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PROFINITE AND FINITE GROUPS ASSOCIATED WITH LOOP AND DIFFEOMORPHISM GROUPS OF NON-ARCHIMEDEAN MANIFOLDS

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We investigate p-adic completions of clopen (i.e., closed and open at the same time) subgroups W of loop groups and diffeomorphism groups G of compact manifolds over non-Archimedean fields. We outline two different compactifications of loop groups and one compactification of diffeomorphism groups, describe associated finite groups in projective limits, and discuss relations with the representation theory.

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1. Introduction. The importance of such groups in the non-Archimedean functional analysis, representation theory, and mathematical physics is clear (see [1, 8, 10, 11, 14, 18, 19]). This paper is devoted to one aspect of such groups: their structure from the point of view of the p-adic compactification (see also about Banaschewski compactification in [18]). The p-adic compactifications are constructed below such that they are also groups. This also opens new possibilities for studying their representations as restrictions of representations of p-adic compactifications.

First, we recall basic facts and notation, which are given in detail in [10, 11, 13, 17, 18]. For a diffeomorphism group $\mathrm{Diff}(M)$ of a Banach manifold over a local field $\mathbf{K}$, there are clopen (i.e., closed and open at the same time) subgroups W such that they contain a sequence of profinite subgroups G_n with $G_n \subset G_{n+1}$ for each $n \in \mathbb{N}$ and $\bigcup_n G_n$ is dense in W, where $\mathbb{N}$ is the set of natural numbers. A loop group $L_t(M,N)$ is defined as a quotient space of a family of mappings $f : M \to N$ of class C^t of one Banach manifold M into another N over the same local field $\mathbf{K}$ such that $\lim_{z\to s}(\bar{\Phi}^v f)(z;h_1,\dots,h_n;\zeta_1,\dots,\zeta_n) = 0$ for each $0 \le v \le t$, where M and N are embedded into the corresponding Banach spaces X and Y, $\mathrm{cl}(M) = M \cup \{s\}$, $\mathrm{cl}(M)$ and N are clopen in X and Y, respectively, $0 \in N$, $(\bar{\Phi}^v f)(z;h_1,\dots,h_n;\zeta_1,\dots,\zeta_n)$ are continuous extensions of difference quotients, $z \in M$, $h_1,\dots,h_n$ are nonzero vectors in X, $\zeta_1,\dots,\zeta_n \in \mathbf{K}$ such that $z+\zeta_1 h_1+\cdots+\zeta_n h_n \in M$, $n = [v]+\mathrm{sign}\{v\}$. As usual, $[v]$ denotes the integral part of v such that $[v] \le v$ and $\{v\} := v-[v]$ denotes the fractional part of v.

The p-adic completions of clopen subgroups W of loop groups G and diffeomorphism groups G are considered. In the case of the diffeomorphism group, the p-adic completion produces weakened topology on W relatively to which it remains a topological group. In the case of the loop group, the p-adic completion produces a new topological group V in which the initial group W is embedded as a dense subgroup such that $V \neq W$. The topology on W inherited from V is weaker than the initial one. For the compact manifold M in the case of the diffeomorphism group, the p-adic completion of W produces the profinite group. For the locally compact manifolds M and N in the case of the loop group $L_t(M,N)$, the p-adic completion of W produces its embedding into $\mathbf{Q}_p^{\mathbb{N}}$, where $\mathbf{Q}_p$ denotes the field of p-adic numbers. When W is bounded relatively to the corresponding metric in $L_t(M,N)$, then W is embedded into $\mathbb{Z}_p^{\mathbb{N}}$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. The group $\mathrm{Diff}(M)$ is perfect and simple, on the other hand, the group $L_t(M,N)$ is commutative. The notation given below and the corresponding definitions are given in detail in [10, 13].

2. p-adic completion of diffeomorphism groups

2.1. Notation and remarks. Let N be a compact manifold over a local field $\mathbf{K}$, that is, $\mathbf{K}$ is a finite algebraic extension of the field of p-adic numbers $\mathbf{Q}_p$ [20]. Let also N be embedded into $B(\mathbf{K}^{\xi},0,1)$ as a clopen subset [2, 9], where $\xi \in \mathbb{N}$, $B(X,y,r) := \{z : z \in X; d_X(y,z) \leq r\}$ denotes a clopen ball in a space X with an ultrametric d_X. The ball $B(\mathbf{K}^{\xi},0,1)$ has the ring structure with coordinatewise addition and multiplication, in particular, $B(\mathbf{Q}_p,0,1) = \mathbb{Z}_p$ is the ring of entire p-adic numbers. The ring $B(\mathbf{K}^{\xi},0,1)$ is homeomorphic with the projective limit $B(\mathbf{K}^{\xi},0,1) = \text{pr-lim}_k \mathbf{S}^{\xi}_{|\pi|^{-k}}$, where $\mathbf{S}_{|\pi|^{-k}}$ is a finite ring consisting of $|\pi|^{-kc}$ elements such that $\mathbf{S}_{|\pi|^{-k}}$ is equal to the quotient ring $B(\mathbf{K},0,1)/B(\mathbf{K},0,|\pi|^k)$, $\mathbf{S}^{\xi}_{|\pi|^{-k}}$ is a product of ξ copies of $\mathbf{S}_{|\pi|^{-k}}$, c is a dimension $\dim_{\mathbf{Q}_p}\mathbf{K}$ of $\mathbf{K}$ over $\mathbf{Q}_p$ as a $\mathbf{Q}_p$-linear space, π is an element of $\mathbf{K}$ such that $p^{-1} \leq |\pi| < 1$ and $|\pi|$ is the generator of the valuation group of $\mathbf{K}$ (see also about local fields in [20]). In particular, $B(\mathbf{Q}_p,0,1)/B(\mathbf{Q}_p,0,p^{-k}) = \mathbb{Z}_p/p^k\mathbb{Z}_p = \mathbf{F}_{p^k}$ is a finite ring consisting of p^k elements, $aB := \{x : x = ab, b \in B\}$ for a multiplicative group E and its element $a \in E$ and a subset $B \subset E$, $k \in \mathbb{N}$ [18, 20]. For each $m \geq k$ there are the following quotient mappings (ring homomorphisms): $\tilde{\pi}_m : B(\mathbf{K},0,1) \to \mathbf{S}_{|\pi|^{-m}}$ and $\tilde{\pi}^m_k : \mathbf{S}_{|\pi|^{-m}} \to \mathbf{S}_{|\pi|^{-k}}$. This induces the quotient mappings $\tilde{\pi}_m : N \to N_m$ and $\tilde{\pi}^m_k : N_m \to N_k$, where $N_m \subset \mathbf{S}_{|\pi|^{-m}}$, $\tilde{\pi}^m_k \circ \tilde{\pi}_m = \tilde{\pi}_k$, $\tilde{\pi}_m(B(\mathbf{K}^{\xi},0,1)) = \mathbf{S}^{\xi}_{|\pi|^{-m}}$ for each $\xi \in \mathbb{N}$.

Let now M and N be two analytic compact manifolds embedded into $B(\mathbf{K}^{\psi},0,1)$ and $B(\mathbf{K}^{\xi},0,1)$, respectively, as clopen subsets and $f \in C^t(M,N)$, where $C^t(M,N)$ denotes the space of functions $f : M \to N$ of class C^t, $t \geq 0$. For an integer t it is the space of t-times continuously differentiable functions in the sense of partial difference quotients (see [10, 13, 17]). Then $f = \text{pr-lim}_k f_k$, where $f_k := \tilde{\pi}_k \circ f$. We introduce the notation $C^t(M,N_k) := \tilde{\pi}_k \circ C^t(M,N) = \{f_k : f \in C^t(M,N)\}$, hence $C^t(M,N) = \text{pr-lim}_k C^t(M,N_k)$ algebraically without taking into account topologies (or the limit of the inverse sequence, see [5, Section 2.5] and [15, Sections 3.3, 12.202]. Each function $f \in C^t(M,N)$ has a

$C^t(B(\mathbf{K}^\psi,0,1),\mathbf{K}^\xi)$-extension by zero on $B(\mathbf{K}^\psi,0,1)$, hence it has the decomposition $f = \sum_{l,m} f^l_m \bar{Q}_m e_l$ in the Amice polynomial basis $\bar{Q}_m$, where e_l is the standard orthonormal basis in $\mathbf{K}^\xi$ such that $e_l = (0,\dots,0,1,0,\dots)$ with 1 in the lth place, $\mathbb{Z} \ni m_l \geq 0$ for each l, $m = (m_1,\dots,m_\xi)$, $f^l_m \in \mathbf{K}$ are expansion coefficients such that $\lim_{l+|m|\to\infty} |f^l_m|_{\mathbf{K}} J(t,m) = 0$, $\bar{Q}_m$ are polynomials on $B(\mathbf{K}^\psi,0,1)$ with values in $\mathbf{K}$, $J(t,m) := \|\bar{Q}_m\|_{C^t(B(\mathbf{K}^\psi,0,1),\mathbf{K})}$. The space $C^t(M,N)$ is supplied with the uniformity inherited from the Banach space $C^t(\mathbf{K}^\psi,\mathbf{K}^\xi)$.

Let M_ξ denote $\tilde{\pi}_\xi(M)$ and N_ξ denote $\tilde{\pi}_\xi(N)$. For two sets E and F, as usual E^F is the set of all mappings $f : F \to E$.

Lemma 2.1. *Each $f \in C^t(M,N)$ is a projective limit $f = \text{pr-lim}_k f_k$ of polynomials $f_k = \sum_{l,m} f^l_{m,k} \bar{Q}_{m,k} e_l$ on rings $\mathbf{S}^\psi_{|\pi|^{-k}}$ with values in $\mathbf{S}^\xi_{|\pi|^{-k}}$, where $f^l_{m,k} \in \mathbf{S}_{|\pi|^{-k}}$ and $\bar{Q}_{m,k}$ are polynomials on $\mathbf{S}^\psi_{|\pi|^{-k}}$ with values in $\mathbf{S}_{|\pi|^{-k}}$.*

Proof. In view of Section 2.1,

$$f_k = \tilde{\pi}_k \circ f, \qquad \tilde{\pi}_k \circ f(x) = \sum_{l,m} (\tilde{\pi}_k(f^l_m)) \times (\tilde{\pi}_k \bar{Q}_m(x)) e_l, \tag{2.1}$$

since $\tilde{\pi}_k$ is a ring homomorphism and $\tilde{\pi}_k(e_l) = e_l$. Then $\tilde{\pi}_k(ax^m) = a_k x^m(k)$ for each $a \in \mathbf{K}$ and $x \in B(\mathbf{K}^\psi,0,1)$, where $x^m := x_1^{m_1},\dots,x_\psi^{m_\psi}$, $x_1,\dots,x_\psi \in B(\mathbf{K},0,1)$; $m := (m_1,\dots,m_\psi)$, $\mathbb{Z} \ni m_l \geq 0$ for each $l = 1,\dots,\psi$, $x = (x_1,\dots,x_\psi)$, $x(k) := \tilde{\pi}_k(x)$, $a_k = \tilde{\pi}_k(a)$ with $a_k \in \mathbf{S}_{|\pi|^{-k}}$ and $x^m(k) = \tilde{\pi}_k(x^m)$ with $x(k) \in \mathbf{S}^\psi_{|\pi|^{-k}}$, consequently, $\tilde{\pi}_k(\bar{Q}_m(x)) = \bar{Q}_{m,k}(x(k))$. The series for f_k is finite since $\tilde{\pi}_k(a) = 0$ for each $a \in \mathbf{K}$ with $|a| < |\pi|^k$ and $\lim_{l+|m|\to\infty} |f^l_m|_{\mathbf{K}} J(t,m) = 0$. ☐

Corollary 2.2. *The uniform space $C^t(M,N_k)$ is isomorphic with the space $N_k^{M_k}$ of all mappings from M_k into N_k. Moreover, $(\mathbf{S}^\xi_{|\pi|^{-k}})^{(\mathbf{S}^\psi_{|\pi|^{-k}})}$ is a finite-dimensional module over the ring $\mathbf{S}_{|\pi|^{-k}}$.*

Proof. From the proof of Lemma 2.1, there is only a finite number of $\mathbf{S}_{|\pi|^{-k}}$-linearly independent polynomials $\bar{Q}_{m,k}(x(k))$ (i.e., in the module of the ring $\mathbf{S}_{|\pi|^{-k}}$), since the rings $\mathbf{S}^\psi_{|\pi|^{-k}}$ and $\mathbf{S}_{|\pi|^{-k}}$ are finite, also $z^a = z^b$ for each natural numbers a and b such that $a \equiv b \pmod{(p^k)}$ and each $z \in \mathbf{S}_{|\pi|^{-k}}$. The space $C^t(M,N_k)$ is discrete and isomorphic with $N_k^{M_k}$, since M_k and N_k are discrete. ☐

Corollary 2.3. *The quotient group $\tilde{\pi}_k \circ \mathrm{Diff}^t(M)$ is isomorphic with the symmetric group S_{ξ_k}, where ξ_k is the cardinality of M_k.*

Proof. If $h \in \mathrm{Diff}^t(M)$, then $h_k(M_k) = M_k$ since $h(M) = M$. In view of Corollary 2.2, $\tilde{\pi}_k \circ \mathrm{Diff}^t(M)$ is isomorphic with the following group $\mathrm{Hom}(M_k)$ of all homeomorphisms h_k of M_k, that is, bijective surjective mappings $h_k : M_k \to M_k$. Using an enumeration of elements of M_k, we get an isomorphism of $\mathrm{Hom}(M_k)$ with S_{ξ_k}. ☐

2.2. Let $C_w(M,N) := \text{pr-lim}_k N_k^{M_k}$ be a uniform space of continuous mappings $f : M \to N$ supplied with a uniformity inherited from products of uniform spaces $\prod_{k=1}^{\infty} N_k^{M_k}$ (see also [5, Section 8.2]). The uniform spaces $C^t(M,N)$ and $C_w(M,N)$ are subsets of $\mathbf{K}$-linear spaces $C^t(M,\mathbf{K}^\xi)$ and $C^0(M,\mathbf{K}^\xi)$, respectively. We consider algebraic structures of subsets of the latter $\mathbf{K}$-linear spaces as inherited from them.

Corollary 2.4. *The space $C^t(M,N)$ is not algebraically isomorphic with $C_w(M,N)$, when $t > 0$. The uniform space $C_w(M,N)$ is uniformly isomorphic with $C^0(M,N)$, when the latter space is supplied with a weak uniformity inherited from $C^0(M,\mathbf{K}^\xi)$. The space $C_w(M,N)$ is compact.*

Proof. In view of [5, Section 2.5], $C^0(M,N)$ and $C_w(M,N)$ coincide algebraically since the connecting mappings $\tilde{\pi}_n^m$ are uniformly continuous for each $m \geq n$. The space $C^0(M,\mathbf{K}^\xi)$ is $\mathbf{K}$-linear and its uniformity is completely defined by a neighbourhood base of zero. The set of all evaluation mappings in points of M produces the base of the topology of $C^0(M,\mathbf{K}^\xi)$. In its weak topology, the latter space is isomorphic with the product $\prod_{x\in M}\mathbf{K}^\xi = \mathbf{K}^{\text{card}(M)}$, since $\text{card}(M) = \text{card}(\mathbb{R}) = \Upsilon$, where $\text{card}(M)$ denotes the cardinality of M. Then $C^0(M,N)$ and $C_w(M,N)$ have embeddings into $B(\mathbf{K},0,1)^{\text{card}(M)}$ as closed bounded subspaces. The latter space is uniformly homeomorphic with pr-$\lim_k(\mathbf{S}_{|\pi|^{-k}})^{M_k}$, which is compact by the Tychonoff theorem [5, Theorem 3.2.4]. Since $C^0(M,N) \neq C^t(M,N)$ for $t > 0$, then $C_w(M,N)$ and $C^t(M,N)$ are different algebraically. □

2.3. Let $\text{Diff}_w(M) := \text{pr-lim}_k \text{Hom}(M_k)$ be supplied with the uniformity inherited from $C_w(M,M)$. The group $\text{Diff}_w(M)$ is called the p-adic compactification of $\text{Diff}^t(M)$. The following theorem shows that this terminology is justified.

Theorem 2.5. *The group $\text{Diff}_w(M)$ is a compact topological group and it is the compactification of $\text{Diff}^t(M)$ in the weak topology. If $t > 0$, then $\text{Diff}^t(M)$ does not coincide with $\text{Diff}_w(M)$.*

Proof. Since $\text{Diff}^t(M) \subset C^t(M,M)$, then $\text{Diff}^t(M)$ has the corresponding embedding into $C_w(M,M)$. Since $C_w(M,M)$ is compact and $\text{Hom}(M)$ is a closed subset in $C_w(M,M)$, then due to Corollary 2.4, $\text{Hom}(M) \cap C_w(M,M) = \text{Diff}_w(M)$ is compact. The space $C^t(M,M)$ is dense in $C^0(M,M)$, consequently, $\text{Diff}^t(M)$ is dense in $\text{Diff}_w(M)$. If $t > 0$, then $\text{Diff}^t(M) \neq \text{Hom}(M)$, hence the two groups $\text{Diff}^t(M)$ and $\text{Diff}_w(M)$ do not coincide algebraically. It remains to verify that $\text{Diff}_w(M)$ is the topological group in its weak topology. If $f,g \in C^t(M,N)$, then $\tilde{\pi}_k(\bar{Q}_m(g(x))) = \bar{Q}_{m,k}(g_k(x(k)))$, consequently,

$$\tilde{\pi}_k(f \circ g) = \sum_{l,m} \tilde{\pi}_k(f_m^l)\bar{Q}_{m,k}(g_k(x(k)))e_l \tag{2.2}$$

and inevitably $(f \circ g)_k = f_k \circ g_k$. On the other hand, $\tilde{\pi}_k(x) = x(k)$ hence

$\tilde{\pi}_k(\mathrm{id}(x)) = \mathrm{id}_k(x(k))$, where $\mathrm{id}(x) = x$ for each $x \in M$. Therefore, for $f = g^{-1}$ we have $(f \circ g)_k = f_k \circ g_k = \mathrm{id}_k$, hence $\tilde{\pi}_k(g^{-1}) = g_k^{-1}$. The associativity of the composition $(f_k \circ g_k) \circ h_k = f_k \circ (g_k \circ h_k)$ of all functions $f_k, g_k, h_k \in \mathrm{Hom}(M_k)$, together with other properties given above, means that $\mathrm{Diff}_w(M)$ is the algebraic group, since $f = \text{pr-lim}_k f_k$, $g = \text{pr-lim}_k g_k$, and $h = \text{pr-lim}_k h_k$ also satisfy the associativity axiom, each f has the inverse element $f^{-1}(f(x)) = \mathrm{id}$, and $e = \mathrm{id}$ is the unit element. By the definition of the weak topology in $\mathrm{Diff}_w(M)$, for each neighbourhood of $e = \mathrm{id}$ in $\mathrm{Diff}_w(M)$ there exist $k \in \mathbb{N}$ and a subset $W_k \subset \mathrm{Hom}(M_k)$ such that $e_k \in W_k$ and $e \in \tilde{\pi}_k^{-1}(W_k) \subset W$. But $\mathrm{Hom}(M_k)$ is discrete, hence there are neighbourhoods $V_k \subset \mathrm{Hom}(M_k)$ and $U_k \subset \mathrm{Hom}(M_k)$ of e_k such that $V_k U_k \subset W_k$, for example, $V_k = \{e_k\}$ and $U_k = \{e_k\}$, since $e_k \in W_k$, hence there are neighbourhoods $e \in V \subset \mathrm{Diff}_w(M)$ and $e \in U \subset \mathrm{Diff}_w(M)$ such that $VU \subset W$, where $V = \tilde{\pi}_k^{-1}(V)$, $U = \tilde{\pi}_k^{-1}(U)$, and $VU = \{h : h = f \circ g,\ f \in V,\ g \in U\}$. If W' is a neighbourhood of f^{-1}, then $V := W' f^{-1}$ is the neighbourhood of e and there exists $k \in \mathbb{N}$ such that $\tilde{\pi}_k^{-1}(e_k) =: U \subset V^{-1}$ since $e_k^{-1} = e_k$ and $\tilde{\pi}_k$ is the homomorphism. Therefore, $fU := W$ is the neighbourhood of f such that $W^{-1} \subset W'$, which demonstrates the continuity of the inversion operation $f \mapsto f^{-1}$. □

2.4. Notes. Each projection $\tilde{\pi}_k : B(C^t(M, \mathbf{K}^\xi), 0, 1) \to (\mathbf{S}^\xi_{|\pi|^{-k}})^{M_k}$ produces the quotient metric ρ_k in the $\mathbf{S}_{|\pi|^{-k}}$-module $(\mathbf{S}^\xi_{|\pi|^{-k}})^{M_k}$ such that

$$\rho_k(f_k, g_k) := \inf_{z, \tilde{\pi}_k(z)=0} \|f - g + z\|_{C^t(M, \mathbf{K}^\xi)}, \tag{2.3}$$

where $\mathbf{S}_{|\pi|^{-k}} := B(\mathbf{K}, 0, 1)/B(\mathbf{K}, 0, |\pi|^k)$ is the quotient ring and $\tilde{\pi}_k$ is induced by such quotient mapping from $B(\mathbf{K}, 0, 1)$ onto $\mathbf{S}_{|\pi|^{-k}}$. If $B(C^t(M, \mathbf{K}^\xi), 0, 1)$ embeds into $\prod_k \tilde{\pi}_k(B(C^t(M, \mathbf{K}^\xi), 0, 1))$ and supplies the latter space with the box topology given by the following norm $\|f - g\|' := \sup_k \rho_k(f_k, g_k)$, then it produces the uniformity in $B(C^t(M, \mathbf{K}^\xi), 0, 1)$ equivalent with the initial one.

Theorem 2.5 means that the p-adic completion $\mathrm{Diff}_w(M)$ is a profinite group. It is the projective limit of the finite groups $\mathrm{Hom}(M_k)$. If the compact manifold M is decomposed into the disjoint union $M = \bigcup_i B(\mathbf{K}^\Psi, x_i, r_i)$ of clopen balls, then orders of the latter groups are divisible by $(|\pi|^{-a})!$, where $a = \sum_i l_i$, $l_i = k - \max_l\{l : |\pi|^{-l} \le r_i\}$, $x_i \in B(\mathbf{K}^\Psi, 0, 1)$, $0 < r_i \le 1$, since $\mathrm{card}(M_k)$ is divisible by $|\pi|^{-a}$. Then the representations of symmetric groups known from the classical works of Littlewood and Weyl [7, 21] with the help of the projective limit decompositions produce finite-dimensional representations of the diffeomorphism groups.

3. p-adic completion of loop groups. At first, we recall shortly the main details of definitions from [13].

3.1. Definitions and notes. Let X be a Banach space over $\mathbf{K}$. Suppose that M is an analytic manifold modeled on X with an atlas $\mathrm{At}(M)$ consisting of disjoint clopen charts (U_j,ϕ_j), $j\in\Lambda_M$, $\Lambda_M\subset\mathbb{N}$. That is, U_j and $\phi_j(U_j)$ are clopen in M and X, respectively, $\phi_j : U_j\to\phi_j(U_j)$ are homeomorphisms, $\phi_j(U_j)$ are bounded in X.

Then $C^t(M,Y)$ for M with a finite atlas $\mathrm{At}(M)$, $\mathrm{card}(\Lambda_M)<\aleph_0$, denotes a Banach space of functions $f: M\to Y$ with an ultranorm

$$\|f\|_t=\sup_{j\in\Lambda_M}\big\|f|_{U_j}\big\|_{C^t(U_j,Y)}<\infty, \tag{3.1}$$

where Y is the Banach space over $\mathbf{K}$, $0\le t\in\mathbb{R}$, their restrictions $f|_{U_j}$ are in $C^t(U_j,Y)$ for each j.

By $C_0^t(M,Y)$ we denote a completion of a subspace of cylindrical functions restrictions of which on each chart $f|_{U_l}$ are finite $\mathbf{K}$-linear combinations of functions $\{\bar{Q}_{\bar m}(x_{\bar m})q_i|_{U_l}: i\in\beta, m\}$ relatively to the following norm:

$$\|f\|_{C_0^t(M,Y)}:=\sup_{i,m,l}\big|a(m,f^i|_{U_l})\big|J_l(t,m), \tag{3.2}$$

where multipliers $J_l(t,m)$ are defined as follows:

$$J_l(t,m):=\big\|\bar{Q}_{\bar m}|_{U_l}\big\|_{C^t(\phi_l(U_l)\cap\mathbf{K}^n,\mathbf{K})}, \tag{3.3}$$

$m=(m_i:i)$ with components $m_i\in\mathbb{N}_0$, nonzero components of m are $m_{i_1},\dots,$ m_{i_n} with $n\in\mathbb{N}$, $\bar m:=(m_{i_1},\dots,m_{i_n})$ for each $m\ne 0$, $x_{\bar m}:=(x^{i_1},\dots,x^{i_n})\in\mathbf{K}^n\hookrightarrow X$, $\bar{Q}_0:=1$.

Let N be an analytic manifold modeled on Y with an atlas:

$$\mathrm{At}(N)=\{(V_k,\psi_k):k\in\Lambda_N\}, \tag{3.4}$$

such that $\psi_k: V_k\to\psi_k(V_k)\subset Y$ are homeomorphisms, $\mathrm{card}(\Lambda_N)\le\aleph_0$, and $\theta: M\to N$ is a $C^{t'}$-mapping, also $\mathrm{card}(\Lambda_M)<\aleph_0$, where V_k are clopen in N, $t'\ge\max(1,t)$ is the index of a class of smoothness, that is, for each admissible (i,j)

$$\theta_{i,j}\in C_*^{t'}(U_{i,j},Y), \tag{3.5}$$

with $*$ either empty or taking the value 0, respectively,

$$\theta_{i,j}:=\psi_i\circ\theta|_{U_{i,j}}, \tag{3.6}$$

where $U_{i,j}:=[U_j\cap\theta^{-1}(V_i)]$ are nonvoid clopen subsets. We denote by $C_*^{\theta,\xi}(M,N)$, for $0\le\xi\le\infty$, a space of mappings $f: M\to N$ such that

$$f_{i,j}-\theta_{i,j}\in C_*^{\xi}(U_{i,j},Y). \tag{3.7}$$

In view of formulas (3.4), (3.5), (3.6), and (3.7), we supply it with an ultrametric

$$\rho_*^\xi(f,g) = \sup_{i,j} \|f_{i,j} - g_{i,j}\|_{C_*^\xi(U_j,Y)}, \tag{3.8}$$

for each $0 \le \xi < \infty$.

3.2. For infinite atlases we use the traditional procedure of inductive limits of spaces. For M with the infinite atlas, $\operatorname{card}(\Lambda_M) = \aleph_0$, and Y is the Banach space over $\mathbf{K}$; we denote by $C_*^{\theta,\xi}(M,Y)$, for $0 \le \xi \le \infty$, a locally $\mathbf{K}$-convex space, which is the strict inductive limit

$$C_*^{\theta,\xi}(M,Y) := \text{str-ind}\left\{C_*^{\theta,\xi}(U^E,Y), \pi_E^F, \Sigma\right\}, \tag{3.9}$$

where $E \in \Sigma$, Σ is the family of all finite subsets of Λ_M directed by the inclusion $E < F$ if $E \subset F$, $U^E := \bigcup_{j\in E} U_j$.

For mappings from one manifold into another $f : M \to N$ we therefore get the corresponding uniform spaces denoted by $C_*^{\theta,\xi}(M,N)$.

We introduce the notation

$$\begin{aligned} G(\xi,M) &:= C_0^{\theta,\xi}(M,M) \cap \operatorname{Hom}(M), \\ \operatorname{Diff}^\xi(M) &= C^{\theta,\xi}(M,M) \cap \operatorname{Hom}(M), \end{aligned} \tag{3.10}$$

which are called groups of diffeomorphisms (and homeomorphisms for $0 \le \xi < 1$), $\theta = \mathrm{id}$, $\mathrm{id}(x) = x$ for each $x \in M$, where $\operatorname{Hom}(M) := \{f : f \in C^0(M,M),\ f$ is bijective, $f(M) = M$, f and $f^{-1} \in C^0(M,M)\}$ denotes the usual homomorphism group.

3.3. Notes. Henceforth, ultrametrizable separable complete manifolds $\bar{M}$ and N are considered. Since a large inductive dimension $\operatorname{Ind}(\bar{M}) = 0$ (see [5, Theorem 7.3.3]), $\bar{M}$ does not have boundaries in the usual sense. Therefore,

$$\operatorname{At}(\bar{M}) = \{(\bar{U}_j, \bar{\phi}_j) : j \in \Lambda_{\bar{M}}\} \tag{3.11}$$

has a refinement $\operatorname{At}'(\bar{M})$, which is countable, and its charts $(\bar{U}'_j, \bar{\phi}'_j)$ are clopen, disjoint, and homeomorphic with the corresponding balls $B(X, y_j, \bar{r}'_j)$, where

$$\bar{\phi}'_j : \bar{U}'_j \longrightarrow B(X, y'_j, \bar{r}'_j) \quad \forall j \in \Lambda'_{\bar{M}} \tag{3.12}$$

are homeomorphisms (see [5, 9]). For $\bar{M}$ we fix such $\operatorname{At}'(\bar{M})$.

We define topologies of groups $G(\xi,\bar{M})$ and locally $\mathbf{K}$-convex spaces $C_*^\xi(\bar{M},Y)$ relatively to $\operatorname{At}'(\bar{M})$, where Y is the Banach space over $\mathbf{K}$. Therefore, we suppose also that $\bar{M}$ and N are clopen subsets of the Banach spaces X and Y, respectively. Up to the isomorphism of loop semigroups, we can suppose that $s_0 = 0 \in \bar{M}$ and $y_0 = 0 \in N$.

For $M = \bar{M}\setminus\{0\}$ let $\mathrm{At}(M)$ be consisted of charts (U_j,ϕ_j), $j\in\Lambda_M$, while $\mathrm{At}'(M)$ consists of charts (U'_j,ϕ'_j), $j\in\Lambda'_M$, where due to formulas (3.11) and (3.12) we define

$$\begin{gathered} U_1=\bar{U}_1\setminus\{0\},\quad \phi_1=\bar{\phi}_1\big|_{U_1},\qquad U_j=\bar{U}_j,\quad \phi_j=\bar{\phi}_j,\quad \forall j>1,\\ 0\in\bar{U}_1,\ \Lambda_M=\Lambda_{\bar{M}},\qquad U'_1=\bar{U}'_1\setminus\{0\},\qquad \phi'_1=\bar{\phi}'_1\big|_{U'_1},\\ U'_j=\bar{U}'_j,\quad \phi'_j=\bar{\phi}'_j,\quad \forall j>1,\ j\in\Lambda'_M=\Lambda'_{\bar{M}},\ \bar{U}'_1\ni 0. \end{gathered} \tag{3.13}$$

3.4. Definitions and notes. Let the spaces be the same as in Section 3.2 (see formulas (3.9) and (3.10)) with the atlas of M defined by conditions (3.13). Then we consider their subspaces of mappings preserving marked points:

$$\begin{aligned} &C_0^{\theta,\xi}((M,s_0),(N,y_0))\\ &\quad:=\Big\{f\in C_0^{\theta,\xi}(\bar{M},N):\lim_{|\zeta_1|+\cdots+|\zeta_k|\to 0}\bar{\Phi}^v(f-\theta)(s_0;h_1,\dots,h_k;\zeta_1,\dots,\zeta_k)=0\\ &\qquad\qquad\forall v\in\{0,1,\dots,[t],t\},\ k=[v]+\operatorname{sign}\{v\}\Big\}, \end{aligned} \tag{3.14}$$

for each $v\in\{[t]+n\gamma,t+n\gamma\}$, and the following subgroup:

$$G_0(\xi,M):=\{f\in G(\xi,\bar{M}):f(s_0)=s_0\} \tag{3.15}$$

of the diffeomorphism group.

With the help of them we define the following equivalence relations K_ξ: $fK_\xi g$ if and only if the following sequences exist:

$$\{\psi_n\in G_0(\xi,M):n\in\mathbb{N}\}, \tag{3.16}$$

$$\{f_n\in C_0^{\theta,\xi}(M,N):n\in\mathbb{N}\}, \tag{3.17}$$

$$\{g_n\in C_0^{\theta,\xi}(M,N):n\in\mathbb{N}\}, \tag{3.18}$$

such that

$$f_n(x)=g_n(\psi_n(x))\quad\forall x\in M,\qquad \lim_{n\to\infty}f_n=f,\qquad \lim_{n\to\infty}g_n=g. \tag{3.19}$$

Due to condition (3.19) these equivalence classes are closed, since $(g(\psi(x))'=g'(\psi(x))\psi'(x)$, $\psi(s_0)=s_0$, $g'(s_0)=0$ for $t+s\ge 1$. We denote them by $\langle f\rangle_{K,\xi}$. Then for $g\in\langle f\rangle_{K,\xi}$ we write $gK_\xi f$ also. We denote the quotient space $C_0^{\theta,\xi}((M,s_0),(N,y_0))/K_\xi$ by $\Omega_\xi(M,N)$, where $\theta(M)=\{y_0\}$.

3.5. Let as usually $A\vee B:=A\times\{b_0\}\cup\{a_0\}\times B\subset A\times B$ be the wedge product of pointed spaces (A,a_0) and (B,b_0), where A and B are topological spaces with marked points $a_0 \in A$ and $b_0 \in B$. Then the composition $g \circ f$ of two elements $f, g \in C_0^{\theta,\xi}((M,s_0),(N,y_0))$ is defined on the domain $\bar{M} \vee \bar{M} \setminus \{s_0 \times s_0\} =: M \vee M$.

Let $M = \bar{M} \setminus \{0\}$ be as in Section 3.3. We fix an infinite atlas $\tilde{A}t'(M) := \{(\tilde{U}'_j, \phi'_j) : j \in \mathbb{N}\}$ such that $\phi'_j : \tilde{U}'_j \to B(X, y'_j, r'_j)$ are homeomorphisms,

$$\lim_{k\to\infty} r'_{j(k)} = 0, \qquad \lim_{k\to\infty} y'_{j(k)} = 0, \tag{3.20}$$

for an infinite sequence $\{j(k) \in \mathbb{N} : k \in \mathbb{N}\}$ such that $\mathrm{cl}_{\bar{M}}[\bigcup_{k=1}^{\infty} \tilde{U}'_{j(k)}]$ is a clopen neighbourhood of 0 in $\bar{M}$, where $\mathrm{cl}_{\bar{M}} A$ denotes the closure of a subset A in $\bar{M}$. In $M \vee M$ we choose the following atlas $\tilde{A}t'(M \vee M) = \{(W_l, \xi_l) : l \in \mathbb{N}\}$ such that $\xi_l : W_l \to B(X, z_l, a_l)$ are homeomorphisms,

$$\lim_{k\to\infty} a_{l(k)} = 0, \qquad \lim_{k\to\infty} z_{l(k)} = 0, \tag{3.21}$$

for an infinite sequence $\{l(k) \in \mathbb{N} : k \in \mathbb{N}\}$ such that $\mathrm{cl}_{\bar{M}\vee\bar{M}}[\bigcup_{k=1}^{\infty} W_{l(k)}]$ is a clopen neighbourhood of 0×0 in $\bar{M} \vee \bar{M}$ and

$$\mathrm{card}\,(\mathbb{N} \setminus \{l(k) : k \in \mathbb{N}\}) = \mathrm{card}\,(\mathbb{N} \setminus \{j(k) : k \in \mathbb{N}\}). \tag{3.22}$$

Then we fix a $C(\infty)$-diffeomorphism $\chi : M \vee M \to M$ such that

$$\begin{gathered} \chi(W_{l(k)}) = \tilde{U}'_{j(k)} \quad \forall k \in \mathbb{N}, \\ \chi(W_l) = \tilde{U}'_{\kappa(l)} \quad \forall l \in (\mathbb{N} \setminus \{l(k) : k \in \mathbb{N}\}), \end{gathered} \tag{3.23}$$

where

$$\kappa : (\mathbb{N} \setminus \{l(k) : k \in \mathbb{N}\}) \longrightarrow (\mathbb{N} \setminus \{j(k) : k \in \mathbb{N}\}) \tag{3.24}$$

is a bijective mapping for which

$$|\pi| \le \frac{a_{l(k)}}{r'_{j(k)}} \le |\pi|^{-1}, \qquad |\pi| \le \frac{a_l}{r'_{\kappa(l)}} \le |\pi|^{-1}. \tag{3.25}$$

This induces the continuous injective homomorphism

$$\chi^* : C_0^{\theta,\xi}((M \vee M, s_0 \times s_0), (N, y_0)) \longrightarrow C_0^{\theta,\xi}((M, s_0), (N, y_0)) \tag{3.26}$$

such that

$$\chi^*(g \vee f)(x) = (g \vee f)(\chi^{-1}(x)) \quad \forall x \in M, \tag{3.27}$$

where $(g \vee f)(y) = f(y)$ for $y \in M_2$ and $(g \vee f)(y) = g(y)$ for $y \in M_1$, $M_1 \vee M_2 = M \vee M$, $M_i = M$ for $i = 1, 2$. Therefore,

$$g \circ f := \chi^*(g \vee f) \tag{3.28}$$

may be considered as defined on M also, that is, to $g\circ f$ there corresponds the unique element in $C_0^{\theta,\xi}((M,s_0),(N,y_0))$.

3.6. The composition in $\Omega_\xi(M,N)$ is defined due to the following inclusion $g\circ f\in C_0^{\theta,\xi}((M,s_0),(N,y_0))$ (see formulas (3.23), (3.24), (3.25), (3.26), (3.27), and (3.28)) and then using the equivalence relations K_ξ (see condition (3.19)).

It is shown below that $\Omega_\xi(M,N)$ is the monoid, which we call the loop monoid.

3.7. Note and definition. For a commutative monoid $\Omega_\xi(M,N)$ with the unity and the cancellation property there exists a commutative group $L_\xi(M,N)$ equal to the Grothendieck group. This group is the quotient group $F/\mathcal{B}$, where F is a free abelian group generated by $\Omega_\xi(M,N)$ and $\mathcal{B}$ is a closed subgroup of F generated by elements $[f+g]-[f]-[g]$, f and $g\in\Omega_\xi(M,N)$, $[f]$ denotes an element of F corresponding to f. The natural mapping

$$\gamma:\Omega_\xi(M,N)\longrightarrow L_\xi(M,N) \tag{3.29}$$

is injective. We supply F with a topology inherited from the Tychonoff product topology of $\Omega_\xi(M,N)^{\mathbb{Z}}$, where each element z of F is

$$z=\sum_f n_{f,z}[f], \tag{3.30}$$

$n_{f,z}\in\mathbb{Z}$ for each $f\in\Omega_\xi(M,N)$,

$$\sum_f |n_{f,z}|<\infty. \tag{3.31}$$

In particular $[nf]-n[f]\in\mathcal{B}$, where $1f=f$, $nf=f\circ(n-1)f$ for each $1<n\in\mathbb{N}$, $f+g:=f\circ g$. We call $L_\xi(M,N)$ the loop group.

3.8. Let, as in Sections 2.1 and 3.3, $\bar{M}$ and N be two compact manifolds.

Theorem 3.1. *Let $\Omega_\xi(M,N)$ be the commutative loop monoids, then the quotient mappings $\tilde{\pi}_k$ induce the corresponding inverse sequence $\{\Omega(M_k,N_k):k\in\mathbb{N}\}$ such that $\Omega^w(M,N):=\text{pr-lim}_k\,\Omega(M_k,N_k)$ is a commutative compact topological monoid, where $\tilde{\pi}_k:\Omega_\xi(M,N)\to\Omega(M_k,N_k)$, $\tilde{\pi}_k^l:\Omega(M_l,N_l)\to\Omega(M_k,N_k)$ are surjective mappings for each $l\geq k$, $\Omega(M_k,N_k)=\{f_k:f_k\in N_k^{M_k},f_k(s_{0,k})=y_{0,k}\}/K_{\xi,k}$, $K_{\xi,k}$ is an equivalence relation induced by an equivalence relation K_ξ. Moreover, $\Omega^w(M,N)$ is a compactification of $\Omega_\xi(M,N)$.*

Proof. In view of Corollary 2.2, $\tilde{\pi}_k(C_0^\xi(M,N))$ is isomorphic with $\{f_k:f_k\in N_k^{M_k},f_k(s_{0,k})=y_{0,k}\}$, where the quotient mapping is denoted by $\tilde{\pi}_k$ for both M and N, since it is induced by the same ring homomorphism $\tilde{\pi}_k:B(\mathbf{K},0,1)\to B(\mathbf{K},0,1)/B(\mathbf{K},0,|\pi|^k)$, $s_{0,k}:=\tilde{\pi}_k(s_0)$ and $y_{0,k}:=\tilde{\pi}_k(y_0)$. Then

$\tilde{\pi}_k(G_0(t,M))$ is isomorphic with $\mathrm{Hom}_0(M_k) := \{\psi_k : \psi_k \in \mathrm{Hom}(M_k), \psi_k(s_{0,k}) = s_{0,k}\}$ (see Section 3.4). All of this is also applicable with the corresponding changes to classes of smoothness C^ξ (or $C(\xi)$ in the notation of [13], where $\xi = (t,s)$). If f and g are two K_ξ-equivalent elements in $C_0^\xi(M,N)$, that is, there are sequences f_n and g_n in $C_0^\xi(M,N)$ converging to f and g, respectively, and also a sequence $\psi_n \in \mathrm{Diff}_0^\xi(M)$ such that $f_n(x) = g_n(\psi_n(x))$ for each $x \in M$, then $\tilde{\pi}_k(f_n) =: f_{n,k}$ and $g_{n,k} := \tilde{\pi}_k(g_n)$ converge to $\tilde{\pi}_k(f)$ and $\tilde{\pi}_k(g)$, respectively, and also $\psi_{n,k} := \tilde{\pi}_k(\psi_n) \in \mathrm{Hom}_0(M_k)$. From the equality $f_{n,k}(x(k)) = g_{n,k}(\psi_{n,k}(x(k)))$ for each $n \in \mathbb{N}$ and $x(k) \in M_k$, it follows that the equivalence relation K_ξ induces the corresponding equivalence relation $K_{\xi,k}$ in $\tilde{\pi}_k(C_0^t(M,N))$ such that the classes $\langle \tilde{\pi}_k(f) \rangle_{K,\xi,k}$ of $K_{\xi,k}$-equivalent elements are closed. Each element $f_k \in \tilde{\pi}_k(C_0^\xi(M,N))$ is characterized by the equality $f_k(s_{0,k}) = y_{0,k}$. Each $\Omega(M_k,N_k)$ is the finite discrete set, since each $N_k^{M_k}$ is the finite discrete set. This induces the quotient mapping $\tilde{\pi}_k : \Omega_t(M,N) \to \Omega(M_k,N_k)$ and surjective mappings $\tilde{\pi}_k^l : \Omega(M_l,N_l) \to \Omega(M_k,N_k)$ for each $l \geq k$. It produces the inverse sequence of finite discrete spaces, hence the limit of the inverse sequence is compact and totally disconnected. It remains to verify that $\Omega^w(M,N)$ is a commutative topological monoid with unit element and the cancellation property.

From the equality $M = \bar{M} \setminus \{s_0\}$, it follows that $M_k = \bar{M}_k$, since for each $k \in \mathbb{N}$ there exists $x \in M$ such that $x + B(\mathbf{K}^\psi, 0, |\pi|^k) \ni s_0$. Moreover, M_k and N_k are finite discrete spaces. Then $\tilde{\pi}_k(M \vee M) = M_k \vee M_k$ (see Section 3.5). The composition operation is defined on threads $\{\langle f_k \rangle_{K,\xi,k} : k \in \mathbb{N}\}$ of the inverse sequence in the following way. There is a fixed C^∞-diffeomorphism $\chi : M \vee M \to M$. Let $x \in M$, then $\tilde{\pi}_k(x) \in M_k$ and $\chi^{-1}(U) \in M \vee M$, where $U := \tilde{\pi}_k^{-1}(x + B(\mathbf{K}, 0, |\pi|^k) \cap M$. On the other hand, $\chi^{-1}(U)$ is a disjoint union of balls of radius $|\pi|^{2k}$ in $B(\mathbf{K}^{2m}, 0, 1)$, hence there is defined a surjective mapping $\chi_k : M_{2k} \vee M_{2k} \to M_k$ induced by χ, $\tilde{\pi}_k$, and $\tilde{\pi}_{2k}$ such that $\chi_k(\chi^{-1}(U)) = \tilde{\pi}_k(x)$. If f and $g \in C^\xi(M,N)$, then $f \vee g \in C^\xi((M \vee M), N)$ and $\chi(f \vee g) \in C^\xi(M,N)$ as in [13, Section 2.6]. Hence $\chi_k(f_{2k} \vee g_{2k}) \in C^\xi(M_k,N_k)$ and inevitably $\chi_k(\langle f_{2k} \vee g_{2k} \rangle_{K,\xi,2k}) = \chi_k(\langle f_{2k} \rangle_{K,\xi,2k} \vee \langle g_{2k} \rangle_{K,\xi,2k}) \in \Omega(M_k,N_k)$.

There exists one-to-one correspondence between the elements $f \in C_w(\bar{M},N)$ and $\{f_k : k\} \in \{N_k^{M_k} : k\}$. Therefore, $\text{pr-lim}_k \Omega(M_k,N_k)$ algebraically is the commutative monoid with the cancellation property. Let U be a neighbourhood of e in $\Omega^w(M,N)$, then there exists $U_k = \tilde{\pi}_k^{-1}(V_k)$ such that V_k is open in $\Omega(M_k,N_k)$, $e \in U_k$, and $U_k \subset U$. On the other hand, there exists $U_{2k} = \tilde{\pi}_{2k}^{-1}(V_{2k})$ such that V_{2k} is open in $\Omega(M_{2k},N_{2k})$, $e \in U_{2k}$, and $U_{2k} + U_{2k} \subset U_k$. Therefore, $(f + U_{2k}) + (g + U_{2k}) \subset f + g + U_k \subset f + g + U$ for each $f, g \in \Omega^w(M,N)$, consequently, the composition in $\Omega^w(M,N)$ is continuous. Since $C_0^\xi(M,N)$ is dense in $C_{0,w}(\bar{M},N)$, then $\Omega_\xi(M,N)$ is dense in $\Omega^w(M,N)$. □

3.9. Note. The compactification of $\Omega_\xi(M,N)$ given above is not unique. Another compactification is given below. The second is larger than the first one.

Using the Grothendieck construction, we get a compactification $L^w(M,N) = \bar{F}/\bar{B}$ of a loop group $L_\xi(M,N)$, where $\bar{F}$ is a closure in $(\Omega^w(M,N))^{\mathbb{Z}}$ of a free commutative group F generated by $\Omega^w(M,N)$ and $\bar{B}$ is a closure of a subgroup B generated by all elements $[a+b]-[a]-[b]$, since the product of compact spaces is compact by the Tychonoff theorem [5].

3.10. Let now $s_0 = 0$ and $y_0 = 0$ be two marked points in the compact manifolds $\bar{M}$ and N embedded into $\mathbf{K}^\psi$ and $\mathbf{K}^\xi$, respectively. Define the following C^∞-diffeomorphism $\mathrm{inv} : (\mathbf{K}^\psi)' \to (\mathbf{K}^\psi)'$ for $(\mathbf{K}^\psi)' := \mathbf{K}^\psi \setminus \{x : \text{there exists } j \text{ with } x_j = 0\}$ such that $\mathrm{inv}(x_1,\dots,x_\psi) := (x_1^{-1},\dots,x_\psi^{-1})$, where $x_j \in \mathbf{K}$, $j = 1,\dots,\psi$. Let $M' = M \cap (\mathbf{K}^\psi)'$, where $M = \bar{M}\setminus\{s_0\}$ as in Section 3.3. Then $\mathrm{inv}(M')$ is locally compact, noncompact, and unbounded in $\mathbf{K}^\psi$, since M' is locally compact and noncompact. Let $\mathbf{K}^* = \mathbf{K}\setminus\{0\}$ then evidently $(\mathbf{K}^\psi)'$ is equal to $(\mathbf{K}^*)^\psi$. Let the disjoint union of $\bar{x}_j + \mathbf{S}^\psi_{|\pi|^{-\mathbf{k}}}$ be chosen equal to $\tilde{\pi}_k((\mathbf{K}^\psi)') := (\mathbf{K}^\psi)'_k$ for each $k\in\mathbb{N}$, where $\{B(\mathbf{K}^\psi, x_j, 1) : j\}$ is the disjoint covering of $(\mathbf{K}^\psi)'$ and $\bar{x}_j = x_j + B(\mathbf{K}, 0, |\pi|^{-k}) = \tilde{\pi}_k(x_j)$. Therefore, $\tilde{\pi}_k(\mathrm{inv}(M')) = (\mathrm{inv}(M'))_k$ is a discrete infinite subset in $\tilde{\pi}_k((\mathbf{K}^\psi)')$ for each $k\in\mathbb{N}$. Analogously, $\tilde{\pi}_k(\mathrm{inv}(M'\vee M')) = (\mathrm{inv}(M'\vee M'))_k \subset [\tilde{\pi}_k((\mathbf{K}^\psi)')]^2$. There exists a C^∞-diffeomorphism $\chi : M\vee M \to M$ such that $\mathrm{inv}\circ\chi\circ\mathrm{inv}$ is the C^∞-diffeomorphism of $\mathrm{inv}(M'\vee M')$ with $\mathrm{inv}(M')$ and it induces bijective mappings χ_k of $\mathrm{inv}((\mathrm{inv}(M'\vee M'))_k)$ with $\mathrm{inv}((\mathrm{inv}(M'))_k)$ for each $k\in\mathbb{N}$ such that $\hat{\pi}^l_k\circ\chi_l = \chi_k$ for each $l \ge k$, where $\hat{\pi}^l_k := \mathrm{inv}\circ\tilde{\pi}^l_k\circ\mathrm{inv}$. This produces inverse sequences of discrete spaces $\mathrm{inv}((\mathrm{inv}(M'))_k) =: \hat{M}_k$, $\mathrm{inv}((\mathrm{inv}(M'\vee M'))_k) = \hat{M}_k\vee\hat{M}_k$ and their bijections χ_k such that $\text{pr-lim}_k \hat{M}_k$ is homeomorphic with M' and $\text{pr-lim}_k \chi_k$ is equal to χ up to the homomorphism, since $\text{pr-lim}_k \mathbf{S}^\psi_{|\pi|^{-\mathbf{k}}} = B(\mathbf{K}^\psi, 0, 1)$ (see also about admissible modifications and polyhedral expansions in [12]). If $\psi \in G_0(\xi,\bar{M})$, then $\hat{\psi}\in \mathrm{Diff}^\xi(\hat{M})$. Let $J_{f,k} := \{h_k : h_k = f_k\circ\psi_k,\ \psi_k \in \mathrm{Hom}(\hat{M}_k),\ \psi_k(s_{0,k}) = s_{0,k}\}$ for $f_k \in N_k^{\hat{M}_k}$ with $\lim_{x\to 0} f_k(x) = 0$, then $J_{f,k}$ is closed and $\hat{\pi}_k(\langle f\rangle_{K,\xi}) \subset J_{f,k}$. Therefore, g_k and f_k are $\hat{K}_{\xi,k}$-equivalent if and only if there exists $\psi_k \in \mathrm{Hom}(\hat{M}_k)$ such that $\psi_k(s_{0,k}) = s_{0,k}$ and $g_k(x) = f_k(\psi_k(x))$ for each $x\in\hat{M}_k$. Let $\Omega(\hat{M}_k, N_k) := \hat{\pi}_k(\Omega_\xi(M,N))$.

Theorem 3.2. *The set of $\Omega(\hat{M}_k,N_k)$ forms an inverse sequence $S = \{\Omega(\hat{M}_k,N_k); \hat{\pi}^l_k; k\in\mathbb{N}\}$ such that* $\text{pr-lim}\, S =: \Omega^{i,w}(M,N)$ *is an associative topological loop monoid with the cancellation property and unit element e. There exists an embedding of $\Omega_\xi(M,N)$ into $\Omega^{i,w}(M,N)$ such that $\Omega_\xi(M,N)$ is dense in $\Omega^{i,w}(M,N)$.*

Proof. Let U'_i be an analytic disjoint atlas of $\mathrm{inv}(M')$, $f\in C^\xi(\mathrm{inv}(M'),\mathbf{K})$, $\psi \in \mathrm{Diff}^\xi(\mathrm{inv}(M'))$, then each restriction $f|_{U'_i}$ has the form $f|_{U'_i}(x) = \sum_m f_{i,m}\bar{Q}_{i,m}(x)$ for each $x\in U'_i$, where $\bar{Q}_{i,m}$ are basic Amice polynomials for U'_i, $f_{i,m}\in\mathbf{K}$. Therefore f is a combination $f = \nabla_i f|_{U'_i}$ of its restrictions $f|_{U'_i}$, hence

$$\hat{\pi}_k(f\circ\psi(x)) = \sum_m \left[\hat{\pi}_k(f_{i,m})\nabla_{(i,\psi_k(x(k))\in\hat{\pi}_k(U'_k))}\bar{Q}_{i,m,k}(\psi_k(x(k)))\right] \tag{3.32}$$

and inevitably

$$\hat{\pi}_k((f\circ\psi)(x)) = f_k\circ\psi_k(x(k)), \tag{3.33}$$

where $\bar{Q}_{i,m,k} := \hat{\pi}_k(\bar{Q}_{i,m})$, $x\in \operatorname{inv}(M')$ and $x(k) = \hat{\pi}_k(x)$.

As in [13, Section 2.6.2] and Section 3.5 we choose an infinite atlas $\mathrm{At}'(M) := \{(U'_j,\phi'_j) : j\in\mathbb{N}\}$ such that $\phi'_j : U'_j \to B(X,y'_j,r'_j)$ are homeomorphisms, $\lim_{k\to\infty} r'_{j(k)} = 0$, $\lim_{k\to\infty} y'_{j(k)} = 0$ for an infinite sequence $\{j(k)\in\mathbb{N} : k\in\mathbb{N}\}$ such that $\mathrm{cl}_{\bar{M}}[\bigcup_{k=1}^{\infty} U'_{j(k)}]$ is a clopen neighbourhood of zero in $\bar{M}$. We take $|y'_{j(k)}| > r'_{j(k)}$ for each k, hence $\operatorname{inv}(B(X,y'_j,r'_j)\cap X') = B(X,y'^{-1}_j,r'^{-1}_j)\cap X'$ and $\bigcup_k \operatorname{inv}(U'_{j(k)}\cap X')$ is open in X'. For an atlas $\mathrm{At}'(M\vee M) := \{(W_l,\xi_l) : l\in\mathbb{N}\}$ with homeomorphisms $\xi_l : W_l \to B(X,z_l,a_l)$, $\lim_{k\to\infty} a_{l(k)} = 0$, $\lim_{k\to\infty} z_{l(k)} = 0$ for an infinite sequence $\{l(k)\in\mathbb{N} : k\in\mathbb{N}\}$ such that $\mathrm{cl}_{\bar{M}\vee\bar{M}}[\bigcup_{k=1}^{\infty} W_{l(k)}]$ is a clopen neighbourhood of 0×0 in $\bar{M}\vee\bar{M}$ we also choose $|z_l| > a_l$ for each l. We can choose the locally affine mapping χ such that $\bar{\Phi}^n\chi \equiv 0$ for each $n\geq 2$ (see the notation of Section 3.5) and $B(X',y'^{-1}_l,r'^{-1}_l)$ are diffeomorphic with $\operatorname{inv}(U'_l\cap X')$ and $B(X'\vee X',z_l^{-1},a_l^{-1})$ are diffeomorphic with $\operatorname{inv}(W_l\cap(X'\vee X'))$.

This induces the diffeomorphisms $\hat{\chi} := \operatorname{inv}\circ\chi\circ\operatorname{inv} : \hat{M}\vee\hat{M}\to\hat{M}$ and $\hat{\chi}^* : C_0^{\xi}((\hat{M}\vee\hat{M},\infty\times\infty),(N,y_0)) \to C_0^{\xi}((\hat{M},\infty),(N,y_0))$, since each $\Phi^n(f\vee g)(\hat{\chi}^{-1})$ has an expression through $\Phi^l(f\vee g)$ and $\Phi^j(\hat{\chi}^{-1})$ with $l,j\leq n$ and n subordinated to ξ, where $\hat{M} := \operatorname{inv}(M')$ and the conditions defining the subspace $C_0^{\xi}((\hat{M},\infty),(N,y_0))$ differ from that of $C_0^{\xi}((M,s_0),(N,y_0))$ by substitution of $\lim_{x\to s_0}$ on $\lim_{|x|\to\infty}$. Then $\lim_{|x|\to\infty}|\hat{\chi}(x)| = \infty$, consequently, there exists $k_0\in\mathbb{N}$ such that $\hat{\chi}_k : \hat{M}_k\vee\hat{M}_k\to\hat{M}_k$ are bijections for each $k\geq k_0$, where $\hat{\chi}_k := \hat{\pi}_k\circ\hat{\chi}$. If $\psi\in\mathrm{Diff}^{\xi}(\bar{M})$ and $\psi(0) = 0$, then $\lim_{|x|\to\infty}\hat{\psi}(x) = \infty$ and $\lim_{|x|\to\infty}\hat{\psi}^{-1}(x) = \infty$. Then considering $\hat{\psi}_k$ we get an equivalence relation $K_{\xi,k}$ in $\{f_k : f_k\in N_k^{\hat{M}_k}, \lim_{|x|\to\infty} f_k(x) = 0\}$ induced by K_{ξ}, where $\hat{M}_k$ is supplied with the quotient norm induced from the space X, since $X'\subset X$, $x\in\hat{M}_k$. Let J_k denote the quotient mapping corresponding to $K_{\xi,k}$. Therefore, analogously to [13, Section 2.6] we get that $\Omega(\hat{M}_k,N_k)$ are commutative monoids with the cancellation property and the unit elements e_k, since $\Omega(\hat{M}_k,N_k) = \{f_k : f_k\in C^0(\hat{M}_k,N_k), \lim_{|x|\to\infty} f_k(x) = 0\}/\hat{K}_{\xi,k}$ and the mappings $\hat{\pi}_k^l : (\mathbf{K}^{\psi})'_l \to (\mathbf{K}^{\psi})'_k$ and mappings $\tilde{\pi}_k^l : \mathbf{S}^{\xi}_{|\pi|^{-1}} \to \mathbf{S}^{\xi}_{|\pi|^{-k}}$ induce mappings $\hat{\pi}_k^l : \Omega(\hat{M}_l,N_l)\to\Omega(\hat{M}_k,N_k)$ for each $l\geq k$. Let the topology in $\{f_k : f_k\in C^0(\hat{M}_k,N_k), \lim_{|x|\to\infty} f_k(x) = 0\}$ be induced from the Tychonoff product topology in $N_k^{\hat{M}_k}$, and let $\Omega(\hat{M}_k,N_k)$ be in the quotient topology. The space $N_k^{\hat{M}_k}$ is metrizable by the Baire metric $\rho(x,y) := p^{-j}$, where $j = \min\{i : x_i\neq y_i,\ x_1 = y_1,\dots,x_{i-1} = y_{i-1}\}$, $x = (x_l : x_l\in N_k,\ l\in\mathbb{N})$, $\hat{M}_k$ as enumerated as $\mathbb{N}$. Therefore, $\Omega(\hat{M}_k,N_k)$ is metrizable and the mapping $(f_k,g_k)\to f_k\vee g_k$ is continuous, hence the mapping $(J_k(f_k),J_k(g_k))\to J_k(f_k)\circ J_k(g_k)$ is also continuous. Then $J_k(w_{0,k})$ is the unit element, where $w_{0,k}(\hat{M}_k) = 0$. Hence $\Omega^{i,w}(M,N)$ is a commutative monoid with the cancellation property and with unit element. Certainly, $\prod_k\Omega(\hat{M}_k,N_k)$ is a topological monoid and pr-lim S is a closed subset in this topological totally disconnected monoid. For each $f \in C^{\xi}_0(M,N)$ there exists an inverse sequence $\{f_k : f_k = \hat{\pi}_k(f),\ k \in \mathbb{N}\}$ such that $f(x) = \text{pr-lim}_k f_k(x(k))$ for each $x \in M'$, but M' is dense in M. Therefore, there exists an embedding $\Omega^{\xi}(M,N) \hookrightarrow \Omega^{i,w}(M,N)$. Since $C^{\xi}(M,N)$ is dense in $C^0_0(M,N)$, then $\Omega^{\xi}(M,N)$ is dense in $\Omega^{i,w}(M,N)$. □

Corollary 3.3. *The inverse sequence of loop monoids induces the inverse sequence of loop groups* $S_L := \{L(\hat{M}_k,N_k);\hat{\pi}^l_k;\mathbb{N}\}$. *Its projective limit* $L^{i,w}(M,N) :=$ pr-limS_L *is a commutative topological totally disconnected group and* $L_{\xi}(M,N)$ *has an embedding in it as a dense subgroup.*

Proof. Due to the Grothendieck construction, the inversion operation $f_k \mapsto f_k^{-1}$ is continuous in $L(\hat{M}_k,N_k)$, and homomorphisms $\hat{\pi}^l_k$ and $\hat{\pi}_k$ have continuous extensions from loop submonoids onto loop groups $L(\hat{M}_k,N_k)$. Each monoid $\Omega(\hat{M}_k,N_k)$ is totally disconnected, since $N_k^{\hat{M}_k}$ is totally disconnected and $\Omega(\hat{M}_k,N_k)$ is supplied with the quotient ultrametric, hence the free abelian group F_k generated by $\Omega(\hat{M}_k,N_k)$ is also totally disconnected and ultrametrizable, consequently, $L(\hat{M}_k,N_k)$ is ultrametrizable. Evidently, their inverse limit is also ultrametrizable and the equivalent ultrametric can be chosen with values in $\tilde{\Gamma}_{\mathbf{K}} := \{|z| : z \in \mathbf{K}\}$, where $\tilde{\Gamma}_{\mathbf{K}} \cap (0,\infty)$ is discrete in $(0,\infty) := \{x : 0 < x < \infty,\ x \in \mathbb{R}\}$. Then the projective limit (i.e., weak) topology of $L^{i,w}(M,N)$ is induced by the weak topology of $C^0(M,\mathbf{K})$. When M and N are nontrivial, then certainly this weak topology is strictly weaker than that of $L_0(M,N)$. □

Theorem 3.4. *For each prime number* p, *the loop group* $L_{\xi}(M,N)$ *in its weak topology inherited from* $L^{i,w}(M,N)$ *has a* p*-adic completion isomorphic with* $\mathbb{Z}_p^{\aleph_0}$.

Proof. If $\mathbf{K}$ is a finite algebraic extension of the field $\mathbf{Q}_p$, then the projective ring homomorphism $\tilde{\pi}_k : B(\mathbf{K},0,1) \to \mathbf{S}_{|\pi|^{-k}}$ induces the following mapping $\hat{\pi}_k(f(x)) = f_k(x(k))$ for each $f \in B(C^{\xi}(M,\mathbf{K}^{\xi}),0,1)$. Using pavings of $\mathbf{K}$ and $C^{\xi}(M,\mathbf{K}^{\xi})$ by disjoint unions of balls, we get $\tilde{\pi}_k$ on $\mathbf{K}$ and $\hat{\pi}_k$ on $C^{\xi}(M,N)$, respectively, where $\tilde{\pi}_k(x) := \bar{x} := x + B(\mathbf{K},0,|\pi|^k)$ for each $x \in \mathbf{K}$ (see also Sections 3.3 and 3.10). Then the condition

$$\lim_{|x|\to\infty} f(x) = 0 \tag{3.34}$$

implies the condition

$$\lim_{|x(k)|\to\infty} f_k(x(k)) = 0. \tag{3.35}$$

Therefore, $\operatorname{supp}(f_k) := \hat{M}^f_k := \{x(k) : f_k(x(k)) \neq 0\}$ is a finite subset of the discrete space $\hat{M}_k$ for each $k \in \mathbb{N}$. Then evidently, $\hat{\pi}_k(\langle g\rangle_{K,\xi})$ is a closed subset in $N_k^{\hat{M}_k}$ for each $g \in C^{\xi}_0((\hat{M},\infty),(N,0))$, since the support of each limit point f_k of $\hat{\pi}_k(\langle g\rangle_{K,\xi})$ is the finite subset in $\hat{M}_k$. Let k_0 be such that $N_{k_0} \neq \{0\}$, then this is also true for each $k \geq k_0$. If $f_k \notin \hat{\pi}_k(\langle w_0\rangle_{K,\xi})$ and $k \geq k_0$, then $f_k^{\vee n} \notin \hat{\pi}_k(\langle w_0\rangle_{K,\xi})$ for each $n \in \mathbb{N}$, where $f_k^{\vee n} := f_k \vee \cdots \vee f_k$ denotes the n-times wedge product, since

$$||f^{\vee n}||_{C^\xi} \geq \|f\|_{C^\xi} > 0, \qquad ||f_k^{\vee n}||_{C(\mathbf{K}_k^\psi,\mathbf{K}_k^\xi)} \geq \|f\|_{C(\mathbf{K}_k^\psi,\mathbf{K}_k^\xi)} > 0, \tag{3.36}$$

where $C(\mathbf{S}^\psi_{|\pi|-\mathbf{k}},\mathbf{S}^\xi_{|\pi|-\mathbf{k}}) = \tilde{\pi}_k(B((C^\xi(\mathbf{K}^\psi,\mathbf{K}^\xi),0,1))$ is the quotient module over the ring $\mathbf{S}_{|\pi|-\mathbf{k}}$. Each $\hat{\pi}_k(\langle f\rangle_{K,\xi})$ can be presented as the following composition $v_1b_1+\cdots+v_lb_l$ in the additive group $L(\hat{M}_k,N_k)$, where each b_i corresponds to $\hat{\pi}_k(\langle g_i\rangle_{K,\xi})$ and the embedding of $\Omega(\hat{M}_k,N_k)$ into $L(\hat{M}_k,N_k)$, $v_i \in \{-1,0,1\}$, $l = \operatorname{card}(\hat{M}_k^f)$, $\hat{M}_k^{g_i}$ are singletons for each $i = 1,\dots,l$. Using the group $\operatorname{Hom}_0(N_k)$ we get that $L(\hat{M}_k,N_k)$ is isomorphic with $\mathbb{Z}^{n_k}$, where $n_k = \operatorname{card}(N_k) > 1$. In view of Corollary 3.3, $L_\xi(M,N)$ has the p-adic completion isomorphic with $\mathbb{Z}_p^{\aleph_0}$, since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ and $\text{pr-lim}_k \mathbb{Z}^{n_k} = \mathbb{Z}^{\aleph_0}$. □

3.11. Note. Using quotient mappings $\eta_{p,s} : \mathbb{Z} \to \mathbb{Z}/p^s\mathbb{Z}$ we get that $L_\xi(M,N)^{\aleph_0}$ has the compactification equal to $\prod_{p\in\mathcal{P}} \mathbb{Z}_p^{\aleph_0}$, where $\mathcal{P}$ denotes the set of all prime numbers $p > 1$, $s \in \mathbb{N}$. These compactifications produce characters of $L_\xi(M,N)$, since each compact abelian group has only one-dimensional irreducible unitary representations [6]. On the other hand, there are irreducible continuous representations of compact groups in non-Archimedean Banach spaces [19]. Among them there are infinite-dimensional [3, 4, 16]. Moreover, in their initial topologies diffeomorphism and loop groups also have infinite-dimensional irreducible unitary representations [13, 11].

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