AN EQUALITY FOR THE CURVATURE FUNCTION OF A SIMPLE AND CLOSED CURVE ON THE PLANE

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We prove an equality for the curvature function of a simple and closed curve on the plane. This equality leads to another proof of the four-vertex theorem in differential geometry. While examining the regularity assumption on the curve for the equality, we make comments on the relation between the boundary regularity of a Riemann mapping and two important subjects, the Schauder theory and the strong maximum principle, for elliptic partial differential equations of second order. We take a note on the curvature function itself in recognizing people's handwriting by a calculating device, as an afterthought on the equality.

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1. The equality for the curvature function. Let C be a simple and closed curve on the plane. For the convenience of presentation we assume in the beginning that C is analytic. Let $G(z)$ be a one-to-one conformal mapping from the unit disk centered at the origin to the bounded domain enclosed by C. Standard complex analysis tells us that $G(z)$ is analytic and

$$\left| G'(z) \right| \geq \mu > 0 \tag{1.1}$$

for a constant μ on the closed unit disk (cf. [1] or other textbooks). Let

$$F(z) = G\left(\frac{z-i}{z+i}\right) \tag{1.2}$$

and let

$$\begin{aligned}
u(z) &= \mathrm{Re}\left(\ln\left(F'(z)\right)\right) = \ln\left|F'(z)\right|, \\
v(z) &= \mathrm{Im}\left(\ln\left(F'(z)\right)\right) = \arg\left(F'(z)\right).
\end{aligned} \tag{1.3}$$

Then $F(z)$ maps one-to-one the upper half-plane

$$R_+^2 = \left\{z = (x_1, x_2) = x_1 + ix_2 \mid x_2 \geq 0\right\} \tag{1.4}$$

to the bounded domain enclosed by C, and $u(z)$, $v(z)$ are harmonic functions on R_+^2 functions.

On the boundary $x_2 = 0$, $u(z)$ satisfies

$$u_{x_2}(x_1, 0) = -k(x_1)e^{u(x_1,0)}, \tag{1.5}$$

where $k(x_1)$ is the curvature of C at $F(x_1, 0)$. This boundary condition follows from the following calculation of the curvature of C:

$$
\begin{aligned}
k(x_1) &= \frac{\operatorname{Im}\left(\overline{F'(z)}F''(z)\right)}{|F'(z)|^3} \quad (z = x_1 = (x_1, 0)) \\
&= \operatorname{Im} \frac{\exp\left(u(z) - iv(z)\right)\exp\left(u(z) + iv(z)\right)(u_{x_1} + iv_{x_1})}{\exp\left(3u(z)\right)} \\
&= \operatorname{Im} \frac{1}{\exp\left(u(z)\right)}(u_{x_1} + iv_{x_1}) \\
&= \frac{1}{\exp\left(u(z)\right)}(v_{x_1}) \\
&= \frac{1}{\exp\left(u(z)\right)}(-u_{x_2}).
\end{aligned} \tag{1.6}
$$

Our object is to prove the following equality.

THEOREM 1.1. *With $u(x_1, x_2)$ and $k(x_1)$ defined as above,*

$$\int_{-\infty}^{\infty} x_1 k'(x_1) e^{u(x_1,0)} dx_1 = 0. \tag{1.7}$$

PROOF. We first establish the asymptotic behavior of $u(z)$, $v(z)$, and $\nabla u(z)$ at infinity by showing that as $z \to \infty$,

$$
\begin{aligned}
u(z) &= -2\ln|z| + O(1), \\
v(z) &= -2\arg(z) + \arg\left(G'(1)\right) + \frac{\pi}{2} + o(1), \\
\nabla u(z) &= -\frac{2z}{|z|^2} + o\left(\frac{1}{|z|}\right).
\end{aligned} \tag{1.8}
$$

For this purpose, note that by (1.2)

$$F'(z) = G'\left(\frac{z - i}{z + i}\right)\frac{2i}{(z + i)^2} \tag{1.9}$$

and that as z approaches infinity,

$$G'\left(\frac{z - i}{z + i}\right) \longrightarrow G'(1), \tag{1.10}$$

with $G'(1)$ being nonzero. Thus

$$
\begin{aligned}
u(z) &= \ln |F'(z)| = \ln \left| 2G'\left(\frac{z-i}{z+i}\right) \right| - 2\ln |z+i| \\
&= -2\ln |z| - 2\ln \left| 1 + \frac{i}{z} \right| + \ln \left| 2G'\left(\frac{z-i}{z+i}\right) \right| \\
&= -2\ln |z| + O(1),
\end{aligned}
$$

$$
\begin{aligned}
v(z) &= \arg \left(F'(z) \right) = \arg \left(G'\left(\frac{z-i}{z+i}\right) \right) + \frac{\pi}{2} - 2\arg(z+i) \qquad (1.11) \\
&= -2\arg(z) + \arg \left(G'(1) \right) + \frac{\pi}{2} - 2\arg \left(1 + \frac{i}{z} \right) \\
&\quad + \left(\arg \left(G'\left(\frac{z-i}{z+i}\right) \right) - \arg \left(G'(1) \right) \right) \\
&= -2\arg(z) + \arg \left(G'(1) \right) + \frac{\pi}{2} + o(1).
\end{aligned}
$$

As for $\nabla u(z)$, we have

$$
\begin{aligned}
\frac{\partial u}{\partial x_1} - i\frac{\partial u}{\partial x_2} &= \frac{\partial u}{\partial x_1} + i\frac{\partial v}{\partial x_1} = \left(\ln F'(z) \right)' = \frac{1}{F'(z)} F''(z) \\
&= \frac{1}{F'(z)} \left(G'\left(\frac{z-i}{z+i}\right)\frac{-4i}{(z+i)^3} + G''\left(\frac{z-i}{z+i}\right)\left(\frac{2i}{(z+i)^2}\right)^2 \right) \\
&= \frac{-2}{z+i} + \frac{G''((z-i)/(z+i))}{G'((z-i)/(z+i))}\frac{2i}{(z+i)^2} \qquad (1.12) \\
&= -\frac{2}{z} + \frac{2i}{z(z+i)} + \frac{G''((z-i)/(z+i))}{G'((z-i)/(z+i))}\frac{2i}{(z+i)^2} \\
&= -\frac{2}{z} + o\left(\frac{1}{|z|}\right) = -\frac{2\bar{z}}{|z|^2} + o\left(\frac{1}{|z|}\right).
\end{aligned}
$$

Thus we have completed the proof of (1.8).

Next, we recall that $u(z)$ being a harmonic function implies

$$
\frac{\partial}{\partial x_j}\left(|\nabla u|^2 \delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \right) = 0 \quad \text{for each } i = 1, 2. \qquad (1.13)
$$

Here and later the convention of summing over a repeated index is assumed. Even though it is straightforward to verify (1.13), this equation comes out of a well-known argument of calculus of variations. We describe the argument briefly. Let ϕ_1 and ϕ_2 be two functions that are smooth and have a bounded support on R_+^2 and let $\phi = (\phi_1, \phi_2)$. For $u_\epsilon(z) = u(z + \epsilon\phi(z))$, consider the integral $\int |\nabla u_\epsilon|^2 dx_1 dx_2$ on a bounded domain that contains the support of ϕ.

The derivative of this integral with respect to ϵ at $\epsilon = 0$ equals

$$\int 2\nabla u \cdot \nabla\left(\frac{\partial u}{\partial x_i}\phi_i\right)dx_1\,dx_2$$
$$= \int \frac{\partial}{\partial x_j}\left(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)\phi_i dx_1\,dx_2, \tag{1.14}$$

after a few elementary manipulations. On the other hand, it equals zero because it is the first variation of the integral involving a harmonic function. Equation (1.13) then follows.

The essential step to equality (1.7) is an integration involving (1.13). Let B_R^+ be the upper-half disk centered at the origin with radius R. By (1.13) and Green's theorem,

$$0 = \int_{B_R^+} x_i\frac{\partial}{\partial x_j}\left(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)dx_1\,dx_2$$
$$= \oint_{\partial B_R^+} x_i\left(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)\nu_j\,dl$$
$$- \int_{B_R^+}\delta_{ij}\left(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)dx_1\,dx_2 \tag{1.15}$$
$$= \oint_{\partial B_R^+} x_i\left(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)\nu_j\,dl.$$

Above $\nu = (\nu_1,\nu_2)$ is the outward unit normal vector to ∂B_R^+, dl is the line integral element, and the line integral is in the counterclockwise direction. Let I_1 be the line integral on the line segment $\{(x_1,0) \mid -R \le x_1 \le R\}$ and let I_2 be the line integral on the upper-half circle $\{(R\cos\theta, R\sin\theta) \mid 0 \le \theta \le \pi\}$. For I_1, we have $x = (x_1,0)$ and $\nu = (0,-1)$; hence

$$I_1 = \int_{-R}^{R} x_1\left(|\nabla u|^2\delta_{12} - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\right)(-1)dx_1$$
$$= \int_{-R}^{R} x_1 2\frac{\partial u}{\partial x_1}\left(-k(x_1)e^{u(x_1,0)}\right)dx_1 \quad \text{(by (1.5))}$$
$$= -2x_1 k(x_1)e^{u(x_1,0)}\,|_{-R}^{R} + \int_{-R}^{R}\left(2k(x_1) + 2x_1 k'(x_1)\right)e^{u(x_1,0)}dx_1$$
$$= -2x_1 k(x_1)e^{u(x_1,0)}\,|_{-R}^{R} - \int_{-R}^{R} 2\frac{\partial u}{\partial x_2}(x_1,0)dx_1 + \int_{-R}^{R} 2x_1 k'(x_1)e^{u(x_1,0)}dx_1$$
$$= \left(-2Rk(R)e^{u(R,0)} - 2Rk(-R)e^{u(-R,0)}\right) + 2\left(v(R,0) - v(-R,0)\right)$$
$$+ \int_{-R}^{R} 2x_1 k'(x_1)e^{u(x_1,0)}dx_1. \tag{1.16}$$

For I_2, we have $x = (R\cos\theta, R\sin\theta)$ and $\nu = x/R$; hence

$$I_2 = \int_0^\pi \left(|\nabla u|^2 R^2 - 2\frac{\partial u}{\partial x_i} x_i \frac{\partial u}{\partial x_j} x_j \right) d\theta. \tag{1.17}$$

As R approaches infinity, because of (1.8) and that $k(R)$ and $k(-R)$ are bounded, we conclude that

$$\left(-2Rk(R)e^{u(R,0)} - 2Rk(-R)e^{u(-R,0)} \right)$$
$$= -2Rk(R)\frac{1}{R^2}e^{O(1)} - 2Rk(-R)\frac{1}{R^2}e^{O(1)} \longrightarrow 0,$$
$$2(v(R,0) - v(-R,0))$$
$$= 2\left(\left(\arg(G'(1)) + \frac{\pi}{2} + o(1) \right) - \left(-2\pi + \arg(G'(1)) + \frac{\pi}{2} + o(1) \right) \right) \longrightarrow 4\pi, \tag{1.18}$$

and consequently

$$I_1 \longrightarrow 4\pi + \int_{-\infty}^\infty 2x_1 k'(x_1)e^{u(x_1,0)}dx_1. \tag{1.19}$$

Similarly, as R approaches infinity,

$$I_2 = \int_0^\pi \left(\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}R^2 - 2\frac{\partial u}{\partial x_i} x_i \frac{\partial u}{\partial x_j} x_j \right) d\theta$$
$$= \int_0^\pi \left(\left(-\frac{2x_i}{R^2} + o\left(\frac{1}{R}\right) \right)\left(-\frac{2x_i}{R^2} + o\left(\frac{1}{R}\right) \right) R^2 \right.$$
$$\left. - 2\left(-\frac{2x_i}{R^2} + o\left(\frac{1}{R}\right) \right)x_i\left(-\frac{2x_j}{R^2} + o\left(\frac{1}{R}\right) \right)x_j \right) d\theta \tag{1.20}$$
$$= \int_0^\pi \left(\left(\frac{4}{R^2} + o\left(\frac{1}{R^2}\right) \right)R^2 - 2(-2+o(1))(-2+o(1)) \right) d\theta$$
$$= \int_0^\pi (-4 + o(1)) d\theta \longrightarrow -4\pi.$$

Adding the limits of I_1 and I_2, we obtain equality (1.7). $\square$

2. Remarks on the equality.

We make several remarks on the equality.
First, the integral in (1.7) is a proper integral. To see this, let

$$s = \int_{-\infty}^{x_1} |F'(x_1,0)| dx_1 = \int_{-\infty}^{x_1} e^{u(x_1,0)}dx_1 \tag{2.1}$$

be a length parameter of C. Then

$$k'(x_1) = \frac{dk}{ds}e^{u(x_1,0)}. \tag{2.2}$$

Since dk/ds is bounded,

$$x_1 k'(x_1) e^{u(x_1,0)} = x_1 \frac{dk}{ds} e^{2u(x_1,0)} \tag{2.3}$$

and is, by (1.8), bounded by a constant multiple of $1/(1+|x_1|)^3$. That is, the integral in (1.7) converges absolutely.

Next, we see how equality (1.7) leads to another proof of the four-vertex theorem in differential geometry. Since we may choose $G(z)$ and $F(z)$ such that $k(0)$ equals the maximum and $k(\pm\infty)$ equal the minimum of the curvature function of C, equality (1.7) implies that it is impossible that $x_1 k'(x_1) \le 0$ for all x_1. That is, it is impossible that $k(x_1)$ is an increasing function on $-\infty < x_1 < 0$ and a decreasing function on $0 < x_1 < \infty$. Thus $k(x_1)$ has at least two more local maximum or minimum points in addition to 0 and $\pm\infty$. It follows that C has at least four critical points, or vertices. We refer to the papers of Osserman [7] and Tabachnikov [10] for proofs of the four-vertex theorem in differential geometry as well as references to other relevant works.

At this point, we examine the regularity assumption on C. Since equality (1.7) involves the first derivative of the curvature function of C, the natural assumption is that C is C^3. We will see that this is also sufficient for (1.7). With (2.3) in mind, we recognize that it is sufficient for all our computations that $G(z)$ be $C^{2,\alpha}$ for some α satisfying $0 < \alpha < 1$ and (1.1) be satisfied on the closed unit disk. Here we need to mention that although (1.13) involves the third derivative of $G(z)$, we have only used (1.13) in its integral form. Thus by integrating on a half disk slightly above the real line and then taking a limit, we may carry out all the same computations.

Now since C is assumed to be C^3, it is $C^{2,\alpha}$ for any constant α satisfying $0 < \alpha < 1$. We need the case $m = 2$ of the following theorem.

Theorem 2.1. *If C is $C^{m,\alpha}$ smooth, where m is a positive integer and $0 < \alpha < 1$, then $G(z)$ is $C^{m,\alpha}$ and $|G'(z)| > 0$ on the closed unit disk.*

Theorem 2.1 is called Kellogg's theorem in complex analysis. In [11] there is a proof of Theorem 2.1 for the most difficult case $m = 1$. It is also known to specialists that Kellogg's theorem can be proved by using the Schauder theory and the strong maximum principle for linear elliptic partial differential equations of second order. However, it is not easy to find a handy reference. My colleague Professor N. V. Rao (Rao Nagisetty) supplied the following proof.

Consider $\ln|G^{-1}(z)|$, which is a Green's function on the domain enclosed by C with a singularity at $G(0)$. In case $m \ge 2$, applying the standard Schauder theory and the strong maximum principle, as presented in [6], we conclude that $G^{-1}(z)$ is $C^{m,\alpha}$ smooth and the gradient of $G^{-1}(z)$ does not vanish on the closed domain enclosed by C. Then the same holds for $G(z)$ on the unit disk by the inverse function theorem. In case $m = 1$, the proof goes the same

way because both the Schauder theory and the strong maximum principle are still valid. Nevertheless, one has to look harder in the literature for a proof in this case. Gilbarg and Trudinger discussed these subjects in the notes of their book [6], and they particularly mentioned works of Finn-Gilbarg and Widman. It seems to be true that it was the study of the boundary regularity of a Riemann mapping that led to the development of the Schauder theory and the strong maximum principle. We refer to the books of Chen and Wu [3] and Giaquinta [5] for different ways to treat the Schauder theory. In addition, there are a lot of works on the Schauder theory for more general elliptic PDEs and on the boundary regularity for disk-type minimal surfaces bounded by a simple and closed space curve. In a series of papers, Professor Friedman extended the Schauder theory to parabolic PDEs, and his work was summarized in [4].

From these discussions, we see that the assumption of C being C^3 is natural and sufficient for equality (1.7). Consequently, our proof works for the four-vertex theorem for a simple and closed curve that is C^3 smooth.

With less regularity assumption on the curve, it would be interesting to see what would come out to replace equality (1.7). For example, when a sequence of simple and closed curves that are at least C^3 smooth converges to a polygon, it remains a problem that what the corresponding equalities (1.7) would converge to.

The higher-dimensional counterpart of equality (1.7) should be the Kazdan-Warner equality. As a matter of fact, our equality (1.7) is inspired by similar equalities in different contexts. We refer the reader to [2, 9, 12], among many contributions.

In a relevant work [8], the author proved a uniqueness theorem for a harmonic function $u(x_1, x_2)$ on the upper half-plane satisfying the boundary condition $u_{x_2}(x_1, 0) = -e^{u(x_1, 0)}$ and the constraint $\int_{R_2^+} e^{2u} dx_1 \, dx_2 < \infty$. We also refer the reader to a recent paper of Zhang [13].

3. A note on the curvature function itself in the recognition of people's handwriting. The author does not know any other use of equality (1.7). However, while pondering on equality (1.7), I came to think about the problem of recognizing people's handwriting by a calculating device, with the help of the curvature function of a curve.

Let $\{(x(t), y(t)) \mid t_1 \le t \le t_2\}$ be a parametric curve representing a stroke of our handwriting. After discarding what we meant to be a dot or breaking what we wrote into a few segments, we can assume that the curve is smooth. We rescale the curve so that the length is one, and then we calculate the curvature function $k(s)$, $0 \le s \le 1$, where s is the length parameter. The very first theorem of differential geometry tells us that two curves are similar if and only if their curvature functions are identical. Also, it would not be hard to show that two curves resemble each other if their curvature functions are close, a mere corollary of the stability for a linear ordinary differential equation. Thus with

the help of the curvature function we can make a calculating device to identify what we write! That is, we could build a calculating peripheral that would record what we write as parametric curves and the accompanying software would then recognize what we write. Such a device would help mathematicians to put in writing more easily mathematical symbols and characters that are not convenient to be put into an electronic file on using a keyboard or a mouse.

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