)}_{(\alpha)|j}|^{(\gamma)}_{(k)} - X^{(i)}_{(\alpha)}|^{(\gamma)}_{(k)|j} &= X^{(m)}_{(\alpha)} P^{i(\gamma)}_{mj(k)} - X^{(i)}_{(\alpha)|m} C^{m(\gamma)}_{j(k)} - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} P^{(m)(\gamma)}_{(\mu)j(k)},\\
X^{(i)}_{(\alpha)}|^{(\beta)}_{(j)}|^{(\gamma)}_{(k)} - X^{(i)}_{(\alpha)}|^{(\gamma)}_{(k)}|^{(\beta)}_{(j)} &= X^{(m)}_{(\alpha)} S^{i(\beta)(\gamma)}_{m(j)(k)} - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)},
\end{aligned}
\tag{4.2}
$$

where $X = X^\alpha(\delta/\delta t^\alpha) + X^i(\delta/\delta x^i) + X^{(i)}_{(\alpha)}(\partial/\partial x^i_\alpha)$ is an arbitrary d-vector field on $J^1(T,M)$.

Proof. Using the local Ricci identities, described in the large context of a Γ-linear connection [10], together with particular features of an h-normal Γ-linear connection of Cartan type $C\nabla\Gamma$ described in Table 3.1 and Remark 4.2(ii) (i.e., the torsion d-components $\bar{T}^{\mu}_{\alpha\beta}$, $\bar{T}^{\mu}_{\alpha j}$, $\bar{P}^{\mu(\beta)}_{i(j)}$, and T^i_{jk} vanish), we obtain what we were looking for. □

In order to find an interesting application of preceding Ricci identities, consider $\mathbf{C} = x^i_\alpha(\partial/\partial x^i_\alpha)$ the *canonical Liouville d-vector field* on $E = J^1(T,M)$, together with an h-normal Γ-linear connection of Cartan type $C\nabla\Gamma$. In this context, we construct the *nonmetrical deflection d-tensors associated to* $C\nabla\Gamma$, setting

$$
\bar{D}^{(i)}_{(\alpha)\beta} = x^i_{\alpha/\beta}, \qquad D^{(i)}_{(\alpha)k} = x^i_{\alpha|k}, \qquad d^{(i)(\gamma)}_{(\alpha)(k)} = x^i_\alpha|^{(\gamma)}_{(k)}, \tag{4.3}
$$

where "$_{/\beta}$", "$_{|k}$", and "$|^{(\gamma)}_{(k)}$" are the local covariant derivatives produced by $C\nabla\Gamma$. By direct local computations, we deduce that the nonmetrical deflection d-tensors of $C\nabla\Gamma$ have the expressions:

$$
\begin{aligned}
\bar{D}^{(i)}_{(\alpha)\beta} &= -M^{(i)}_{(\alpha)\beta} + G^i_{m\beta} x^m_\alpha - H^{\mu}_{\alpha\beta} x^i_\mu,\\
D^{(i)}_{(\alpha)j} &= -N^{(i)}_{(\alpha)j} + L^i_{mj} x^m_\alpha,\\
d^{(i)(\beta)}_{(\alpha)(j)} &= \delta^i_j \delta^\beta_\alpha + C^{i(\beta)}_{m(j)} x^m_\alpha.
\end{aligned}
\tag{4.4}
$$

Applying now the (v)-set of Ricci identities to the components of Liouville vector field $\mathbf{C}$, we obtain the following important result.

Corollary 4.4. *The nonmetrical deflection d-tensors attached to the $C\nabla\Gamma$ verify the following local identities:*

$$
\begin{aligned}
\bar{D}^{(i)}_{(\alpha)\beta/\gamma}-\bar{D}^{(i)}_{(\alpha)\gamma/\beta}&=x^m_\alpha R^i_{m\beta\gamma}-x^i_\mu H^\mu_{\alpha\beta\gamma}-d^{(i)(\mu)}_{(\alpha)(m)}R^{(m)}_{(\mu)\beta\gamma},\\
\bar{D}^{(i)}_{(\alpha)\beta|k}-D^{(i)}_{(\alpha)k/\beta}&=x^m_\alpha R^i_{m\beta k}-D^{(i)}_{(\alpha)m}T^m_{\beta k}-d^{(i)(\mu)}_{(\alpha)(m)}R^{(m)}_{(\mu)\beta k},\\
D^{(i)}_{(\alpha)j|k}-D^{(i)}_{(\alpha)k|j}&=x^m_\alpha R^i_{mjk}-d^{(i)(\mu)}_{(\alpha)(m)}R^{(m)}_{(\mu)jk},\\
\bar{D}^{(i)}_{(\alpha)\beta}|^{(\gamma)}_{(k)}-d^{(i)(\gamma)}_{(\alpha)(k)/\beta}&=x^m_\alpha P^{i(\gamma)}_{m\beta(k)}-d^{(i)(\mu)}_{(\alpha)(m)}P^{(m)(\gamma)}_{(\mu)\beta(k)},\\
D^{(i)}_{(\alpha)j}|^{(\gamma)}_{(k)}-d^{(i)(\gamma)}_{(\alpha)(k)|j}&=x^m_\alpha P^{i(\gamma)}_{mj(k)}-D^{(i)}_{(\alpha)m}C^{m(\gamma)}_{j(k)}-d^{(i)(\mu)}_{(\alpha)(m)}P^{(m)(\gamma)}_{(\mu)j(k)},\\
d^{(i)(\beta)}_{(\alpha)(j)}|^{(\gamma)}_{(k)}-d^{(i)(\gamma)}_{(\alpha)(k)}|^{(\beta)}_{(j)}&=x^m_\alpha S^{i(\beta)(\gamma)}_{m(j)(k)}-d^{(i)(\mu)}_{(\alpha)(m)}S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)}.
\end{aligned}
\tag{4.5}
$$

Remark 4.5. The importance of nonmetrical deflection d-tensors identities comes from their using in the description of *generalized Maxwell equations* which govern the multitime electromagnetism constructed by Riemann-Lagrange geometrical instruments on the jet fibre bundle $J^1(T,M)$. For more details, please consult [7].

5. Local Bianchi identities for $C\nabla\Gamma$ connections on first jet bundle $J^1(T,M)$. From the general theory of linear connections on a vector bundle E, it is known that the torsions $\mathbf{T}$ and the curvature $\mathbf{R}$ of a linear connection ∇ are not independent. In other words, they are connected by the general Bianchi identities:

$$
\begin{aligned}
\sum_{\{X,Y,Z\}}\left\{(\nabla_X\mathbf{T})(Y,Z)-\mathbf{R}(X,Y)Z+\mathbf{T}(\mathbf{T}(X,Y),Z)\right\}&=0,\quad\forall X,Y,Z\in\mathcal{X}(E),\\
\sum_{\{X,Y,Z\}}\left\{(\nabla_X\mathbf{R})(U,Y,Z)+\mathbf{R}(\mathbf{T}(X,Y),Z)U\right\}&=0,\quad\forall X,Y,Z,U\in\mathcal{X}(E),
\end{aligned}
\tag{5.1}
$$

where $\{X,Y,Z\}$ means cyclic sum. Obviously, using a nonlinear connection Γ on a general vector bundle E, together with its local adapted basis of d-vector fields $(X_A)\subset\mathcal{X}(E)$, the Bianchi identities attached to a Γ-linear connection ∇ (i.e., a d-connection) on E can be locally described by the relations:

$$
\sum_{\{A,B,C\}}\left\{R^F_{ABC}-T^F_{AB:C}-T^G_{AB}T^F_{CG}\right\}=0,\qquad\sum_{\{A,B,C\}}\left\{R^F_{DAB:C}+T^G_{AB}R^F_{DAG}\right\}=0,\tag{5.2}
$$

where $\mathbf{R}(X_A,X_B)X_C=R^D_{CBA}X_D$, $\mathbf{T}(X_A,X_B)=T^D_{BA}X_D$, and "$_{:C}$" represent the horizontal or vertical local covariant derivatives produced by the d-connection ∇. For more details, see [4]. Applying these results to our particular 1-jet vector bundle $E=J^1(T,M)$, endowed with an h-normal Γ-linear connection of Cartan type $C\nabla\Gamma$, we find the next important local Bianchi identities.

Theorem 5.1. *The following thirty effective local Bianchi identities for an h-normal Γ-linear connection of Cartan type $C\nabla\Gamma$ on first jet bundle $J^1(T,M)$ are true:*

$$\begin{aligned}
&\sum_{\{\alpha,\beta,\gamma\}} H^{\delta}_{\alpha\beta\gamma}=0,\\
&\mathcal{A}_{\{\alpha,\beta\}}\left\{T^{l}_{\alpha m}T^{m}_{\beta k}-T^{l}_{\alpha k/\beta}\right\}=R^{l}_{k\alpha\beta}-C^{l(\mu)}_{k(m)}R^{(m)}_{(\mu)\alpha\beta},\\
&\mathcal{A}_{\{j,k\}}\left\{C^{l(\mu)}_{k(m)}R^{(m)}_{(\mu)\alpha j}+R^{l}_{j\alpha k}+T^{l}_{\alpha j|k}\right\}=0,\\
&\sum_{\{i,j,k\}}\left\{C^{l(\mu)}_{k(m)}R^{(m)}_{(\mu)ij}-R^{l}_{ijk}\right\}=0,
\end{aligned}\tag{5.3}$$

$$\begin{aligned}
&\sum_{\{\alpha,\beta,\gamma\}}\left\{R^{(l)}_{(\delta)\alpha\beta/\gamma}+P^{(l)(\mu)}_{(\delta)\gamma(m)}R^{(m)}_{(\mu)\alpha\beta}\right\}=0,\\
&\mathcal{A}_{\{\alpha,\beta\}}\left\{R^{(l)}_{(\delta)\alpha k/\beta}+P^{(l)(\mu)}_{(\delta)\beta(m)}R^{(m)}_{(\mu)\alpha k}+R^{(l)}_{(\delta)\beta m}T^{m}_{\alpha k}\right\}=R^{(l)}_{(\delta)\alpha\beta|k}+P^{(l)(\mu)}_{(\delta)k(m)}R^{(m)}_{(\mu)\alpha\beta},\\
&\mathcal{A}_{\{j,k\}}\left\{R^{(l)}_{(\delta)\alpha j|k}+P^{(l)(\mu)}_{(\delta)k(m)}R^{(m)}_{(\mu)\alpha j}+R^{(l)}_{(\delta)km}T^{m}_{\alpha j}\right\}=-R^{(l)}_{(\delta)\alpha j|k}-P^{(l)(\mu)}_{(\delta)\alpha(m)}R^{(m)}_{(\mu)jk},\\
&\sum_{\{i,j,k\}}\left\{R^{(l)}_{(\delta)ij|k}+P^{(l)(\mu)}_{(\delta)k(m)}R^{(m)}_{(\mu)ij}\right\}=0,
\end{aligned}\tag{5.4}$$

$$\begin{aligned}
&T^{l}_{\alpha k}\big|^{(\varepsilon)}_{(p)}-C^{l(\varepsilon)}_{m(p)}T^{m}_{\alpha k}+P^{l(\varepsilon)}_{k\alpha(p)}-C^{l(\varepsilon)}_{k(p)/\alpha}-C^{l(\mu)}_{k(m)}P^{(m)(\varepsilon)}_{(\mu)\alpha(p)}=0,\\
&\mathcal{A}_{\{j,k\}}\left\{C^{l(\varepsilon)}_{j(p)|k}+C^{l(\mu)}_{k(m)}P^{(m)(\varepsilon)}_{(\mu)j(p)}+P^{l(\varepsilon)}_{jk(p)}\right\}=0,
\end{aligned}\tag{5.5}$$

$$\begin{aligned}
&\mathcal{A}_{\{\alpha,\beta\}}\left\{P^{(l)(\varepsilon)}_{(\delta)\alpha(p)/\beta}+P^{(l)(\mu)}_{(\delta)\beta(m)}P^{(m)(\varepsilon)}_{(\mu)\alpha(p)}\right\}=R^{(l)}_{(\delta)\alpha\beta}\big|^{(\varepsilon)}_{(p)}-R^{(l)(\varepsilon)}_{(\delta)(p)\alpha\beta}+S^{(l)(\varepsilon)(\mu)}_{(\delta)(p)(m)}R^{(m)}_{(\mu)\alpha\beta},\\
&\mathcal{A}_{\{\alpha,k\}}\left\{P^{(l)(\varepsilon)}_{(\delta)\alpha(p)|k}+P^{(l)(\mu)}_{(\delta)k(m)}P^{(m)(\varepsilon)}_{(\mu)\alpha(p)}\right\}=R^{(l)}_{(\delta)\alpha k}\big|^{(\varepsilon)}_{(p)}-R^{(l)(\varepsilon)}_{(\delta)(p)\alpha k}+S^{(l)(\varepsilon)(\mu)}_{(\delta)(p)(m)}R^{(m)}_{(\mu)\alpha k}\\
&\qquad\qquad+R^{(l)}_{(\delta)\alpha m}C^{m(\varepsilon)}_{k(p)}-T^{m}_{\alpha k}P^{(l)(\varepsilon)}_{(\delta)m(p)},\\
&\mathcal{A}_{\{j,k\}}\left\{P^{(l)(\varepsilon)}_{(\delta)j(p)|k}+P^{(l)(\mu)}_{(\delta)k(m)}P^{(m)(\varepsilon)}_{(\mu)j(p)}+R^{(l)}_{(\delta)km}C^{m(\varepsilon)}_{j(p)}\right\}\\
&\qquad=R^{(l)}_{(\delta)jk}\big|^{(\varepsilon)}_{(p)}-R^{(l)(\varepsilon)}_{(\delta)(p)jk}+S^{(l)(\varepsilon)(\mu)}_{(\delta)(p)(m)}R^{(m)}_{(\mu)jk},
\end{aligned}\tag{5.6}$$

$$\mathcal{A}_{\left\{\binom{\beta}{j},\binom{\gamma}{k}\right\}}\left\{C^{l(\beta)}_{i(j)}\big|^{(\gamma)}_{(k)}+C^{m(\gamma)}_{i(k)}C^{l(\beta)}_{m(j)}\right\}=S^{l(\beta)(\gamma)}_{i(j)(k)}-C^{l(\mu)}_{i(m)}S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)},\tag{5.7}$$

$$\begin{aligned}
&\mathcal{A}_{\left\{\binom{\beta}{j},\binom{\gamma}{k}\right\}}\left\{P^{(l)(\beta)}_{(\delta)\alpha(j)}\big|^{(\gamma)}_{(k)}+P^{(m)(\beta)}_{(\mu)\alpha(j)}S^{(l)(\gamma)(\mu)}_{(\delta)(k)(m)}+P^{(l)(\beta)(\gamma)}_{(\delta)(j)\alpha(k)}\right\}\\
&\qquad=-S^{(l)(\beta)(\gamma)}_{(\delta)(j)(k)/\alpha}-S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)}P^{(l)(\mu)}_{(\delta)\alpha(m)},\\
&\mathcal{A}_{\left\{\binom{\beta}{j},\binom{\gamma}{k}\right\}}\left\{P^{(l)(\beta)}_{(\delta)i(j)}\big|^{(\gamma)}_{(k)}+P^{(m)(\beta)}_{(\mu)i(j)}S^{(l)(\gamma)(\mu)}_{(\delta)(k)(m)}+P^{(l)(\beta)(\gamma)}_{(\delta)(j)i(k)}\right\}\\
&\qquad=-S^{(l)(\beta)(\gamma)}_{(\delta)(j)(k)|i}-S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)}P^{(l)(\mu)}_{(\delta)i(m)},
\end{aligned}\tag{5.8}$$

$$\sum_{\left\{\binom{\alpha}{i},\binom{\beta}{j},\binom{\gamma}{k}\right\}}\left\{S^{(l)(\alpha)(\beta)}_{(\delta)(i)(j)}\big|^{(\gamma)}_{(k)}+S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)}S^{(l)(\gamma)(\mu)}_{(\delta)(k)(m)}-S^{(l)(\alpha)(\beta)(\gamma)}_{(\delta)(i)(j)(k)}\right\}=0,\tag{5.9}$$

$$\sum_{\{\alpha,\beta,\gamma\}} H^{\delta}_{\varepsilon\alpha\beta/\gamma}=0,\qquad H^{\delta}_{\varepsilon\alpha\beta|k}=0,$$

$$\sum_{\{i,j,k\}} R^{(m)}_{(\mu)ij}P^{(\delta)(\mu)}_{(\varepsilon)k(m)}=0,$$

$$\sum_{\{\alpha,\beta,\gamma\}}\left\{R^{l}_{p\alpha\beta/\gamma}-R^{(m)}_{(\mu)\alpha\beta}P^{l(\mu)}_{p\gamma(m)}\right\}=0,$$

$$\mathcal{A}_{\{\alpha,\beta\}}\left\{R^{l}_{p\alpha k/\beta}+R^{(m)}_{(\mu)\alpha k}P^{l(\mu)}_{p\beta(m)}-T^{m}_{\alpha k}R^{l}_{p\beta m}\right\}=R^{l}_{p\alpha\beta}+R^{(m)}_{(\mu)\alpha\beta}P^{l(\mu)}_{pk(m)},\tag{5.10}$$

$$\mathcal{A}_{\{j,k\}}\left\{R^{l}_{p\alpha j|k}+R^{(m)}_{(\mu)\alpha j}P^{l(\mu)}_{pk(m)}-T^{m}_{\alpha j}R^{l}_{pkm}\right\}=-R^{l}_{pjk/\alpha}+R^{(m)}_{(\mu)\alpha k}P^{l(\mu)}_{pj(m)},$$

$$\sum_{\{i,j,k\}}\left\{R^{l}_{pij|k}-R^{(m)}_{(\mu)ij}P^{l(\mu)}_{pk(m)}\right\}=0,$$

$$\mathcal{A}_{\{\alpha,\beta\}}\left\{P^{l(\varepsilon)}_{i\alpha(p)/\beta}-P^{(m)(\varepsilon)}_{(\mu)\alpha(p)}P^{l(\mu)}_{i\beta(m)}\right\}=R^{l}_{i\alpha\beta}\big|^{(\varepsilon)}_{(p)}+R^{(m)}_{(\mu)\alpha\beta}S^{l(\varepsilon)(\mu)}_{i(p)(m)},$$

$$\begin{aligned}\mathcal{A}_{\{\alpha,k\}}\left\{P^{l(\varepsilon)}_{i\alpha(p)|k}-P^{(m)(\varepsilon)}_{(\mu)\alpha(p)}P^{l(\mu)}_{ik(m)}\right\}&=R^{l}_{i\alpha k}\big|^{(\varepsilon)}_{(p)}-R^{(m)}_{(\mu)\alpha k}S^{l(\varepsilon)(\mu)}_{i(p)(m)}\\&\quad-C^{m(\varepsilon)}_{k(p)}R^{l}_{i\alpha m}+T^{m}_{\alpha k}P^{l(\varepsilon)}_{im(p)},\end{aligned}\tag{5.11}$$

$$\mathcal{A}_{\{j,k\}}\left\{P^{l(\varepsilon)}_{ij(p)|k}-P^{(m)(\varepsilon)}_{(\mu)j(p)}P^{l(\mu)}_{ik(m)}-C^{m(\varepsilon)}_{j(p)}R^{l}_{ikm}\right\}=R^{l}_{ijk}\big|^{(\varepsilon)}_{(p)}+R^{(m)}_{(\mu)jk}S^{l(\varepsilon)(\mu)}_{i(p)(m)},$$

$$\mathcal{A}_{\left\{{}^{(\beta)}_{(j)},{}^{(\gamma)}_{(k)}\right\}}\left\{P^{l(\beta)}_{p\alpha(j)}\big|^{(\gamma)}_{(k)}-P^{(m)(\beta)}_{(\mu)\alpha(j)}S^{l(\gamma)(\mu)}_{p(k)(m)}\right\}=S^{l(\beta)(\gamma)}_{p(j)(k)/\alpha}+S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)}P^{l(\mu)}_{p\alpha(m)},$$

$$\mathcal{A}_{\left\{{}^{(\beta)}_{(j)},{}^{(\gamma)}_{(k)}\right\}}\left\{P^{l(\beta)}_{pi(j)}\big|^{(\gamma)}_{(k)}-P^{(m)(\beta)}_{(\mu)i(j)}S^{l(\gamma)(\mu)}_{p(k)(m)}+C^{m(\beta)}_{i(j)}P^{l(\gamma)}_{pm(k)}\right\}=S^{l(\beta)(\gamma)}_{p(j)(k)|i}+S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)}P^{l(\mu)}_{pi(m)},\tag{5.12}$$

$$\sum_{\left\{{}^{(\alpha)}_{(i)},{}^{(\beta)}_{(j)},{}^{(\gamma)}_{(k)}\right\}}\left\{S^{l(\alpha)(\beta)}_{p(i)(j)}\big|^{(\gamma)}_{(k)}+S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)}S^{l(\gamma)(\mu)}_{p(k)(m)}\right\}=0,\tag{5.13}$$

where, if $\{A,B,C\}$ *are indices of type* $\{\alpha,i,{}^{(\alpha)}_{(i)}\}$, *then* $\sum_{\{A,B,C\}}$ *represents a cyclic sum, and* $\mathcal{A}_{\{A,B\}}$ *represents an alternate sum.*

Proof. Let $(X_A)=(\delta/\delta t^{\alpha},\delta/\delta x^{i},\partial/\partial x^{i}_{\alpha})$ be the adapted basis associated to the nonlinear connection $\Gamma=(M^{(i)}_{(\alpha)\beta},N^{(i)}_{(\alpha)j})$ on the 1-jet vector bundle $E=J^1(T,M)$. Taking into account, on the one hand, that the indices $A,B,\dots$ are of type $\{\alpha,i,{}^{(\alpha)}_{(i)}\}$, and, on the other hand, that the torsion T^{C}_{AB} and curvature R^{D}_{ABC} adapted components are given by Tables 3.1 and 3.2, after laborious local computations, formulas (5.2) imply the required Bianchi identities. □

Remark 5.2. (i) Although the author hopes that there is no mistakes in the preceding local expressions of Bianchi identities, he thanks in advance for any correction coming from readers. However, we should like to point out that, in the particular case $(T,h)=(\mathbb{R},\delta)$, the last identity of each set of local Bianchi identities reduces to one of classical eleven Bianchi identities that characterize the *N-linear connections* from Lagrangian geometry [4]. (ii) The importance of local Bianchi identities of an h-normal Γ-linear connection of Cartan type $C\nabla\Gamma$ on 1-jet bundles comes from their use in the description of generalized Maxwell equations of the multitime electromagnetic field, and the description of generalized conservation laws of the multitime stress-energy d-tensor from the Riemann-Lagrange geometry of multitime physical fields on $J^1(T,M)$, developed in [5, 7].

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