RING HOMOMORPHISMS ON REAL BANACH ALGEBRAS

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Let B be a strictly real commutative real Banach algebra with the carrier space Φ_B. If A is a commutative real Banach algebra, then we give a representation of a ring homomorphism $\rho : A \to B$, which needs not be linear nor continuous. If A is a commutative complex Banach algebra, then $\rho(A)$ is contained in the radical of B.

1. Introduction and results. Ring homomorphisms are mappings between two rings that preserve both addition and multiplication. In particular, we are concerned with ring homomorphisms between two commutative Banach algebras. If $\mathbb{R}$ is the real number field, then the zero map and the identity are typical examples of ring homomorphisms on $\mathbb{R}$. Furthermore, the converse is true: if ρ is a nonzero ring homomorphism on $\mathbb{R}$, then $\rho(t) = t$ for every $t \in \mathbb{R}$. For if ρ is nonzero, then $\rho(1) = \rho(1)^2$ implies $\rho(1) = 1$, and hence ρ preserves every rational number. Suppose that $a \geq 0$. Then we have $\rho(a) = \rho(\sqrt{a})^2 \geq 0$. It follows that ρ preserves the order. Fix $t \in \mathbb{R}$ and choose rational sequences $\{p_n\}$ and $\{q_n\}$ converging to t such that $p_n \leq t \leq q_n$. Since ρ preserves both rational numbers and the order $p_n \leq \rho(t) \leq q_n$, thus $\rho(t) = t$.

Let $C_{\mathbb{R}}(K)$ be the commutative real Banach algebra of all real-valued continuous functions on a compact Hausdorff space K. In the proof of [11, Theorem 3.1], Šemrl essentially gave a representation of a ring homomorphism on $C_{\mathbb{R}}(X)$ into $C_{\mathbb{R}}(Y)$ which states that ring homomorphisms preserve scalar multiplication automatically.

THEOREM 1.1 (Šemrl [11]). *If $\rho : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$ is a ring homomorphism, then there exist a closed and open subset $Y_0 \subset Y$ and a continuous map $\varphi : Y \setminus Y_0 \to X$ such that*

$$\rho(f)(y) = \begin{cases} 0, & y \in Y_0, \\ f(\varphi(y)), & y \in Y \setminus Y_0, \end{cases} \tag{1.1}$$

for every $f \in C_{\mathbb{R}}(X)$.

Recall that a commutative real Banach algebra A is said to be strictly real if $\phi(A) \subset \mathbb{R}$ for all $\phi \in \Phi_A$ (cf. [4]), where Φ_A denotes the carrier space of A. We generalize the above result as follows.

Theorem 1.2. *Suppose that A is a commutative real Banach algebra with carrier space Φ_A and that B is a commutative strictly real Banach algebra with carrier space Φ_B. If ρ is a ring homomorphism on A into B, then there exist a closed subset $\Phi_0 \subset \Phi_B$ and a continuous map $\varphi : \Phi_B \setminus \Phi_0 \to \Phi_A$ such that*

$$\rho(a)\hat{\ }(\psi) = \begin{cases} 0, & \psi \in \Phi_0, \\ \hat{a}(\varphi(\psi)), & \psi \in \Phi_B \setminus \Phi_0, \end{cases} \tag{1.2}$$

for every $a \in A$, where $\hat{\ }$ denotes the Gelfand transform.

  If, in addition, A is unital, then the above Φ_0 is closed and open.

Let $C(K)$ be the commutative complex Banach algebra of all complex-valued continuous functions on a compact Hausdorff space K. One might expect that a similar result holds for ring homomorphisms on $C(X)$ into $C(Y)$. Unfortunately, this is not the case. Indeed, there exists a nonzero ring homomorphism τ on $\mathbb{C}$ such that τ is not the identity nor complex conjugate (cf. [6]); such a map is called nontrivial. More precisely, there exist $2^{\mathfrak{c}}$ nontrivial ring homomorphisms on $\mathbb{C}$ (cf. [2]), where $\mathfrak{c}$ denotes the cardinal number of the continuum. However, many authors treat ring homomorphisms between two complex Banach algebras (cf. [1, 3, 5, 7, 8, 9, 10, 11, 12]).

On the other hand, it is easy to see that the zero map is the only ring homomorphism on $\mathbb{C}$ into $\mathbb{R}$. This fact can be generalized as follows.

Theorem 1.3. *Suppose that A is a commutative complex Banach algebra and that B is a commutative strictly real Banach algebra with carrier space Φ_B. If $\rho : A \to B$ is a ring homomorphism, then $\rho(a)\hat{\ } = 0$ for all $a \in A$, or equivalently, A is mapped into the radical of B.*

2. Proof of results. Suppose that $\mathcal{A}$ is a commutative algebra. We define $\mathcal{A}_e = \mathcal{A}$ if $\mathcal{A}$ is unital; otherwise, $\mathcal{A}_e$ denotes the commutative algebra adjoining a unit element e to $\mathcal{A}$.

Lemma 2.1. *If $\mathcal{A}$ is a commutative algebra over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and if ϕ is a nonzero ring homomorphism on $\mathcal{A}$ into $\mathbb{C}$, then ϕ can be extended to a unique ring homomorphism $\tilde{\phi}$ on $\mathcal{A}_e$ into $\mathbb{C}$.*

Proof. Choose $a \in \mathcal{A}$ so that $\phi(a) \neq 0$. If we define $\tilde{\phi} : \mathcal{A}_e \to \mathbb{C}$ by

$$\tilde{\phi}(f, \lambda) \overset{\text{def}}{=} \phi(f) + \frac{\phi(\lambda a)}{\phi(a)}, \quad (f, \lambda) \in \mathcal{A}_e, \tag{2.1}$$

it is trivial to verify that $\tilde{\phi}$ is additive. Identifying f with $(f, 0)$, we obtain $\tilde{\phi}|_{\mathcal{A}} = \phi$. We show that $\tilde{\phi}$ is multiplicative. For every $\nu, \lambda, \mu \in \mathbb{F}$ and $h \in \mathcal{A}$,

we have

$$\phi(vh) = \phi(h)\frac{\phi(va)}{\phi(a)}, \qquad \frac{\phi(\lambda\mu a)}{\phi(a)} = \frac{\phi(\lambda a)}{\phi(a)}\frac{\phi(\mu a)}{\phi(a)}. \tag{2.2}$$

Hence

$$\begin{aligned}
\tilde{\phi}((f,\lambda)(g,\mu)) &= \tilde{\phi}(fg + \mu f + \lambda g, \lambda\mu) \\
&= \phi(f)\phi(g) + \phi(\mu f) + \phi(\lambda g) + \frac{\phi(\lambda\mu a)}{\phi(a)} \\
&= \left\{\phi(f) + \frac{\phi(\lambda a)}{\phi(a)}\right\}\left\{\phi(g) + \frac{\phi(\mu a)}{\phi(a)}\right\} \\
&= \tilde{\phi}(f,\lambda)\tilde{\phi}(g,\mu)
\end{aligned} \tag{2.3}$$

whenever $(f,\lambda),(g,\mu) \in \mathcal{A}_e$, and thus $\tilde{\phi}$ is multiplicative.

We have now proved that there exists an extension $\tilde{\phi}$ of ϕ on $\mathcal{A}_e$.

It remains to prove that $\tilde{\phi} = \tilde{\phi}'$ whenever $\tilde{\phi}'$ is a ring homomorphism which extends ϕ on $\mathcal{A}_e$. So, fix $(f,\lambda) \in \mathcal{A}_e$. Since

$$\phi(\lambda a) = \tilde{\phi}'(\lambda a) = \tilde{\phi}'(\lambda e)\phi(a), \tag{2.4}$$

it follows that

$$\tilde{\phi}'(f,\lambda) = \tilde{\phi}'(f) + \tilde{\phi}'(\lambda e) = \phi(f) + \frac{\phi(\lambda a)}{\phi(a)} = \tilde{\phi}(f,\lambda), \tag{2.5}$$

and the uniqueness is proved.   □

DEFINITION 2.2. Let A be a commutative Banach algebra over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let B be a commutative real or complex Banach algebra with carrier space Φ_B. If ρ is a ring homomorphism on A into B, then the formula

$$\rho_\psi(f) \overset{\text{def}}{=} \rho(f)\Upsilon(\psi), \quad f \in A, \tag{2.6}$$

assigns to each $\psi \in \Phi_B$ a ring homomorphism $\rho_\psi : A \to \mathbb{C}$.

If ρ_ψ is nonzero, then there is a unique extension $\widetilde{\rho_\psi}$ of ρ_ψ on A_e (Lemma 2.1). We define a ring homomorphism $\sigma_\psi : \mathbb{F} \to \mathbb{C}$ by

$$\sigma_\psi(\lambda) = \widetilde{\rho_\psi}(\lambda e), \quad \lambda \in \mathbb{F}. \tag{2.7}$$

It follows from this definition that

$$\rho_\psi(\lambda f) = \widetilde{\rho_\psi}(\lambda f) = \widetilde{\rho_\psi}(\lambda e)\widetilde{\rho_\psi}(f) = \sigma_\psi(\lambda)\rho_\psi(f) \tag{2.8}$$

whenever ρ_ψ is nonzero, $\lambda \in \mathbb{F}$, and $f \in A$.

PROOF OF THEOREM 1.2. Put $\Phi_0 \overset{\text{def}}{=} \{\psi \in \Phi_B : \ker \rho_\psi = A\}$ (possibly empty), and let $\{\psi_\alpha\} \subset \Phi_0$ be a net converging to $\psi_0 \in \Phi_B$. Fix $a \in A$. Since $\rho(a)\check{\ }$ is a continuous function on Φ_B, then $0 = \rho(a)\check{\ }(\psi_\alpha) \to \rho(a)\check{\ }(\psi_0)$. Since a was arbitrary, it follows that $\psi_0 \in \Phi_0$, and hence, Φ_0 is a closed subset of Φ_B. Moreover, if A has a unit e, then $\rho_\psi(e) = 0$ or 1, and hence,

$$\Phi_0 = \{\psi \in \Phi_B : \rho_\psi(e) = 0\} = \{\psi \in \Phi_B : \rho_\psi(e) < 2^{-1}\}. \tag{2.9}$$

It follows that Φ_0 is closed and open whenever A is unital.

Pick $\psi \in \Phi_B \setminus \Phi_0$. Since B is strictly real, the map σ_ψ as in Definition 2.2 is a nonzero ring homomorphism on $\mathbb{R}$ into $\mathbb{R}$ so that σ_ψ is the identity map on $\mathbb{R}$. It follows from (2.8) that

$$\rho_\psi(ta) = \sigma_\psi(t)\rho_\psi(a) = t\rho_\psi(a), \quad t \in \mathbb{R},\ a \in A, \tag{2.10}$$

proving that $\rho_\psi \in \Phi_A$ for every $\psi \in \Phi_B \setminus \Phi_0$. Let $\varphi : \Phi_B \setminus \Phi_0 \to \Phi_A$ be the map defined by $\varphi(\psi) \overset{\text{def}}{=} \rho_\psi$. Then we have (1.2) for every $a \in A$. Finally, we show the continuity of φ. Suppose that $\{\psi_\beta\} \subset \Phi_B \setminus \Phi_0$ is a net converging to $\psi_1 \in \Phi_B \setminus \Phi_0$. Then (1.2) gives

$$\hat{a}(\varphi(\psi_\beta)) = \rho(a)\check{\ }(\psi_\beta) \longrightarrow \rho(a)\check{\ }(\psi_1) = \hat{a}(\varphi(\psi_1)) \tag{2.11}$$

for every $a \in A$. Hence $\varphi(\psi_\beta)$ converges to $\varphi(\psi_1)$. This implies that φ is continuous on $\Phi_B \setminus \Phi_0$. $\qquad\square$

PROOF OF THEOREM 1.3. Pick $a \in A$ and $\psi \in \Phi_B$. If $\rho(a)\check{\ }(\psi) \neq 0$, then $\sigma_\psi(i) = \pm i$, and hence, $\rho(ia)\check{\ }(\psi)$ would be a nonzero pure imaginary number by (2.8), in contradiction to B being strictly real. $\qquad\square$

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