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PROPERTIES OF CERTAIN p-VALENTLY CONVEX FUNCTIONS

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A subclass $\mathcal{C}_p(\lambda,\mu)$ ($p \in \mathbb{N}$, $0 < \lambda < 1$, $-\lambda \leqq \mu < 1$) of p-valently convex functions in the open unit disk $\mathbb{U}$ is introduced. The object of the present paper is to discuss some interesting properties of functions belonging to the class $\mathcal{C}_p(\lambda,\mu)$.

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1. Introduction. Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in $\mathcal{A}_p$ is said to be p-valently convex of order α if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > p\alpha \quad (z \in \mathbb{U}) \tag{1.2}$$

for some α ($0 \leqq \alpha < 1$). We denote by $\mathcal{K}_p(\alpha)$ the subclass of $\mathcal{A}_p$ consisting of functions which are p-valently convex of order α in $\mathbb{U}$. In particular, we denote $\mathcal{K}_1(0) = \mathcal{K}$.

A function $f(z) \in \mathcal{A}_1$ is said to be uniformly convex in $\mathbb{U}$ if $f(z)$ is in the class $\mathcal{K}$ and has the property that the image arc $f(\gamma)$ is convex for every circular arc γ contained in $\mathbb{U}$ with center at $t \in \mathbb{U}$. We also denote by $\mathcal{UK}$ the subclass of $\mathcal{A}_1$ consisting of all uniformly convex functions in $\mathbb{U}$. Goodman [2] has introduced the class $\mathcal{UK}$ and given that $f(z) \in \mathcal{A}_1$ belongs to the class $\mathcal{UK}$ if and only if

$$\operatorname{Re}\left\{1 + (z-t)\frac{f''(z)}{f'(z)}\right\} \geqq 0 \quad ((z,t) \in \mathbb{U}\times\mathbb{U}). \tag{1.3}$$

Ma and Minda [3] and Rønning [5] have showed a more applicable characterization for $\mathcal{UK}$. We state this as the following theorem.

Theorem 1.1. *Let $f(z) \in \mathcal{A}_1$. Then $f(z) \in \mathcal{UK}$ if and only if*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U}). \tag{1.4}$$

In view of Theorem 1.1, Owa [4] considered a subclass $\mathcal{UK}(\mu)$ $(-1<\mu<1)$ of $\mathcal{A}_1$. A function $f(z)\in\mathcal{A}_1$ is said to be a member of the class $\mathcal{UK}(\mu)$ $(-1<\mu<1)$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}-\mu>\left|\frac{zf''(z)}{f'(z)}\right| \quad (z\in\mathbb{U}). \tag{1.5}$$

In this paper, we investigate the following subclass of $\mathcal{A}_p$.

Definition 1.2. A function $f(z)\in\mathcal{A}_p$ is said to be a member of the class $\mathcal{C}_p(\lambda,\mu)$ if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}-p\mu>\lambda\left|1+\frac{zf''(z)}{f'(z)}-p\right| \quad (z\in\mathbb{U}) \tag{1.6}$$

for some λ $(0<\lambda<1)$ and μ $(-\lambda\leqq\mu<1)$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written $f(z)\prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)|\leqq|z|$ and $f(z)=g(w(z))$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$.

2. Subordination properties.

Our first result for properties of functions $f(z)\in\mathcal{A}_p$ is contained in the following theorem.

Theorem 2.1. *A function $f(z)\in\mathcal{C}_p(\lambda,\mu)$ if and only if*

$$1+\frac{zf''(z)}{f'(z)}\prec h(z) \tag{2.1}$$

with

$$h(z)=p+\frac{p(1-\mu)}{2\sin^2\beta}\left\{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2\beta/\pi}+\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{2\beta/\pi}-2\right\} \quad (\beta=\arccos\lambda). \tag{2.2}$$

Proof. Let $1+zf''(z)/f'(z)=w$ and $w=u+iv$. Then inequality (1.6) can be written as

$$u-p\mu>\lambda\sqrt{(u-p)^2+v^2}. \tag{2.3}$$

By computing, we find that inequality (2.3) is equivalent to

$$\left(u+\frac{p(\lambda^2-\mu)}{1-\lambda^2}\right)^2-\frac{\lambda^2}{1-\lambda^2}v^2>\left(\frac{p\lambda(1-\mu)}{1-\lambda^2}\right)^2, \tag{2.4}$$

$$u>\frac{p(\lambda+\mu)}{1+\lambda}. \tag{2.5}$$

Thus the domain of the values of $1+zf''(z)/f'(z)$ for $z\in\mathbb{U}$ is contained in

$$\mathbb{D}=\{w=u+iv : u \text{ and } v \text{ satisfy (2.4) and (2.5)}\}. \tag{2.6}$$

In order to prove our theorem, it suffices to show that the function $h(z)$ given by (2.2) maps $\mathbb{U}$ conformally onto the domain $\mathbb{D}$.

Consider the transformations

$$w_1 = \frac{1-\lambda^2}{p(1-\mu)}w + \frac{\lambda^2-\mu}{1-\mu}, \qquad t = \frac{1}{2}\left(w_2^{\pi/\beta} + w_2^{-\pi/\beta}\right), \tag{2.7}$$

where $\beta = \arccos\lambda$ and $w_2 = w_1 + \sqrt{w_1^2 - 1}$ is the inverse function of

$$w_1 = \frac{w_2 + 1/w_2}{2}. \tag{2.8}$$

It is easy to verify that composite function $t = t(w)$ maps $\mathbb{D}^+$ defined by

$$\mathbb{D}^+ = \{w = u + iv : u \text{ and } v \text{ satisfy (2.4), (2.5), and } v > 0\} \tag{2.9}$$

conformally onto the upper-half plane $\operatorname{Im}(t) > 0$ so that $w = p$ corresponds to $t = 1$ and $w = p(\lambda+\mu)/(1+\lambda)$ to $t = -1$. With the help of the symmetry principle, this function $t = t(w)$ maps $\mathbb{D}$ conformally onto the domain

$$\mathbb{G} = \{t : |\arg(t+1)| < \pi\}. \tag{2.10}$$

Since

$$t = 2\left(\frac{1+z}{1-z}\right)^2 - 1 \tag{2.11}$$

maps $\mathbb{U}$ onto $\mathbb{G}$, we see that

$$\begin{aligned} w &= p + \frac{p(1-\mu)}{2(1-\lambda^2)}\left\{\left(t+\sqrt{t^2-1}\right)^{\beta/\pi} + \left(t+\sqrt{t^2-1}\right)^{-\beta/\pi} - 2\right\} \\ &= p + \frac{p(1-\mu)}{2\sin^2\beta}\left\{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2(\beta/\pi)} + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{2(\beta/\pi)} - 2\right\} \\ &= h(z) \end{aligned} \tag{2.12}$$

maps $\mathbb{U}$ onto $\mathbb{D}$ with $h(0) = p$. Hence the proof of the theorem is completed. □

Theorem 2.1 gives the following corollaries.

Corollary 2.2. *If $f(z) \in \mathcal{C}_p(\lambda,\mu)$, then $f(z) \in \mathcal{K}_p((\lambda+\mu)/(1+\lambda))$ and the order $(\lambda+\mu)/(1+\lambda)$ is sharp with the extremal function*

$$f_0(z) = p\int_0^z \left(t_2^{p-1}\exp\int_0^{t_2}\frac{h(t_1)-p}{t_1}dt_1\right)dt_2, \tag{2.13}$$

where $h(z)$ is given by (2.2).

Proof. Using (2.5) in the proof of Theorem 2.1 and noting that

$$\operatorname{Re}\left(1+\frac{zf_0''(z)}{f_0'(z)}\right)=\operatorname{Re}(h(z))\longrightarrow p\frac{\lambda+\mu}{1+\lambda} \tag{2.14}$$

as $z=\operatorname{Re}(z)\to -1$, we have the corollary. □

Corollary 2.3. *If $f(z)\in\mathbb{C}_p(\lambda,\mu)$ and $-\lambda<\mu<\lambda<1$, then*

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\arctan\left(\frac{1-\mu}{\sqrt{\lambda^2-\mu^2}}\right)\quad (z\in\mathbb{U}). \tag{2.15}$$

The bound in (2.15) is sharp with the extremal function $f_0(z)$ given by (2.13).

Proof. Let the function $h(z)$ be defined by (2.4). Then $h(\mathbb{U})=\mathbb{D}$ and an easy calculation yields that

$$\min\{\theta:|\arg(h(z))|<\theta(z\in\mathbb{U})\}=\arctan\left(\frac{1-\mu}{\sqrt{\lambda^2-\mu^2}}\right) \tag{2.16}$$

for $-\lambda<\mu<\lambda<1$. Therefore, the corollary follows immediately from Theorem 2.1. □

Next we derive the following theorem.

Theorem 2.4. *Let $f(z)\in\mathscr{C}_p(\lambda,\mu)$ and let $h(z)$ be defined by (2.2). Then*

$$\frac{f'(z)}{pz^{p-1}}\prec\exp\int_0^z\frac{h(t)-p}{t}dt, \tag{2.17}$$

$$\left|\frac{f'(z)}{pz^{p-1}}\right|\leqq\exp\int_0^1\frac{h(\rho)-p}{\rho}d\rho\quad (z\in\mathbb{U}). \tag{2.18}$$

The bound in (2.18) is sharp with the extremal function $f_0(z)$ given by (2.13).

Proof. Since the function $h(z)-p$ is univalent and starlike (with respect to the origin), by Theorem 2.1 and the result due to Suffridge [6, Theorem 3], we have

$$\log\left(\frac{f'(z)}{pz^{p-1}}\right)=\int_0^z\left(\frac{f''(t)}{f'(t)}-\frac{p-1}{t}\right)dt\prec\int_0^z\frac{h(t)-p}{t}dt, \tag{2.19}$$

which implies the subordination (2.17).

Furthermore, noting that the univalent function $h(z)$ maps the disk $|z|<\rho$ $(0<\rho\leqq 1)$ onto the domain which is convex and symmetric with respect to the real axis, we deduce that

$$\operatorname{Re}\int_0^z\frac{h(t)-p}{t}dt=\int_0^1\frac{\operatorname{Re}\{h(\rho z)-p\}}{\rho}d\rho\leqq\int_0^1\frac{h(\rho)-p}{\rho}d\rho \tag{2.20}$$

for $z\in\mathbb{U}$. Thus inequality (2.18) follows from (2.19) and (2.20). □

Remark 2.5. If we let $\beta = \pi/4$ and $x = ((1+\sqrt{\rho})/(1-\sqrt{\rho}))^{1/2}$ $(0 \leqq \rho < 1)$, then

$$\begin{aligned}\int_0^1 \left\{\left(\frac{1+\sqrt{\rho}}{1-\sqrt{\rho}}\right)^{2(\beta/\pi)} + \left(\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right)^{2(\beta/\pi)} - 2\right\}\frac{d\rho}{\rho} \\ = 8\int_1^{+\infty}\left(\frac{x}{x^2+1} - \frac{1}{x+1}\right)dx = 4\log 2.\end{aligned} \tag{2.21}$$

Thus, as the special case of Theorem 2.4, we have that if $f(z) \in \mathscr{C}_p(1/\sqrt{2},\mu)$ $(-1/\sqrt{2} \leqq \mu < 1)$, then

$$\left|\frac{f'(z)}{pz^{p-1}}\right| \leqq 16^{p(1-\mu)} \quad (z \in \mathbb{U}) \tag{2.22}$$

and the result is sharp.

3. Coefficient inequalities

Theorem 3.1. *If*

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \tag{3.1}$$

belongs to $\mathscr{C}_p(\lambda,\mu)$, *then*

$$|a_{p+1}| \leqq \frac{8p^2(1-\mu)}{p+1}\left(\frac{\beta}{\pi\sin\beta}\right)^2 \quad (\beta = \arccos\lambda). \tag{3.2}$$

Proof. It can be easily verified that

$$\begin{aligned}1+\frac{zf''(z)}{f'(z)} &= p + \left(1+\frac{1}{p}\right)a_{p+1}z + \cdots, \\ h(z) &= p + \frac{p(1-\mu)}{2\sin^2\beta}\left(\frac{8\beta}{\pi} + \frac{8\beta}{\pi}\left(\frac{2\beta}{\pi}-1\right)\right)z + \cdots \\ &= p + 8p(1-\mu)\left(\frac{\beta}{\pi\sin\beta}\right)^2 z + \cdots,\end{aligned} \tag{3.3}$$

where $h(z)$ is given by (2.2). Since

$$f(z) = z^p + a_{p+1}z^{p+1} + \cdots \in \mathscr{C}_p(\lambda,\mu), \tag{3.4}$$

it follows from (3.3) and Theorem 2.1 that

$$\begin{aligned}\frac{\pi^2}{8p(1-\mu)}\left(\frac{\sin\beta}{\beta}\right)^2\left(1+\frac{zf''(z)}{f'(z)} - p\right) \\ = \frac{p+1}{8p^2(1-\mu)}\left(\frac{\pi\sin\beta}{\beta}\right)^2 a_{p+1}z + \cdots \\ \prec \frac{\pi^2}{8p(1-\mu)}\left(\frac{\sin\beta}{\beta}\right)^2 (h(z)-p).\end{aligned} \tag{3.5}$$

It is well known that if

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \prec g(z) \tag{3.6}$$

for $g(z) \in \mathcal{K}$, then (cf. Duren [1])

$$|a_n| \leqq 1 \quad (n = 1, 2, 3, \ldots). \tag{3.7}$$

Noting that

$$\frac{\pi^2}{8p(1-\mu)} \left(\frac{\sin\beta}{\beta}\right)^2 (h(z) - p) \in \mathcal{K}, \tag{3.8}$$

we get (3.2). Also the bound in (3.2) is sharp for the function $f_0(z)$ given by (2.13). □

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