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A NEW CHARACTERIZATION OF SOME ALTERNATING AND SYMMETRIC GROUPS

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We suppose that $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Then we prove that the simple groups A_n, where $n = p, p+1$, or $p+2$, and finite groups S_n, where $n = p, p+1$, are also uniquely determined by their order components. As corollaries of these results, the validity of a conjecture of J. G. Thompson and a conjecture of Shi and Bi (1990) both on A_n, where $n = p, p+1$, or $p+2$, is obtained. Also we generalize these conjectures for the groups S_n, where $n = p, p+1$.

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1. Introduction. Let G be a finite group. We denote by $\pi(G)$ the set of all prime divisors of $|G|$. We construct the prime graph of G as follows. *The prime graph* $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose that $2 \in \pi_1$.

Now $|G|$ can be expressed as a product of coprime positive integers m_i, $i = 1, 2, \ldots, t(G)$, where $\pi(m_i) = \pi_i$. These integers are called *the order components* of G. The set of order components of G will be denoted by $\mathrm{OC}(G)$. Also we call $m_2, \ldots, m_{t(G)}$ *the odd-order components* of G. The order components of non-abelian simple groups having at least three prime graph components are obtained by Chen [7, Tables 1, 2, 3]. Similarly, the order components of non-abelian simple groups with two-order components can be obtained by using the tables in [18, 28].

The following groups are uniquely determined by their order components: Suzuki-Ree groups [6], Sporadic simple groups [4], $\mathrm{PSL}_2(q)$ [7], $E_8(q)$ [2], $G_2(q)$, where $q \equiv 0 \pmod 3$ [3], $F_4(q)$, where q is even [15], $\mathrm{PSL}_3(q)$, where q is an odd prime power [14], $\mathrm{PSL}_3(q)$, where $q = 2^n$ [13], $\mathrm{PSU}_3(q)$, where $q > 5$ [16], and A_p, where p and $p-2$ are primes [12].

It was proved by Oyama [20] that a finite group which has the same table of characters as an alternating group A_n is isomorphic to A_n. It was also proved by Koike [17] that a finite group which has the isomorphic subgroup-lattice as an alternating group A_n is isomorphic to A_n.

Let $\pi_e(G)$ denote the set of orders of elements in G. Shi and Bi [27] proved that if $\pi_e(G) = \pi_e(A_n)$ and $|G| = |A_n|$, then $G \cong A_n$. Iranmanesh and Alavi [12] proved that if p and $p-2$ are primes and $\mathrm{OC}(G) = \mathrm{OC}(A_p)$, then $G \cong A_p$. Praeger and Shi [21] and Shi and Bi [26] proved that A_8, A_9, A_{11}, A_{13}, S_7, and S_8 are characterizable by their element orders. Also recently, Kondrat'ev and Mazurov [19] and Zavarnitsin [29] proved that if $\pi_e(G) = \pi_e(A_n)$, where $n = s, s+1, s+2$ and s is a prime number, then $G \cong A_n$.

Now we prove the following theorems.

Theorem 1.1. *Let $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = A_n$, where $n = p, p+1, p+2$. Then $\mathrm{OC}(G) = \mathrm{OC}(M)$ if and only if $G \cong M$.*

Theorem 1.2. *Let $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = S_n$, where $n = p, p+1$. Then $\mathrm{OC}(G) = \mathrm{OC}(M)$ if and only if $G \cong M$.*

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and we refer, for example, to [10]. Also frequently we use the results of Williams [28] and Kondrat'ev [18] about the prime graph of simple groups.

2. Preliminary results

Remark 2.1. Let N be a normal subgroup of G and $p \sim q$ in $\Gamma(G/N)$. Then $p \sim q$ in $\Gamma(G)$. In fact if $xN \in G/N$ has order pq, then there is a power of x which has order pq.

Definition 2.2 (see [11])**.** A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.3 (see [28, Theorem A])**.** *If G is a finite group with its prime graph having more than one component, then G is one of the following groups:*

(a) *a Frobenius or 2-Frobenius group;*
(b) *a simple group;*
(c) *an extension of a π_1-group by a simple group;*
(d) *an extension of a simple group by a π_1-solvable group;*
(e) *an extension of a π_1-group by a simple group by a π_1-group.*

Lemma 2.4 (see [28, Lemma 3])**.** *If G is a finite group with more than one prime graph component and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple group, then H is a nilpotent group.*

The next lemma follows from [1, Theorem 2].

Lemma 2.5. *Let G be a Frobenius group of even order and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(\Gamma(G)) = 2$,* *and the prime graph components of G are $\pi(H)$, $\pi(K)$ and G has one of the following structures:*

(a) $2 \in \pi(K)$ *and all Sylow subgroups of H are cyclic;*

(b) $2 \in \pi(H)$, *K is an abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic groups, and the 2-Sylow subgroups of H are cyclic or generalized quaternion groups;*

(c) $2 \in \pi(H)$, *K is an abelian group, and there exists $H_0 \le H$ such that $|H : H_0| \le 2$, $H_0 = Z \times \mathrm{SL}(2,5)$, $(|Z|, 2.3.5) = 1$, and the Sylow subgroups of Z are cyclic.*

The next lemma follows from [1, Theorem 2] and Lemma 2.4.

Lemma 2.6. *Let G be a 2-Frobenius group of even order. Then $t(\Gamma(G)) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that*

(a) $\pi_1 = \pi(G/K) \cup \pi(H)$ *and* $\pi(K/H) = \pi_2$;

(b) *G/K and K/H are cyclic, $|G/K|$ divides $|\mathrm{Aut}(K/H)|$, $(|G/K|, |K/H|) = 1$, and $|G/K| < |K/H|$;*

(c) *H is nilpotent and G is a solvable group.*

Lemma 2.7 (see [8, Lemma 8]). *Let G be a finite group with $t(\Gamma(G)) \ge 2$ and let N be a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some order components of G but not a π_i-number, then $m_1 m_2 \cdots m_r$ is a divisor of $|N| - 1$.*

The next lemma follows from [5, Lemma 1.4].

Lemma 2.8. *Suppose that G and M are two finite groups satisfying $t(\Gamma(M)) \ge 2$, $N(G) = N(M)$, where $N(G) = \{n \mid G$ has a conjugacy class of size $n\}$, and $Z(G) = 1$. Then $|G| = |M|$.*

Lemma 2.9 (see [5, Lemma 1.5]). *Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(\Gamma(G_1)) = t(\Gamma(G_2))$ and $\mathrm{OC}(G_1) = \mathrm{OC}(G_2)$.*

Lemma 2.10. *Let G be a finite group and let M be a non-abelian finite group with $t(M) = 2$ satisfying $\mathrm{OC}(G) = \mathrm{OC}(M)$.*

(1) *Let $|M| = m_1 m_2$, $\mathrm{OC}(M) = \{m_1, m_2\}$, and $\pi(m_i) = \pi_i$ for $i = 1, 2$. Then $|G| = m_1 m_2$ and one of the following holds:*

  (a) *G is a Frobenius or 2-Frobenius group;*

  (b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group. Moreover, $\mathrm{OC}(K/H) = \{m'_1, m'_2, \ldots, m'_s, m_2\}$, $|K/H| = m'_1 m'_2 \cdots m'_s m_2$, and $m'_1 m'_2 \cdots m'_s \mid m_1$, where $\pi(m'_j) = \pi'_j$, $1 \le j \le s$.*

(2) *In case (b), $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Proof. The proof of (1) follows from the above lemmas. Since $t(G) \ge 2$, we have $t(G/H) \ge 2$. Otherwise $t(G/H) = 1$, so that $t(G) = 1$. Since H is a π_i-group,

TABLE 2.1

p	Finite simple C_{pp} groups
2	A_5, A_6; $L_2(q)$, where q is a Fermat prime, a Mersenne prime, or $q = 2^n$, $n \geq 3$, $L_3(2^2)$, $Sz(2^{2n+1})$, $n \geq 1$
3	A_5, A_6; $L_2(q)$, where $q = 2^3$, 3^{n+1}, or $2.3^n \pm 1$ which is a prime, $n \geq 1$, $L_3(2^2)$
5	A_5, A_6, A_7; M_{11}, M_{22}; $L_2(q)$, where $q = 7^2$, 5^n, or $2.5^n \pm 1$ which is a prime, $n \geq 1$, $L_3(2^2)$, $S_4(q)$, $q = 3, 7$, $U_4(3)$; $Sz(q)$, $q = 2^3, 2^5$
7	A_7, A_8, A_9; M_{22}, J_1, J_2, HS; $L_2(q)$, $q = 2^3$, 7^n, or $2.7^n - 1$ which is a prime, $n \geq 1$, $L_3(2^2)$, $S_6(2)$, $O_8^+(2)$, $G_2(q)$, $q = 3, 19$; $U_3(q)$, $q = 3, 5, 19$; $U_4(3)$, $U_6(2)$, $Sz(2^3)$
13	A_{13}, A_{14}, A_{15}; Suz, Fi_{22}; $L_2(q)$, $q = 3^3$, 5^2, 13^n, or $2.13^n - 1$ which is a prime, $n \geq 1$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $S_6(3)$, $O_8^+(3)$, $G_2(q)$, $q = 2^3, 3$; $F_4(2)$, $U_3(q)$, $q = 2^2, 23$, $Sz(2^3)$, ${}^3D_4(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$
17	A_{17}, A_{18}, A_{19}; J_3, He, Fi_{23}, Fi'_{24}; $L_2(q)$, $q = 2^4$, 17^n, $2.17^n \pm 1$ which is a prime, $n \geq 1$, $S_4(4)$, $S_8(2)$, $F_4(2)$, $O_8^-(2)$, $O_{10}^-(2)$, ${}^2E_6(2)$
19	A_{19}, A_{20}, A_{21}; J_1, J_3, $O'N$, Th, HN; $L_2(q)$, $q = 19^n$, $2.19^n - 1$ which is a prime, $n \geq 1$, $L_3(7)$, $U_3(2^3)$, $R(3^3)$, ${}^2E_6(2)$
37	A_{37}, A_{38}, A_{39}; J_4, Ly; $L_2(q)$, $q = 37^n$, $2.37^n - 1$ which is a prime, $n \geq 1$, $U_3(11)$, $R(3^3)$, ${}^2F_4(2^3)$
73	A_{73}, A_{74}, A_{75}; $L_2(q)$, $q = 73^n$, $2.73^n - 1$ which is a prime, $n \geq 1$, $L_3(2^3)$, $S_6(2^3)$, $G_2(q)$, $q = 2^3, 3^2$; $F_4(3)$, $E_6(2)$, $E_7(2)$, $U_3(3^2)$, ${}^3D_4(3)$
109	A_{109}, A_{110}, A_{111}; $L_2(q)$, $q = 109^n$, $2.109^n - 1$ which is a prime, $n \geq 1$, ${}^2F_4(2^3)$
$p = 2^m + 1$, $m = 2^s$	A_p, A_{p+1}, A_{p+2}; $L_2(q)$, $q = 2^m$, p^k, $2 \cdot p^k \pm 1$ which is a prime, $s \geq k \geq 1$, $S_a(2^b)$, $a = 2^{c+1}$ and $b = 2^d$, $c \geq 1$, $c + d = s$, $F_4(2^e)$, $e \geq 1$, $4e = 2^s$, $O_{2(m+1)}^-(2)$, $s \geq 2$, $O_a^-(2^b)$, $c \geq 2$, $c + d = s$
Other	A_p, A_{p+1}, A_{p+2}; $L_2(q)$, $q = p^k$, $2 \cdot p^k - 1$ which is a prime, $k \geq 1$

we arrive at a contradiction. Moreover, we have $Z(G/H) = 1$. For any $xH \in G/H$ and $xH \notin K/H$, xH induces an automorphism of K/H and this automorphism is trivial if and only if $xH \in Z(G/H)$. Therefore, $G/K \le \mathrm{Out}(K/H)$ and since $Z(G/H) = 1$, (2) follows. □

Definition 2.11. A group G is called a C_{pp} group if the centralizers of its elements of order p in G are p-groups.

Lemma 2.12 (see [9]). *Let p be a prime and $p = 2^\alpha 3^\beta + 1$, $\alpha \ge 0$ and $\beta \ge 0$. Then any finite simple C_{pp} group is given by Table 2.1.*

3. Characterization of some alternating and symmetric groups. In the sequel, we suppose that $p = 2^\alpha 3^\beta + 1$, where $\alpha \ge 1$, $\beta \ge 0$, and $p \ge 7$ is a prime number.

Lemma 3.1. *Let G be a finite group and let M be A_n, where $n = p, p+1$, or $p+2$, or S_n, where $n = p, p+1$. If $\mathrm{OC}(G) = \mathrm{OC}(M)$, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. If G is a Frobenius group, then by Lemma 2.5, $\mathrm{OC}(G) = \{|H|, |K|\}$, where K and H are Frobenius kernel and Frobenius complement of G, respectively. Since $|H| \mid |K| - 1$, we have $|H| < |K|$. Therefore, $2 \nmid |H|$, and hence $2 \mid |K|$. So, $|H| = p, |K| = |G|/p$. We claim that there exists a prime p' such that $3n/4 < p'$. Note that $p \le n$, and hence $p'^2 \nmid |A_n|$. Let $\beta(n)$ be the number of prime numbers less than or equal to n. In fact, by [22, Theorem 2] we have

$$\frac{n}{\log n - 1/2} < \beta(n) < \frac{n}{\log n - 3/2}, \tag{3.1}$$

where $n \ge 67$. Thus

$$\beta(n) - \beta\left(\frac{3n}{4}\right) > \frac{n}{\log n - 1/2} - \frac{3n/4}{\log(3n/4) - 3/2}. \tag{3.2}$$

When $n \ge 405$, we get $\beta(n) - \beta(3n/4) > 1$, and for $n < 405$, we can immediately obtain the result by checking the table of prime numbers. Now let P' be the p'-Sylow subgroup of K. Since K is nilpotent, $P' \lhd G$. Then $p \mid p' - 1$, by Lemma 2.7, which is a contradiction since $p' < p$. Therefore, G is not a Frobenius group.

Now let G be a 2-Frobenius group. By Lemma 2.6, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $|K/H| = p$ and $|G/K| < p$. So, $|H| \ne 1$ since $|G| = |G/K| \cdot |K/H| \cdot |H|$. Since $2 \mid |H|$, let p' be as above and let P' be the p'-Sylow subgroup of H. Now, $p \mid p' - 1$, which is impossible. Hence, G is not a 2-Frobenius group. □

Lemma 3.2. *Let G be a finite group and $M = A_n$, where $n = p, p+1$, or $p+2$, or S_n, where $n = p, p+1$. If $\mathrm{OC}(G) = \mathrm{OC}(M)$, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple group. Moreover, the odd-order component of M is equal to an odd-order component of K/H. In particular, $t(\Gamma(K/H)) \ge 2$. Also $|G/H|$ divides $|\mathrm{Aut}(K/H)|$, and in fact $G/H \le \mathrm{Aut}(K/H)$.*

Proof. The first part of the lemma follows from the above lemmas since the prime graph of M has two prime graph components. For primes p and q, if K/H has an element of order pq, then G has one. Hence, by the definition of prime graph component, the odd-order component of G must be an odd-order component of K/H. Since $K/H \lhd G/H$ and $C_{G/H}(K/H) = 1$, we have

$$G/H = \frac{N_{G/H}(K/H)}{C_{G/H}(K/H)} \cong T, \quad T \leq \mathrm{Aut}(K/H). \tag{3.3}$$

□

Theorem 3.3. *Let $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = A_n$, where $n = p, p+1, p+2$. Then $\mathrm{OC}(G) = \mathrm{OC}(M)$ if and only if $G \cong M$.*

Proof. By Lemma 3.2, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \bigcup \pi(G/K) \subset \pi_1$, K/H is a non-abelian simple group, $t(\Gamma(K/H)) \geq 2$, and the odd-order component of M is an odd-order component of K/H. Therefore, K/H is a finite simple C_{pp} group. Now using Table 2.1, we consider each possibility of K/H separately.

In the sequel, we frequently use the results of [28, Table I] and [18, Tables 2, 3].

Step 1. Let $p = 7, 13, 17, 19, 37, 73$, or 109.

Since the proofs of these cases are similar, we state only one of them, say $p = 13$. Using Table 2.1, we have

(1) $K/H \cong Suz$ or Fi_{22}. It is a contradiction since $3^7 \mid |Suz|$ and $3^9 \mid |Fi_{22}|$ but $3^7 \nmid |A_n|$, where $n = 13, 14, 15$;

(2) $K/H \cong L_2(27)$, $L_2(25)$, $L_3(3)$, $L_4(3)$, $Sz(8)$, ${}^2F_4(2)'$, or $U_3(4)$. If $K/H \cong L_2(27)$, then $|G|/|K/H| = |H| \cdot |G/K| \neq 1$. By Lemma 2.6, $|G/K| \mid |\mathrm{Out}(K/H)| = 6$. So, $|H| \neq 1$. Let P be the 5-Sylow subgroup of H. But since H is nilpotent, $P \lhd G$. Hence, $13 \mid (|P| - 1)$, which is a contradiction. Other cases are similar;

(3) $K/H \cong L_2(13^r)$ or $L_2(2.13^r - 1)$, where $2.13^r - 1$ is a prime, $r \geq 1$. Note that $13^2 \nmid |G|$, hence $r = 1$. So, $K/H \cong L_2(13)$ or $L_2(25)$, and we can proceed similar to (2);

(4) $K/H \cong O_7(3)$. It is a contradiction since $3^9 \mid |O_7(3)|$ but $3^9 \nmid |A_n|$;

(5) $K/H \cong S_4(5)$ or $S_6(3)$. It is a contradiction since $5^4 \mid |S_4(5)|$ but $5^4 \nmid |A_n|$. Also $3^9 \mid |S_6(3)|$ but $3^9 \nmid |A_n|$;

(6) $K/H \cong O_8^+(3)$. It is a contradiction since $3^{12} \mid |O_8^+(3)|$ but $3^{12} \nmid |A_n|$;

(7) $K/H \cong G_2(3)$ or $G_2(8)$. If $K/H \cong G_2(3)$, then we get a contradiction since for $n = 13, 14$ we have $3^6 \mid |G_2(3)|$ but $3^6 \nmid |A_n|$. For $n = 15$, since $|\mathrm{Out}(G_2(3))| = 2$, we have $|H| \neq 1$. Now we proceed similar to (2). If $K/H \cong G_2(8)$, then we get a contradiction since $2^{18} \mid |G_2(8)|$ but $2^{18} \nmid |A_n|$;

(8) $K/H \cong F_4(2)$. It is a contradiction since $17 \mid |F_4(2)|$ but $17 \nmid |A_n|$;

(9) $K/H \cong U_3(23)$. It is a contradiction since $23 \mid |U_3(23)|$ but $23 \nmid |A_n|$;

(10) $K/H \cong {}^3D_4(2)$ or ${}^2E_6(2)$. It is a contradiction since $2^{12} \nmid |A_n|$. Also $19 \nmid |A_n|$;

(11) K/H is an alternating group, namely A_{13}, A_{14}, or A_{15}.

First suppose that $n = 13$. Since $|K/H| \le |A_{13}|$, $K/H \cong A_{13}$. But $|G| = |A_{13}|$, and hence $H = 1$ and $K = G \cong A_{13}$. If $n = 14$, then $K/H \cong A_{13}$ or A_{14}. But if $r \ne 6$, then $\mathrm{Aut}(A_r) = S_r$, and hence $|\mathrm{Out}(A_r)| = 2$. If $K/H \cong A_{13}$, then $|G/K| \mid 2$, and hence $|H| \ne 1$. Now we get a contradiction similar to (2). Therefore, $K/H \cong A_{14}$, and hence $G \cong A_{14}$. If $n = 15$, we do similarly.

Step 2. Let $p = 2^m + 1$, where $m = 2^s$.

Using Table 2.1, we have

(i) $K/H \cong L_2(2^m)$. Note that for every m we have $|L_2(2^m)| \mid |G|$. Using Lemma 2.6, $|G/K| \mid |\mathrm{Out}(K/H)|$. Also $|\mathrm{Out}(L_2(2^m))| = m$. Hence, $|H| \ne 1$. Now let p' be a prime number less than p such that

$$p' \,\|\, \frac{|A_n|}{m|K/H|}. \tag{3.4}$$

Let P' be the p'-Sylow subgroup of H. Since H is nilpotent, $P' \lhd G$. Hence, $p \mid (|P'| - 1)$, which is a contradiction;

(ii) $K/H \cong L_2(p^k)$ or $L_2(2p^k \pm 1)$, where $2p^k \pm 1$ is a prime and $1 \le k \le s$. We know that $p \| |A_n|$, hence $k = 1$. Now we proceed similar to (i);

(iii) $K/H \cong S_a(2^b)$, where $a = 2^{c+1}$ and $b = 2^d$, $c \ge 1$, $c + d = s$. Let $q = 2^b$ and $f = 2^c$, Then $p = q^f + 1$ and we have

$$|S_a(2^b)| = q^{f^2}(q^f - 1)(q^f + 1)\Pi_{i=1}^{f-1}(q^i - 1)(q^i + 1). \tag{3.5}$$

Each factor of the form $(q^j \pm 1)$ is less than or equal to p and therefore divides $|A_n|$. Also $q^{f^2} = (2^m)^f \le 2^{m2} \le 2^{2^m}$. Hence, $|S_a(2^b)| \mid |A_n|$. But $|\mathrm{Out}(S_a(2^b))| = b$. Then $|H| \ne 1$ and we can proceed similar to (i);

(iv) $K/H \cong F_4(2^e)$, where $e \ge 1$, $4e = 2^s$, or $O^-_{2(m+1)}(2)$, where $s \ge 2$, or $O^-_a(2^b)$, where $c \ge 2$, $c + d = s$. Again this part is similar to (iii);

(v) $K/H \cong A_p, A_{p+1}, A_{p+2}$.

First suppose that $n = p$. Since $|K/H| \le |A_p|$, $K/H \cong A_p$. But $|G| = |A_p|$, and hence $H = 1$ and $K = G \cong A_p$. If $n = p + 1$, then $K/H \cong A_p$ or A_{p+1}. But if $r \ne 6$, then $\mathrm{Aut}(A_r) = S_r$, and hence $|\mathrm{Out}(A_r)| = 2$. If $K/H \cong A_p$, then $|G/K| \mid 2$, and hence $|H| \ne 1$. Now we get a contradiction similar to (i). Therefore, $K/H \cong A_{p+1}$, and hence $G \cong A_{p+1}$. If $n = p + 2$, we do similarly.

Step 3. For other primes p, we have $K/H \cong A_p, A_{p+1}, A_{p+2}$; $L_2(q)$, where $q = p^k, 2p^k - 1$ which is a prime, $k \ge 1$.

In fact the proof of this step is exactly similar to that of Step 2 and we omit it for convenience. □

Theorem 3.4. *If G is a non-abelian finite group with connected prime graph, then G is not characterizable with its order component.*

Proof. Clearly, $\mathrm{OC}(G) = \mathrm{OC}(\mathbb{Z}_{|G|})$, but $G \ncong \mathbb{Z}_{|G|}$. □

Corollary 3.5. *Every simple group with one component (see* [28, Table I]*) is not characterizable with this method.*

Theorem 3.6. *Let n be a positive integer. If there exist at least two non-isomorphic abelian groups of order n, then abelian groups of order n are not characterizable with their order component.*

Proof. The proof is obvious. □

Remark 3.7. It was a conjecture that every finite simple group M, where $\Gamma(M)$ is not connected, is characterizable with its order components. But the following example is a counterexample.

Example 3.8. If q is an odd-prime power and $n = 2^k \geq 4$, then $\mathrm{OC}(S_{2n}(q)) = \mathrm{OC}(O_{2n+1}(q))$, but obviously $S_{2n}(q) \ncong O_{2n+1}(q)$.

Theorem 3.9. *Let $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = S_n$, where $n = p, p+1$. Then $\mathrm{OC}(G) = \mathrm{OC}(M)$ if and only if $G \cong M$.*

Proof. Similar to the proof of Theorem 3.3, since G is a C_{pp} group, we have $K/H \cong A_n$. Now using Lemma 3.2, we have

$$A_n \leq \frac{G}{H} \leq \mathrm{Aut}(A_n) = S_n. \tag{3.6}$$

Therefore, $G/H \cong A_n$ or $\mathrm{Aut}(A_n) = S_n$. If $G/H \cong A_n$, then $|H| = 2$ and $H \lhd G$. Hence, $H \subseteq Z(G) = 1$, which is a contradiction. Therefore, $G/H \cong S_n$, and since $|G| = |S_n|$, we have $G \cong S_n$. □

4. Some related results

Remark 4.1. It is a well known conjecture of J. G. Thompson that if G is a finite group with $Z(G) = 1$ and M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

We can generalize this conjecture for the groups under discussion by our characterization of these groups.

Corollary 4.2. *Let G be a finite group with $Z(G) = 1$ and let M be A_p, A_{p+1}, A_{p+2}, S_p, or S_{p+1}. If $N(G) = N(M)$, then $G \cong M$.*

Proof. By Lemmas 2.8 and 2.9, if G and M are two finite groups satisfying the conditions of Corollary 4.2, then $\mathrm{OC}(G) = \mathrm{OC}(M)$. So, Theorems 3.3 and 3.9 imply this corollary. □

Remark 4.3. Shi and Bi in [26] put forward the following conjecture.

Shi's conjecture. *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if*

(i) $|G| = |M|$,

(ii) $\pi_e(G) = \pi_e(M)$, *where $\pi_e(G)$ denotes the set of orders of elements in G.*

This conjecture is valid for sporadic simple groups [24], groups of alternating type [27], and some simple groups of Lie type [23, 25, 26]. As a consequence of Theorems 3.3 and 3.9, we prove a generalization of this conjecture for the groups under discussion.

Corollary 4.4. *Let G be a finite group and let M be A_p, A_{p+1}, A_{p+2}, S_p, or S_{p+1}. If $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, then $G \cong M$.*

Proof. By assumption, we must have $\mathrm{OC}(G) = \mathrm{OC}(M)$. Thus the corollary follows by Theorems 3.3 and 3.9. □

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