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THE HULL NUMBER OF AN ORIENTED GRAPH

GARY CHARTRAND, JOHN FREDERICK FINK,
and PING ZHANG

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We present characterizations of connected graphs G of order $n \geq 2$ for which $h^+(G) = n$. It is shown that for every two integers n and m with $1 \leq n-1 \leq m \leq \binom{n}{2}$, there exists a connected graph G of order n and size m such that for each integer k with $2 \leq k \leq n$, there exists an orientation of G with hull number k.

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1. Introduction. The (*directed*) *distance* $d(u,v)$ from a vertex u to a vertex v in an oriented graph D is the length of a shortest directed $u-v$ path in D. A directed $u-v$ path of length $d(u,v)$ is referred to as a $u-v$ *geodesic*. A vertex w is said to *lie in* a $u-v$ geodesic P if w is an internal vertex of P, that is, w is a vertex of P distinct from u and v. The *closed interval* $I[u,v]$ consists of u and v together with all vertices lying in a $u-v$ geodesic or in a $v-u$ geodesic in D. Hence, if there is neither a $u-v$ geodesic nor a $v-u$ geodesic in D, then $I[u,v] = \{u,v\}$. For a nonempty subset S of $V(D)$, define

$$I[S] = \bigcup_{u,v \in S} I[u,v]. \tag{1.1}$$

Then certainly $S \subseteq I[S]$. A set S is *convex* if $I[S] = S$. The *convex hull* $[S]$ of S is the smallest convex set containing S. The set $[S]$ is also the intersection of all convex sets containing S. The convex hull $[S]$ of S can also be formed from the sequence $\{I^k[S]\}$, $k \geq 0$, where $I^0[S] = S$, $I^1[S] = I[S]$, and $I^k[S] = I[I^{k-1}[S]]$ for $k \geq 2$. From some term on, this sequence must be constant. Let p be the smallest number such that $I^p[S] = I^{p+1}[S]$. Then $I^p[S]$ is the convex hull $[S]$. A set S of vertices of D is called a *hull set* of D if $[S] = V(D)$. A hull set of minimum cardinality is a *minimum hull set of* D. The cardinality of a minimum hull set in D is called the *hull number* $h(D)$. Certainly, if D is a nontrivial connected oriented graph of order n, then $2 \leq h(D) \leq n$.

Concepts related to hull sets and hull numbers in oriented graphs were studied in [7]. A set S of vertices in an oriented graph D is a *geodetic set* if $I[S] = V(D)$. A geodetic set of minimum cardinality is a *minimum geodetic set*, and this cardinality is the *geodetic number* $g(D)$.

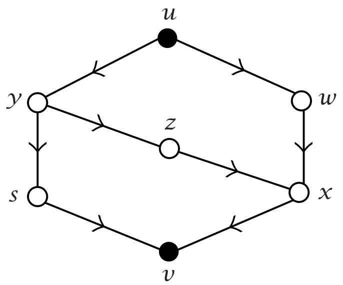

FIGURE 1.1. An oriented graph D with $h(D) = 2$ and $g(D) = 3$.

To illustrate these concepts, consider the oriented graph D of Figure 1.1. Let $S = \{u,v\}$. Since $I[S] = V(D) - \{z\} \neq S$, it follows that S is not convex. However, $[S] = I^2[S] = V(G)$ and so S is a minimum hull set in D. Therefore, $h(D) = 2$. On the other hand, the geodetic number of the oriented graph D of Figure 1.1 is 3 and $\{u,v,z\}$ is a minimum geodetic set in D.

If S is a hull set of an oriented graph D and $u,v \in S$, then each vertex of every $u-v$ geodesic of D belongs to $I[S]$. This observation implies the following lemma.

Lemma 1.1. *Let S be a minimum hull set of an oriented graph D and let $u,v \in S$. If w lies in a $u-v$ geodesic in D, then $w \notin S$.*

The *degree* $\deg v$ of a vertex v in an oriented graph is the sum of its indegree and outdegree, that is, $\deg v = \operatorname{id} v + \operatorname{od} v$. A vertex v is an *endvertex* if $\deg v = 1$. A *transmitter* is a vertex having positive outdegree and indegree 0, while a *receiver* is a vertex having positive indegree and outdegree 0. For a vertex u of D, let

$$N^+(u) = \{x : (u,x) \in E(D)\}, \qquad N^-(u) = \{x : (x,u) \in E(D)\}. \tag{1.2}$$

So if u is a transmitter, then $N^-(u) = \emptyset$; while if v is a receiver, then $N^+(u) = \emptyset$. A vertex u of D is a *transitive vertex* if (1) $\operatorname{od} u > 0$ and $\operatorname{id} u > 0$, (2) for every $v \in N^+(u)$ and $w \in N^-(u)$, $(w,v) \in E(D)$. A vertex v of D is an *extreme* vertex if v is a transmitter, receiver, or transitive vertex (see [2]). If v is an extreme vertex, then v can only be the initial or the terminal vertex of a geodesic containing v. This observation yields the following lemma.

Lemma 1.2. *Every hull set of a connected oriented graph D must contain the extreme vertices of D. In particular, every hull set of D must contain its endvertices. Moreover, if the set of the extreme vertices of D is a hull set, then it is the unique minimum hull set.*

The closed intervals $I[u,v]$ in a connected graph were studied and characterized by Nebeský [12, 13] and were also investigated extensively in Mulder

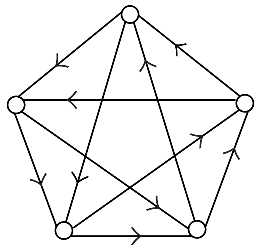

FIGURE 1.2. An oriented graph in which every pair of vertices is a hull set.

[10], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. The sequential construction of a convex hull of a set of vertices in a graph was utilized in [9]. The hull number of a graph was introduced by Everett and Seidman [8] who characterized graphs having some particular hull numbers and who obtained a number of bounds for the hull numbers of graphs. The hull numbers of median graphs was determined by Mulder [11]. (A connected graph G is a *median graph* if for every three vertices u, v, and w of G, there is a unique vertex lying on a geodesic between each pair of u, v, and w.) The hull number of a graph was studied further in [3]. Convexity in graphs and digraphs was studied in [2, 6]. The geodetic number of a graph was introduced in [1] and studied further in [4], while the geodetic number of an oriented graph was studied in [7]. We refer to [1] for concepts and results on distance in graphs.

We have seen that if D is a nontrivial connected oriented graph of order n, then

$$2 \le h(D) \le n. \tag{1.3}$$

The upper and lower bounds in (1.3) are sharp for all $n \ge 2$. For example, the directed path $\overrightarrow{P_n} : v_1, v_2, \dots, v_n$ of order $n \ge 2$ has hull number 2, where the set $\{v_1, v_n\}$ is its *unique* minimum geodetic set. Obviously, the hull number of the directed cycle $\overrightarrow{C_n}$ is 2 as well, but in this case *every* pair of vertices in $\overrightarrow{C_n}$ is a hull set of $\overrightarrow{C_n}$. It was shown in [7] that $\overrightarrow{C_n}$ is the only connected oriented graph of order n such that every pair of its vertices forms a geodetic set. However, this is not true for hull sets. It can be shown that every pair of vertices of the oriented graph in Figure 1.2 is a hull set.

At the other extreme are oriented graphs D of order n for which $h(D) = n$. We need an additional definition. An oriented graph D is *transitive* if whenever (u, v) and (v, w) are arcs of D, then (u, w) is an arc of D. We can now characterize oriented graphs of order n having hull number n.

PROPOSITION 1.3. *Let D be a nontrivial oriented graph of order n. Then $h(D) = n$ if and only if D is transitive.*

Proof. Assume first that $h(D) < n$. Then there exists a vertex v in D such that $S = V(D) - \{v\}$ is a hull set of D. Then $\operatorname{od} v > 0$ and $\operatorname{id} v > 0$. This implies that v lies in some $u-w$ geodesic u, v, w in D, where $u,w \in S$. Therefore, $(u,w) \notin E(D)$ and D is not transitive.

Conversely, assume that D is an oriented graph that is not transitive. Then there exist distinct vertices u, v, and w such that $(u,v),(v,w) \in E(D)$, but $(u,w) \notin E(D)$. Then $S = V(D) - \{v\}$ is a hull set and so $h(D) < n$. □

Next we show that for a given integer $n \geq 2$, every integer k with $2 \leq k \leq n$ is the hull number of some oriented graph of order n.

Proposition 1.4. *For every two integers k and n with $2 \leq k \leq n$, there exists an oriented graph of order n and hull number k.*

Proof. We show, in fact, that there exists an oriented graph with this property having the path P_n of order n as its underlying graph. Let $P_n : v_1, v_2, \ldots, v_n$. We construct an oriented graph D from P_n by directing the two edges incident with v_i towards v_i for all even i with $i < k$. If k is odd, then each edge $v_i v_{i+1}$ with $i \geq k$ is directed as (v_{i+1}, v_i). If k is even, then each edge $v_i v_{i+1}$ for $i \geq k-1$ is directed as (v_i, v_{i+1}). In each case, the set $\{v_1, v_2, \ldots, v_{k-1}, v_n\}$ of extreme vertices is a hull set, so $h(D) = k$. □

Next we provide an upper bound for the hull number of an oriented graph in terms of its order and *diameter* (the length of a longest geodesic).

Proposition 1.5. *If D is a connected oriented graph of order n and diameter d, then*

$$h(D) \leq n - d + 1. \tag{1.4}$$

Proof. Let u and v be vertices of D for which $d(u,v) = d$ and let $u = v_0, v_1, \ldots, v_d = v$ be a $u-v$ geodesic. Let $S = V(D) - \{v_1, v_2, \ldots, v_{d-1}\}$. Then $[S] = V(D)$ and so $h(D) \leq |S| = n - d + 1$. □

Note that, in the proof of Proposition 1.4, the oriented graph D constructed there has order n, diameter $d = n - k + 1$, and hull number $k = n - d + 1$. Therefore, the upper bound for $h(D)$ presented in Proposition 1.5 is sharp.

2. Relating hull number to geodetic number. If D is a nontrivial connected oriented graph with $h(D) = a$ and $g(D) = b$, then necessarily $2 \leq a \leq b$. We now show that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the hull number and geodetic number, respectively, of some oriented graph. The following lemma is analogous to Lemma 1.2.

Lemma 2.1. *Every geodetic set of a connected oriented graph D must contain the extreme vertices of D. In particular, every geodetic set of D must contain its endvertices. Moreover, if the set of the extreme vertices of D is a geodetic set, then it is the unique geodetic set.*

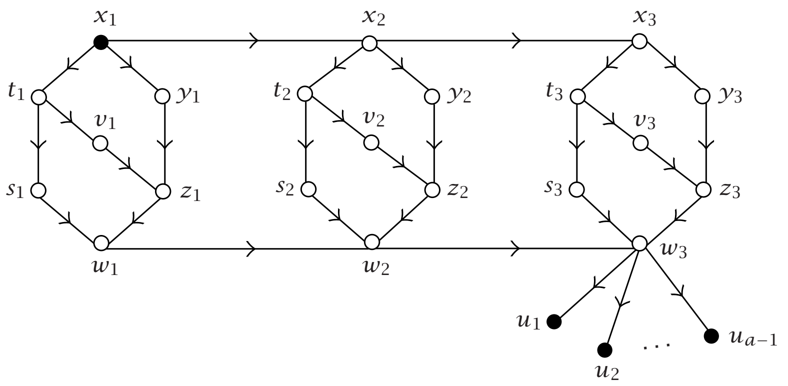

FIGURE 2.1. An oriented graph D with $h(D) = a$ and $g(D) = b = a+3$.

THEOREM 2.2. *For every pair a, b of integers with $2 \le a \le b$, there exists a connected oriented graph D such that $h(D) = a$ and $g(D) = b$.*

PROOF. Assume first that $a = b \ge 2$. Let D be the oriented graph whose underlying graph is the star $K_{1,a}$, where $V(D) = \{v, v_1, v_2, \ldots, v_a\}$ and $\deg v = a$, and such that $(v_1, v) \in E(D)$ and $(v, v_i) \in E(D)$ for $2 \le i \le a$. Then $V(D) - \{v\}$ is the set of extreme vertices of D. Since $V(D) - \{v\}$ is both a hull set and a geodetic set, it follows by Lemmas 1.2 and 2.1 that $h(D) = g(D) = |V(D) - \{v\}| = a$.

Assume next that $2 \le a < b$. We construct an oriented graph D with the required hull and geodetic numbers. For each integer i with $1 \le i \le b-a$, let F_i be the oriented graph whose underlying graph is the 6-cycle x_i, y_i, z_i, w_i, s_i, t_i, x_i and such that F_i contains the two directed $x_i - w_i$ paths x_i, y_i, z_i, w_i and x_i, t_i, s_i, w_i. The oriented graph D_i is produced by adding the vertex v_i and the two arcs (t_i, v_i) and (v_i, z_i) to F_i. Therefore, D_i is isomorphic to the oriented graph of Figure 1.1. The oriented graph D is then obtained from the oriented graphs D_i $(1 \le i \le b-a)$ by adding (1) the $a-1$ new vertices u_j $(1 \le j \le a-1)$, (2) the arcs (w_{b-a}, u_j) for $1 \le j \le a-1$, and (3) the arcs (x_i, x_{i+1}) and (w_i, w_{i+1}) for $1 \le i \le b-a-1$. The oriented graph D is shown in Figure 2.1 for $b-a = 3$.

Let $U = \{u_1, u_2, \ldots, u_{a-1}\}$ and $V = \{v_1, v_2, \ldots, v_{b-a}\}$. Then $\{x_1\} \cup U$ is the set of extreme vertices of D. Since $I[\{x_1\} \cup U] = V(D) - V$ and $[\{x_1\} \cup U] = I^2[\{x_1\} \cup U] = V(D)$, it follows that $\{x_1\} \cup U$ is a hull set of D and so $h(G) = |\{x_1\} \cup U| = a$ by Lemma 1.2.

Next we show that $g(G) = b$. Since $\{x_1\} \cup U \cup V$ is a geodetic set, $g(G) \le |\{x_1\} \cup U \cup V| = b$. It remains to show that $g(G) \ge b$. Let W be a minimum geodetic set of D. Certainly, $\{x_1\} \cup U \subset W$ by Lemma 2.1. Of course, W contains the vertex x_1 in D_1. We claim that W contains at least one vertex in each

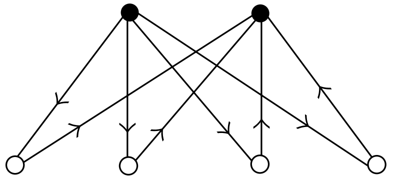

FIGURE 3.1. An oriented graph D of $K_{2,4}$ with $h(D) = 2$.

D_i for all i with $2 \le i \le b-a$. Otherwise, $V(D_i) \cap W = \emptyset$ for some i with $2 \le i \le b-a$. Observe that v_i does not lie on any $x-y$ geodesic in G for $x, y \notin V(D_i)$. This implies that $v_i \notin I[W]$, which contradicts the fact that W is a geodetic set. Therefore, as claimed, W contains at least one vertex from each D_i $(2 \le i \le b-a)$ and so $|W| \ge |\{x_1\} \cup U| + (b-a-1) = b-1$. On the other hand, if $|W| = b-1$, then W contains exactly one vertex from each D_i $(1 \le i \le b-a)$. In particular, x_1 is the only vertex of D_1 belonging to W. Since each vertex v_i $(2 \le i \le b-a)$ only lies on those geodesics having v_i as one of its endvertices or having both endvertices belonging to D_i, it follows that $v_i \in W$. This implies that $W = \{x_1\} \cup U \cup (V - \{v_1\})$. However, then $v_1 \notin I[W]$, which is a contradiction. Therefore, $g(D) = b$. □

3. Orientable hull numbers of graphs. For a connected graph G of order $n \ge 2$, the *lower orientable hull number* $h^-(G)$ of G is defined as the minimum hull number among the orientations of G and the *upper orientable hull number* $h^+(G)$ as the maximum hull number, that is,

$$h^-(G) = \min\{h(D) : D \text{ is an orientation of } G\},$$
$$h^+(G) = \max\{h(D) : D \text{ is an orientation of } G\}. \tag{3.1}$$

Hence, for every connected graph G of order $n \ge 2$, we have $2 \le h^-(G) \le h^+(G) \le n$. First, we present a lemma that gives a sufficient condition for a graph to have lower hull number 2. Observe that if a graph G contains a Hamiltonian path, then there exists an orientation D of G in which the Hamiltonian path in G is oriented as a Hamiltonian path in D such that $h(D) = 2$. Thus we have the following lemma.

LEMMA 3.1. *Let G be a connected graph of order $n \ge 2$. If G contains a Hamiltonian path, then $h^-(G) = 2$.*

The sufficient condition given in Lemma 3.1 for a graph to have lower orientable hull number 2 is not necessary. For example, the graph $K_{2,4}$ contains no Hamiltonian path. Since the orientation D of $K_{2,4}$ of Figure 3.1 has $h(D) = 2$, it follows that $h^-(G) = 2$.

Certainly, if G contains a Hamiltonian path, then G contains a spanning tree with two endvertices. More generally, we have the following lemma.

Lemma 3.2. *Let μ be the minimum number of endvertices among all spanning trees of a connected graph G. Then $h^-(G) \le \mu$.*

Proof. Let T be a spanning tree of G with μ endvertices, say, $v_1, v_2, \dots, v_\mu$. We orient the edges of T so that T is rooted at v_1, that is, so that there is a directed path in T from v_1 to every other vertex of T. For each edge uv of G not in T, if $d_T(v_1,u) \le d_T(v_1,v)$, then orient uv as (v,u), and otherwise, orient uv as (u,v). We denote the resulting digraph by D.

Let $S = \{v_1, v_2, \dots, v_\mu\}$. For distinct integers $i, j \in \{1, 2, \dots, \mu\}$, the only $v_i - v_j$ geodesics in D are the unique $v_1 - v_j$ paths in T with $2 \le j \le \mu$. Thus $I[v_1, v_j]$ contains all vertices of the unique $v_1 - v_j$ path in T. Hence $V(D) = \cup_{j=2}^{\mu} I[v_1, v_j] \subseteq I[S]$ and so $I[S] = V(D)$. Therefore, $h(D) \le |S| = \mu$. □

We now turn to those connected graphs G of order $n \ge 2$ for which $h^+(G) = n$. Recall that a vertex v is an extreme vertex in an oriented graph if v is a transmitter, a receiver, or a transitive vertex. Hence if there exists an orientation D of a connected graph G of order $n \ge 2$ such that every vertex of D is an extreme vertex, then $h^+(G) = n$. The converse is also true for assume that G is a connected graph of order $n \ge 2$ for which every orientation of G has some vertex that is not an extreme vertex. Let D be an orientation of G. Then D contains a vertex v that is not an extreme vertex. Hence v is neither a transmitter, a receiver, nor a transitive vertex. Therefore, there exist two vertices u and w distinct from v such that (u,v) and (v,w) are arcs of D but (u,w) is not an arc of D. Consequently, v lies in a $u - w$ geodesic in D and $V(D) - \{v\}$ is a hull set. Thus $h(D) \le n-1$. Since D is an arbitrary orientation of G, it follows that $h^+(G) \le n-1$. We summarize these observations.

Proposition 3.3. *Let G be a connected graph of order $n \ge 2$. Then $h^+(G) = n$ if and only if there exists an orientation D of G such that every vertex of D is an extreme vertex.*

For example, it is not difficult to show that there exists an orientation of any bipartite graph and any complete multipartite graph in which every vertex is an extreme vertex. Thus if G is a bipartite graph or a complete multipartite graph of order $n \ge 2$, then $h^+(G) = n$.

There is yet another characterization of those connected graphs G of order $n \ge 2$ for which $h^+(G) = n$. Let S be a finite nonempty set of positive integers. The *divisor graph $G(S)$ of S* has S as its vertex set and two vertices i and j are adjacent if either $i \mid j$ or $j \mid i$. A graph G is *a divisor graph* if $G = G(S)$ for some finite nonempty set S of positive integers. The following theorem was proved in [5].

Theorem 3.4. *Let G be a graph. Then G is a divisor graph if and only if there exists an orientation D of G such that every vertex of D is an extreme vertex.*

Combining Proposition 3.3 and Theorem 3.4, we have the following characterization which relates two concepts that, at the outset, would appear to be unrelated.

Theorem 3.5. *Let G be a connected graph of order $n \geq 2$. Then $h^+(G) = n$ if and only if G is a divisor graph.*

There exist connected graphs G and an integer k with $h^-(G) < k < h^+(G)$ such that G has no orientation with hull number k. In order to show this, we first determine the hull number of an orientation of a cycle.

Proposition 3.6. *Let D be an orientation of C_n for $n \geq 3$. Then $h(D) = 3$ or $h(D) = 2t$ for some integer t with $1 \leq t \leq n/2$.*

Proof. Let $C_n : v_1, v_2, \ldots, v_n, v_1$ be a cycle of order n and let D be an orientation of C_n. Then the number t of transmitters of D equals the number of receivers of D. If $t = 0$, then D is a directed cycle and so $h(D) = 2$. For $t \geq 1$, let $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ be the transmitters of D and let $v_{j_1}, v_{j_2}, \ldots, v_{j_t}$ be the receivers of D, where $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_t \leq n$. By Lemma 1.2, $h(D) \geq 2t$. We assume, without loss of generality, that these transmitters and receivers appear around the cycle C_n clockwise as $v_{i_1}, v_{j_1}, v_{i_2}, v_{j_2}, \ldots, v_{i_t}, v_{j_t}$.

Case 1 ($t = 1$). If $d_D(v_{i_1}, v_{j_1}) = n/2$, then $\{v_{i_1}, v_{j_1}\}$ is a hull set and so $h(D) = 2$. If $d_D(v_{i_1}, v_{j_1}) < n/2$, then $\{v_{i_1}, v_{j_1}\}$ is not a hull set and so $h(D) \geq 3$ by Lemma 1.2. On the other hand, $\{v_{i_1}, v_{i_1+1}, v_{j_1}\}$ or $\{v_{i_1-1}, v_{i_1}, v_{j_1}\}$ is a hull set. Thus $h(D) = 3$.

Case 2 ($t \geq 2$). Since $\{v_{i_1}, v_{i_2}, \ldots, v_{i_t}, v_{j_1}, v_{j_2}, \ldots, v_{j_t}\}$ is a hull set of D, it follows by Lemma 1.2 that $h(D) = 2t$. □

By Lemma 3.1 and Proposition 3.3, it is easy to verify that (1) $h^-(C_n) = 2$ for all $n \geq 3$ and (2) $h^+(C_n) = n$ if $n = 3$ or n is even, while $h^+(C_n) = n-1$ if $n \geq 5$ is odd. Thus, by Proposition 3.6, if $n \geq 6$ and $5 \leq k < n$, where k is odd, then there exists no orientation of C_n with hull number k. Therefore, there are connected graphs G such that G has no orientation with hull number k for some integer k with $h^-(G) < k < h^+(G)$.

On the other hand, for every two integers n and m with $1 \leq n-1 \leq m \leq \binom{n}{2}$, there exists a connected graph G of order n and size m such that, for each integer k with $2 \leq k \leq n$, there exists an orientation of G with hull number k. In order to show this, we first present three lemmas. The *converse* D^* of an oriented graph D has the same vertex set as D and the arc (u, v) in D^* if and only if the arc (v, u) is in D. Since the reversal of the edge directions on any $u-v$ geodesic in D yields a $v-u$ geodesic in D^* and vice versa, we have the following lemma.

Lemma 3.7. *If D^* is the converse of an oriented graph D, then $h(D) = h(D^*)$.*

Lemma 3.8. *Let D be an oriented graph obtained from an oriented graph D' by adding a new vertex and joining it to all vertices of D'. Then $h(D) = h(D')+1$.*

Proof. Let v be the vertex of D that is not in D'. Since $\text{id}_D\, v = 0$, it follows that v belongs to every hull set of D by Lemma 1.2. Let S' be a minimum hull set of D'. Then $[S']_{D'} = V(D')$. Since $[S' \cup \{v\}] = V(D') \cup \{v\} = V(D)$, it follows that $S \cup \{v\}$ is a hull set of D and so $h(D) \le |S'| + 1 = h(D') + 1$. On the other hand, let $S = A \cup \{v\}$ be a minimum hull set of D, where $A \subseteq V(D')$. Since every geodesic of D is either a geodesic of D' or an arc (v, v') for some vertex v' in D', it follows that $V(D) = [S] = [A] \cup \{v\}$. Thus $[A] = V(D')$ and so A is a hull set of D'. So, $h(D') \le |A| = |S| - 1 = h(D) - 1$ or $h(D) \ge h(D') + 1$. Therefore, $h(D) = h(D') + 1$. □

Lemma 3.9. *Let D be an oriented graph obtained from an oriented graph D' by adding a new vertex v and joining it to a vertex v' of D'.*

(a) *If v' is a transmitter of D', then $h(D) = h(D')$.*

(b) *If v' is a receiver of D', then $h(D) = h(D') + 1$.*

Proof. First, assume that v' is a transmitter of D'. Let S' be a minimum hull set of D'. Then $v' \in S'$ and $[S']_{D'} = V(D')$. Since every geodesic of D is a geodesic in D' or a geodesic starting with v followed by v', it follows that $(S' - \{v'\}) \cup \{v\}$ is a hull set of D. Thus $h(D) \le |S'| = h(D')$. On the other hand, let S be a minimum hull set of D. Since v is a transmitter of D, it follows by Lemma 1.2 that $v \in S$. Thus $S = \{v\} \cup A$, where $A \subseteq V(D')$. Since every $v - x$ geodesic of D, where $x \in V(D')$, has v' as its second vertex, it follows that $A \cup \{v'\}$ is a hull set of D'. Thus $h(D') \le |A \cup \{v'\}| \le |A \cup \{v\}| = h(D)$. Therefore, $h(D) = h(D')$ and so (a) holds.

Next assume that v' is a receiver of D'. Then v' is also a receiver of D and so v' belongs to every hull set of D. Let S' be a minimum hull set of D'. Then $v' \in S'$. Since $S' \cup \{v\}$ is a hull set of D, it follows that $h(D) \le |S' \cup \{v\}| = h(D') + 1$. On the other hand, let S be a minimum hull set of D. Since v is a transmitter of D, it follows by Lemma 1.2 that $v \in S$. Thus $S = \{v\} \cup A$, where $A \subseteq V(D')$. Since v is on no geodesic joining two vertices of $V(D')$, it follows that A is a hull set of D'. Thus $h(D') \le |A| \le h(D) - 1$ or $h(D') + 1 \le h(D)$. Therefore, $h(D) = h(D') + 1$ and so (b) holds. □

We are now prepared to present the following result.

Theorem 3.10. *For every two integers n and m with $1 \le n - 1 \le m \le \binom{n}{2}$, there exists a connected graph G of order n and size m such that for each integer k with $2 \le k \le n$ there exists an orientation of G with hull number k.*

Proof. We prove the more general statement: for every two integers n and m with $1 \le n - 1 \le m \le \binom{n}{2}$, there exists a connected graph G of order n and size m having a vertex v such that (a) G contains a Hamiltonian path with initial vertex v and (b) for each $3 \le k \le n$, there exists an orientation of G having v as a transmitter and hull number k. By Lemma 3.1, (a) and (b) imply that for each $2 \le k \le n$, there exists an orientation of G having v as a transmitter and hull number k.

We proceed by induction on n. Since the statement is certainly true if $n \le 3$, we assume that $n \ge 4$. Suppose that the statement is true for $n-1$. We consider two cases.

Case 1 $\left(2n-3 \le m \le \binom{n}{2}\right)$. Then $(n-1)-1 \le m-(n-1) \le \binom{n-1}{2}$. By the induction hypothesis, there exists a connected graph G' of order $n-1$ and size $m-(n-1)$ having a vertex v' such that G' contains a Hamiltonian path starting at v' and for each $3 \le k \le n-1$, there exists an orientation of G' having v' as a transmitter and hull number k.

Let $G = G' + K_1$ with $V(K_1) = \{v\}$. Then G is a connected graph of order n and size m. Since G' contains a Hamiltonian path starting at v', it follows that G contains a Hamiltonian path starting at v. Thus G has an orientation with hull number 2 by Lemma 3.1. We now assume that $3 \le k \le n$. Then $2 \le k-1 \le n-1$. By the induction hypothesis, there exists an orientation D' of G' having v' as a transmitter and hull number $k-1$. We extend the orientation D' of G' to an orientation D of G by directing each edge incident with v in G away from v. Then v is a transmitter of D. Since $h(D') = k-1$, it then follows by Lemma 3.8 that $h(D) = k$.

Case 2 $(n-1 \le m \le 2n-4)$. Then $(n-1)-1 \le m-1 \le 2n-5$. By the induction hypothesis, there exists a connected graph G' of order $n-1$ and size $m-1$ having a vertex v' such that G' contains a Hamiltonian path starting at v' and for each $3 \le k \le n-1$, G' has an orientation having v' as a transmitter and hull number k. Let G be the graph obtained from G' by adding a new vertex v and the pendant edge vv'. Then G has order n and size m. Since G' contains a Hamiltonian path starting at v', it follows that G contains a Hamiltonian path starting at v. So, G has an orientation with hull number 2 by Lemma 3.1. For $3 \le k \le n$, we consider two subcases.

Subcase 2.1 $(3 \le k \le n-1)$. Let D' be an orientation of G' having v' as a transmitter and $h(D') = k$. We extend D' to an orientation of G by directing the edge vv' as (v, v'). Then v is a transmitter of D. By Lemma 3.9, $h(D) = h(D') = k$.

Subcase 2.2 $(k = n)$. Let D' be an orientation of G' having v' as a transmitter and $h(D') = n-1$ and let D^* be the converse of D'. Then v' is a receiver of D^* and $h(D^*) = h(D') = n-1$ by Lemma 3.7. We now extend D^* to an orientation D of G by directing the edge vv' as (v, v'). Then v is a transmitter of D. It then follows by Lemma 3.9 that $h(D) = h(D^*)+1 = n$. □

There is reason to believe that if G is *any* connected graph of order $n \ge 3$, then $h^-(G) \ne h^+(G)$. Consequently, we conclude this paper with the following conjecture.

Conjecture 3.11. *For every connected graph G of order at least 3, $h^-(G) \ne h^+(G)$.*

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Gary Chartrand: Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA

E-mail address: chartrand@wmich.edu

John Frederick Fink: Department of Mathematics, The University of Michigan-Dearborn, Dearborn, MI 48128-1491, USA

E-mail address: jffink@umich.edu

Ping Zhang: Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA

E-mail address: ping.zhang@wmich.edu