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NEW APPROACH TO THE FRACTIONAL DERIVATIVES

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We introduce a new approach to the fractional derivatives of the analytical functions using the Taylor series of the functions. In order to calculate the fractional derivatives of f, it is not sufficient to know the Taylor expansion of f, but we should also know the constants of all consecutive integrations of f. For example, any fractional derivative of e^x is e^x only if we assume that the nth consecutive integral of e^x is e^x for each positive integer n. The method of calculating the fractional derivatives very often requires a summation of divergent series, and thus, in this note, we first introduce a method of such summation of series via analytical continuation of functions.

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1. Introduction. The great and famous mathematician L. Euler was criticized by the mathematicians in the following centuries for working very freely with the infinite processes. More concretely, he was criticized for very free calculating with the divergent series. Further, the mathematical analysis was quite strongly founded using $\epsilon-\delta$ criteria. Later in the 20th century, he was partially rehabilitated by the development of the calculus of the divergent series. However, by introducing an axiomatic approach, this note shows that Euler was, indeed, centuries in front of his time.

The theory of the fractional derivatives is an important part of the analysis, and the book of Samko et al. [1] is a basic monograph on that topic. In this note, the summation of series is considered; more precisely, summation of "divergent" series and a method of fractional derivatives. Although both of them are present in the literature, there is a basically new view of these two parts of the analysis. In Section 2, quite a strong method of summation is introduced, which considers a large class of series. It is used in Section 3 in order to effectively calculate the fractional derivatives of a given function. The analytic functions should be treated as given series but not classically according to the set theory of functions. Indeed, this theory should be considered as an axiomatic theory. This new approach can find application in solving equations and systems of equations with fractional derivatives. In [3], the formula for $D^k(f)$ is given, where D is a linear differential operator. That result, together with the present ones, can be used for solving equations and systems with fractional derivatives.

At the end of this section, we present the simple numerical identity:

$$\frac{\pi}{8}=\frac{1}{3}+\frac{1}{1\cdot 3\cdot 5}-\frac{1}{3\cdot 5\cdot 7}+\frac{1}{5\cdot 7\cdot 9}-\frac{1}{7\cdot 9\cdot 11}+\cdots. \tag{1.1}$$

It is easy to verify by a computer that the previous equality is true on $10, 20, \ldots$ decimals, but it is very difficult (or impossible) to prove the previous identity using the methods of the standard analysis. On the other hand, using the theory presented in this note, it is very easy to prove this equality. It shows a necessity of a new theory.

2. Summation of series using analytic continuation.

In this section, we introduce a method of summation of series. This method is very strong, which means that, for given series $\sum_{i=0}^{\infty} a_i$, all possible values (of convergence) are determined; it tends to infinity or the series is unsummable, that is, divergent. We assume the convention that convergent series will mean convergent according to the method that follows, while the convergence in the classical sense will be called "ordinary convergence." Generally, we consider series with complex elements a_i.

Let a series $\sum_{i=0}^{\infty} a_i$ be given such that $\limsup_{n\to\infty} |a_n|^{1/n} < \infty$. Suppose that f is an analytic function regular in a neighborhood of the point z_0 and its expansion is

$$f(z)=b_0+b_1(z-z_0)+b_2(z-z_0)^2+\cdots, \tag{2.1}$$

and suppose also that there exists $z_1 \in \mathbb{C}$ such that

$$b_i(z_1-z_0)^i = a_i \quad (0 \le i < \infty). \tag{2.2}$$

Then, three cases are possible.

(1) If f can be analytically continued to the point z_1, where z_1 is a regular point of f, then $\sum_{i=0}^{\infty} a_i$ converges to any possible value of the analytically continued function f at z_1, that is, $\sum_{i=0}^{\infty} a_i \in A = \{f(z_1)\}$.

(2) If z_1 is a singular point of f, then $\sum_{i=0}^{\infty} a_i$ is said to converge or tend to infinity.

(3) If z_1 does not belong to the domain of analytic continuation of f, then we say that $\sum_{i=0}^{\infty} a_i$ diverges, that is, it is unsummable.

Note that, without loss of generality, we can always assume that $z_0 = 0$, and we should find $f(1)$ where

$$f(z)=a_0+a_1z+a_2z^2+\cdots. \tag{2.3}$$

The condition $\limsup_{n\to\infty} |a_n|^{1/n} < \infty$ provides that the right side of (2.3) defines an analytic function that is regular at $z = 0$.

Example 2.1. The series $1+1+1+\cdots$ tends to ∞ because $f(z)=1+z+z^2+z^3+\cdots=1/(1-z)$ has a singular point $z=1$.

Example 2.2. Find the sum $1+2+2^2+2^3+\cdots$. Since

$$f(z)=1+2z+2^2z^2+2^3z^3+\cdots=\frac{1}{1-2z}, \tag{2.4}$$

we obtain $1+2+2^2+2^3+\cdots=f(1)=1/(1-2)=-1$.

Example 2.3. The radius of ordinary convergence of the series

$$f(z)=z+z^2+0\cdot z^3+z^4+0\cdot z^5+0\cdot z^6+0\cdot z^7+z^8+\cdots=\sum_{n=0}^{\infty}z^{2^n} \tag{2.5}$$

is equal to 1. But $f(z)$ cannot be analytically continued for $|z|>1$. Hence, the series

$$2+2^2+0+2^4+0+0+0+2^8+\cdots \tag{2.6}$$

is "essentially" divergent.

Example 2.4. We consider the series

$$S=1+\frac{1}{2}\frac{1}{2}+\frac{1/2\cdot -1/2}{2!}\frac{1}{2^2}+\frac{1/2\cdot -1/2\cdot -3/2}{3!}\frac{1}{2^3}+\cdots. \tag{2.7}$$

Since

$$f(z)=1+\frac{1}{2}z+\frac{1/2\cdot -1/2}{2!}z^2+\frac{1/2\cdot -1/2\cdot -3/2}{3!}z^3+\cdots=(1+z)^{1/2}, \tag{2.8}$$

the series S tends to both $\sqrt{3/2}$ and $-\sqrt{3/2}$. It seems odd to accept that $S=-\sqrt{3/2}$. But we write f as

$$\begin{aligned}f(z)=\sqrt{1+z}=1^{1/2}+\frac{1}{2}\cdot 1^{-1/2}\cdot z^1+\frac{1/2\cdot -1/2}{2!}\cdot 1^{-3/2}\cdot z^2\\ +\frac{1/2\cdot -1/2\cdot -3/2}{3!}\cdot 1^{-5/2}\cdot z^3+\cdots\end{aligned} \tag{2.9}$$

we notice that both sides of this equality may take two values. Indeed, if we take $1^{1/2}=1$ and hence $1^{-1/2}=1^{1/2}/1=1$, $1^{-3/2}=1^{1/2}/1^2=1$, and so on, then we get $S=\sqrt{3/2}$. If we take $1^{1/2}=-1$ and hence $1^{-1/2}=1^{1/2}/1=-1$, $1^{-3/2}=1^{1/2}/1^2=-1$, and so on, then we get $S=-\sqrt{3/2}$.

Remark 2.5. We considered all the series $\sum_{i=0}^{\infty}a_i$ such that

$$\limsup_{n\to\infty}\left|a_n\right|^{1/n}<\infty. \tag{2.10}$$

But, until now, we are not able to do anything if

$$\limsup_{n\to\infty} |a_n|^{1/n} = \infty. \tag{2.11}$$

In this case, we can require a differential equation which satisfies the function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$.

Example 2.6. Find the sum

$$S = 1! - 2! + 3! - 4! + \cdots. \tag{2.12}$$

We consider the function

$$f(z) = 1!z^2 - 2!z^3 + 3!z^4 - 4z^5! + \cdots. \tag{2.13}$$

Then,

$$\begin{gathered} f'(z) = 2!z^1 - 3!z^2 + 4!z^3 - \cdots = \frac{1}{z^2}\left(z^2 - f(z)\right), \\ f'(z) + \frac{1}{z^2} f(z) - 1 = 0. \end{gathered} \tag{2.14}$$

Hence,

$$f(z) = e^{1/z}\left(C + \int_0^z e^{-1/t}\,dt\right) \tag{2.15}$$

and $C = 0$ because $f(0) = 0$. Thus, we obtain

$$S = e\int_0^1 e^{-1/t}\,dt. \tag{2.16}$$

Now, we give some properties of convergence of series. Most of them follow from the standard results of the continuation of the functions of complex variables.

(1) If $S = a_0 + a_1 + a_2 + \cdots$ and $\limsup_{n\to\infty} |a_n|^{1/n} < \infty$, then the sum does not change if a finite number of summands change their places or arbitrarily are grouped.

Proof. It is sufficient to prove that the following equalities:

$$a_0 + a_1 + a_2 + \cdots = S, \qquad a_1 + a_2 + a_3 + \cdots = S - a_0 \tag{2.17}$$

are equivalent. The functions

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \qquad g(z) = a_1 + a_2 z + a_3 z^2 + \cdots \tag{2.18}$$

have the same domain of continuation and everywhere holds $f(z) = zg(z) + a_0$. Thus,

$$f(1) = S \quad \text{iff } g(1) = S - a_0. \tag{2.19}$$

□

Remark 2.7. Note that property (1) does not hold if the word "finite" is replaced by "infinite." For example, $1-1+0+1-1+0+1-1+0+\cdots = 1/3$, but $1-1+1-1+\cdots = 1/2$.

(2) If the series $\sum_{i=0}^{\infty} a_i$ ordinarily converges to S, then S is one of the values of convergence of the series considered. In other words, ordinary convergence to S implies convergence to S.

(3) If the series $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are such that

$$\limsup_{n\to\infty} |a_n|^{1/n} < \infty, \qquad \limsup_{n\to\infty} |b_n|^{1/n} < \infty \tag{2.20}$$

and the series converge to A and B, respectively, then $\sum_{i=0}^{\infty}(\lambda a_i + \mu b_i)$ converges to $\lambda A + \mu B$.

(4) If the series $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are such that

$$\limsup_{n\to\infty} |a_n|^{1/n} < \infty, \qquad \limsup_{n\to\infty} |b_n|^{1/n} < \infty \tag{2.21}$$

and the series converge to A and B, respectively, then the convolute product of these two series converges to AB.

Theorem 2.8. *Let $\mathbb{O}$ be the domain where the analytic function f can be continued. If f is regular at the point $\alpha \in \mathbb{O}$, then the function can be represented as power series in the form*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n \tag{2.22}$$

in the whole domain $\mathbb{O}$.

In Section 3, series of the type $\sum_{i=-\infty}^{\infty} a_i$ are used. If $\sum_{i=0}^{\infty} a_i$ converges to A and $\sum_{i=1}^{\infty} a_{-i}$ converges to B, then we say that $\sum_{i=-\infty}^{\infty} a_i$ converges to $A+B$. The following generalization of Theorem 2.8 also holds.

Theorem 2.9. *Let $\mathbb{O}$ be the domain where the analytic function f can be continued. If f decomposes in the ring $r_1 < |z-\alpha| < r_2$ in the Laurent's series*

$$f(z) = \sum_{i=-\infty}^{\infty} a_i (z-\alpha)^i, \tag{2.23}$$

then the right-hand side of (2.23) converges to $f(z)$ in the whole domain $\mathbb{O}$.

3. Fractional derivatives.

Now, we are ready to introduce fractional derivatives. First, we consider a class of analytic functions "axiomatically" as the formal series

$$f(z) = \sum_{i=-\infty}^{\infty} a_i \frac{z^{i+\alpha}}{(i+\alpha)!} \quad (a_i \in \mathbb{C}) \tag{3.1}$$

for $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$, where $\beta! = \Gamma(\beta+1)$. Note that two functions

$$\sum_{i=-\infty}^{\infty} a_i \frac{z^{i+\alpha}}{(i+\alpha)!} \qquad \sum_{i=-\infty}^{\infty} b_i \frac{z^{i+\alpha}}{(i+\alpha)} \tag{3.2}$$

are different if there exists an index $i \in \mathbb{Z}$ such that $a_i \neq b_i$.

If $\alpha \in \mathbb{Z}$, then, without loss of generality, we assume that $\alpha = 0$ and consider the Taylor's series

$$f(z) = \sum_{i=-\infty}^{\infty} a_i \frac{z^i}{i!}. \tag{3.3}$$

In this case, we need an additional assumption. Assume that the analytic function f is regular at $z = 0$, and let

$$f(z) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} z^i. \tag{3.4}$$

Formally, we can write it as

$$f(z) = \sum_{i=-\infty}^{\infty} \frac{f^{(i)}(0)}{i!} z^i \tag{3.5}$$

because $(-1)! \cdot 0 = 0!$, $(-2)!(-1) \cdot 0 = 0!, \dots$, $(-1)! = (-2)! = \cdots = \pm\infty$, and $f^{(-1)}(z) = \int_0^z f(t)dt + C_1$, $f^{(-2)}(z) = \int_0^z f^{(-1)}(t)dt + C_2, \dots$. Hence, $f^{(-1)}(0) = C_1$, $f^{(-2)}(0) = C_2$, $f^{(-3)}(0) = C_3, \dots$, and (3.5) can be written as

$$f(z) = \cdots + \frac{C_3}{(-3)!} z^{-3} + \frac{C_2}{(-2)!} z^{-2} + \frac{C_1}{(-1)!} z^{-1} + \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} z^i. \tag{3.6}$$

Note that if we change the constants of integration $C_1, C_2, C_3, \dots$, we obtain the same analytic function according to the classical set theory of analytic functions. We assume (by definition) here that by changing the constants $C_1, C_2, C_3, \dots$ we obtain different functions. The reason will be obvious later. Thus, we can summarize as follows: an analytic function f, regular at $z = 0$, is uniquely determined by $f(0)$, $f^{(i)}(0)$ $(i = 1, 2, 3, \dots)$ and by the constants of integration $C_1, C_2, C_3, \dots$.

Remark 3.1. For the sake of simplicity, we consider the analytic function of type (3.1), but, without loss of generality, we can consider functions of type $\sum_{i=-\infty}^{\infty} a_i (z-z_0)^{i+\alpha}/(i+\alpha)!$.

Now, let f be given by series (3.1). Then, we define pth derivative ($p \in \mathbb{C}$) by

$$f^{(p)}(z) = \sum_{i=-\infty}^{\infty} a_i \frac{z^{i+\alpha-p}}{(i+\alpha-p)!}. \tag{3.7}$$

As a direct consequence of the definition, we have the following property.
(1) For any $p, q \in \mathbb{C}$ and any analytic function f, the following holds

$$\left(f^{(p)}\right)^{(q)} = f^{(p+q)}. \tag{3.8}$$

The Leibniz equality also holds in the following form.
(2) For any $p \in \mathbb{C}$ and any analytic functions f and g, the following holds

$$(fg)^{(p)} = \sum_{i=-\infty}^{\infty} p! \frac{f^{(i+\alpha)}}{(i+\alpha)!} \cdot \frac{g^{(p-i-\alpha)}}{(p-i-\alpha)!}. \tag{3.9}$$

Proof. Because of the linearity of the operator, it is sufficient to assume that

$$f(z) = \frac{z^u}{u!}, \qquad g(z) = \frac{z^v}{v!}. \tag{3.10}$$

Then,

$$(fg)^{(p)} = \left(\frac{(u+v)!}{u!v!} \cdot \frac{z^{u+v}}{(u+v)!}\right)^{(p)} = \frac{(u+v)!}{u!v!} \cdot \frac{z^{u+v-p}}{(u+v-p)!},$$

$$\sum_{i=-\infty}^{\infty} p! \frac{f^{(i+\alpha)}}{(i+\alpha)!} \cdot \frac{g^{(p-i-\alpha)}}{(p-i-\alpha)!} = \sum_{i=-\infty}^{\infty} \frac{p!}{(i+\alpha)!(p-i-\alpha)!} \frac{z^{u-i-\alpha}}{(u-i-\alpha)!} \frac{z^{v+i-p+\alpha}}{(v+i-p+\alpha)!}, \tag{3.11}$$

and we have to prove that

$$\sum_{i=-\infty}^{\infty} \frac{p!}{(u-i-\alpha)!(v+i-p+\alpha)!(i+\alpha)!(p-i-\alpha)!} = \frac{(u+v)!}{(u+v-p)!u!v!}, \tag{3.12}$$

that is,

$$\sum_{i=-\infty}^{\infty} \frac{(u+v-p)!}{(u-i-\alpha)!(v+i-p+\alpha)!} \frac{p!}{(i+\alpha)!(p-i-\alpha)!} = \frac{(u+v)!}{u!v!}. \tag{3.13}$$

We put $a = u-\alpha$, $b = v+\alpha-p$, $c = \alpha$, and $d = p-\alpha$. This equality is equivalent to

$$\sum_{i=-\infty}^{\infty} \frac{(a+b)!}{(a-i)!(b+i)!} \frac{(c+d)!}{(c+i)!(d-i)!} = \frac{(a+b+c+d)!}{(a+c)!(b+d)!}. \tag{3.14}$$

Indeed, comparing the coefficient in front of x^{a+c} in the equality

$$(1+x)^{a+b}\cdot(1+x)^{c+d}=(1+x)^{a+b+c+d}, \tag{3.15}$$

that is,

$$\begin{aligned}\left(\sum_{i=-\infty}^{\infty}\frac{(a+b)!}{(a-i)!(b+i)!}x^{a-i}\right)&\left(\sum_{j=-\infty}^{\infty}\frac{(c+d)!}{(c+j)!(d-j)!}x^{c+j}\right)\\&=\sum_{k=-\infty}^{\infty}\frac{(a+b+c+d)!}{(a+c-k)!(b+d+k)!}x^{a+c-k},\end{aligned} \tag{3.16}$$

we obtain the required equality. □

The left- or the right-hand side of the sum (3.6) is not very often ordinarily convergent, and, then, the method of Section 2 should be applied. Of course, we cannot consider all series by (3.1), but we consider only those for which the series converge.

Note that if we apply fractional derivatives to the analytic function given by (3.5) or (3.6), then the constants of integration C_i play a very important role. For example, if $p=1/2$, we get

$$f^{(1/2)}(z)=\cdots+C_3\frac{z^{-3.5}}{(-3.5)!}+C_2\frac{z^{-2.5}}{(-2.5)!}+C_1\frac{z^{-1.5}}{(-1.5)!}+\sum_{i=0}^{\infty}\frac{f^{(i)}(0)}{(i-0.5)!}z^{i-0.5} \tag{3.17}$$

and $C_1,C_2,C_3,\dots$ are essential because $(-1.5)!,(-2.5)!,\dots$ are different from ∞.

Note that we can develop a theory of fractional derivatives without assuming the importance of the constants of integration $C_1,C_2,\dots$, that is, only using the classical set theory of analytic functions. For example, we can define pth derivative of a periodic function

$$\sum_{n=0}^{\infty}a_n\sin nx+b_n\cos nx \tag{3.18}$$

by

$$\sum_{n=0}^{\infty}\left\{a_n\left[\sin\left(nx+p\frac{\pi}{2}\right)\right]+b_n\left[\cos\left(nx+p\frac{\pi}{2}\right)\right]\right\}n^p, \tag{3.19}$$

generalizing the known equalities

$$(\sin nx)^{(k)}=n^k\sin\left(nx+\frac{k\pi}{2}\right),\qquad(\cos nx)^{(k)}=n^k\cos\left(nx+\frac{k\pi}{2}\right) \tag{3.20}$$

for $k \in \mathbb{N}$. According to this theory, it is all right, but it means that intuitively we have accepted the following expansions of $\sin x$ and $\cos x$:

$$\cos x = \sum_{k=-\infty}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \qquad \sin x = \sum_{k=-\infty}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. \tag{3.21}$$

The following example shows that this theory is not meaningless, but an exact theory.

Example 3.2. In this theory, we show that the exponential function is much better and more naturally defined as

$$e^x = \sum_{n=-\infty}^{\infty} \frac{x^n}{n!} \tag{3.22}$$

instead of being defined by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{3.23}$$

For the half derivative of the right-hand side of (3.22), we obtain

$$(e^x)^{(1/2)} = \cdots + \frac{x^{-1.5}}{(-1.5)!} + \frac{x^{-0.5}}{(-0.5)!} + \frac{x^{0.5}}{(0.5)!} + \frac{x^{1.5}}{(1.5)!} + \cdots. \tag{3.24}$$

The right-hand side of (3.24) is a function f such that $f' = f$ and hence $f(z) = Ce^z$. We verify that $C = 1$, which means that $(e^x)^{(1/2)} = e^x$ and which is natural to expect. Indeed, we verify (3.24) for $x = 1$, that is, we prove that

$$\cdots + \frac{1}{(-1.5)!} + \frac{1}{(-0.5)!} + \frac{1}{(0.5)!} + \frac{1}{(1.5)!} + \frac{1}{(2.5)!} + \cdots = e. \tag{3.25}$$

Using the known equality $(-1/2)! = \sqrt{\pi}$ and the identity $(x+1)! = x!(x+1)$, (3.25) is equivalent to

$$\begin{gathered} \cdots - \frac{1\cdot 3\cdot 5}{2^3} + \frac{1\cdot 3}{2^2} - \frac{1}{2^1} + 1 + \frac{2^1}{1} + \frac{2^2}{1\cdot 3} + \frac{2^3}{1\cdot 3\cdot 5} + \cdots = e\sqrt{\pi}, \\ \frac{1}{2^1} - \frac{1\cdot 3}{2^2} + \frac{1\cdot 3\cdot 5}{2^3} - \frac{1\cdot 3\cdot 5\cdot 7}{2^4} + \cdots = -e\sqrt{\pi} + 1 + \frac{2^1}{1} + \frac{2^2}{1\cdot 3} + \frac{2^3}{1\cdot 3\cdot 5} + \cdots. \end{gathered} \tag{3.26}$$

The right side of (3.26) converges ordinarily, and it can be easily calculated. In order to calculate the sum on the left-hand side of (3.26), we apply the method of Section 2. Let

$$y = \frac{1}{2^1}x^3 - \frac{1\cdot 3}{2^2}x^5 + \frac{1\cdot 3\cdot 5}{2^3}x^7 - \frac{1\cdot 3\cdot 5\cdot 7}{2^4}x^9 + \cdots, \tag{3.27}$$

and we need the value $y(1)$. From (3.27) we obtain

$$\begin{aligned} y'(x) &= \frac{1\cdot 3}{2^1}x^2 - \frac{1\cdot 3\cdot 5}{2^2}x^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2^3}x^6 - \cdots \\ &= \frac{2}{x^3}\left(\frac{1\cdot 3}{2^2}x^5 - \frac{1\cdot 3\cdot 5}{2^3}x^7 + \frac{1\cdot 3\cdot 5\cdot 7}{2^4}x^9 - \cdots\right) \\ &= \frac{2}{x^3}\left(\frac{x^3}{2} - y\right) = 1 - \frac{2}{x^3}y \end{aligned} \tag{3.28}$$

and hence

$$y = e^{1/x^2}\left(C + \int_0^x e^{-1/t^2}\,dt\right). \tag{3.29}$$

Since $y(0) = 0$, we obtain $C = 0$. Thus,

$$\frac{1}{2^1} - \frac{1\cdot 3}{2^2} + \frac{1\cdot 3\cdot 5}{2^3} - \frac{1\cdot 3\cdot 5\cdot 7}{2^4} + \cdots = y(1) = e\left(\int_0^1 e^{-1/t^2}\,dt\right). \tag{3.30}$$

Substituting this equality into (3.26), we get

$$1 + \frac{2^1}{1} + \frac{2^2}{1\cdot 3} + \frac{2^3}{1\cdot 3\cdot 5} + \cdots = e\left(\sqrt{\pi} + \int_0^1 e^{-1/t^2}\,dt\right). \tag{3.31}$$

This identity is proved by classical methods in [4] and the proof there is much more complicated.

If we put $x = -1$ in (3.24), then, instead of (3.31), we can obtain the following equality of complex integration:

$$i\left(1 - \frac{2^1}{1} + \frac{2^2}{1\cdot 3} - \frac{2^3}{1\cdot 3\cdot 5} + \cdots\right) = e^{-1}\left(\sqrt{\pi} + \int_0^i e^{-1/t^2}\,dt\right), \tag{3.32}$$

where the integration is done over a curve with tangent vector at 0 toward the positive part of the x-axis. This identity is proved in [2].

At the end of this example, we consider the equality

$$e^x = \cdots + \frac{x^{-1.5}}{(-1.5)!} + \frac{x^{-0.5}}{(-0.5)!} + \frac{x^{0.5}}{(0.5)!} + \frac{x^{1.5}}{(1.5)!} + \cdots. \tag{3.33}$$

In the left-hand side, the function e^x is a single-valued function. Although the right-hand side seems not to be single-valued because $x^{n/2}$ takes two values; we show that the right-hand side is a single-valued function too. Indeed, considering the functions $x!$ as

$$x! = \Gamma(x+1) = \lim_{n\to\infty}\frac{n^x n!}{(1+x)(2+x)\cdots(n+x)}, \tag{3.34}$$

for $x = k + 1/2$ ($k \in \mathbb{Z}$), we notice that it also takes two values like $x^{n/2}$. Thus, if we choose appropriate signs of $x^{k/2}$ and $(k/2)!$, then equality (3.33) is true.

Example 3.3. The half derivatives of the functions

$$\sin x = \sum_{k=-\infty}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=-\infty}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \tag{3.35}$$

are

$$\begin{aligned} \sin\left(x + \frac{\pi}{4}\right) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{x^{2k+0.5}}{(2k+0.5)!}, \\ \cos\left(x + \frac{\pi}{4}\right) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{x^{2k-0.5}}{(2k-0.5)!}. \end{aligned} \tag{3.36}$$

In order to partially verify (3.36), we assume that $x \to \infty$, and we can numerically verify the following asymptotic convergences:

$$\begin{aligned} &\sum_{i=k}^{\infty} (-1)^i \frac{x^{2i+0.5}}{(2i+0.5)!} \sim \sin\left(x + \frac{\pi}{4}\right) \quad \text{for } x \longrightarrow \infty,\ k \longrightarrow -\infty, \\ &\sum_{i=k}^{\infty} (-1)^i \frac{x^{2i-0.5}}{(2i-0.5)!} \sim \cos\left(x + \frac{\pi}{4}\right) \quad \text{for } x \longrightarrow \infty,\ k \longrightarrow -\infty, \end{aligned} \tag{3.37}$$

or, more precisely,

$$\begin{aligned} &\lim_{k\to-\infty} \left[\lim_{x\to\infty} \left(\sin\left(x + \frac{\pi}{4}\right) - \sum_{i=k}^{\infty} (-1)^i \frac{x^{2i+0.5}}{(2i+0.5)!} \right) \right] = 0, \\ &\lim_{k\to-\infty} \left[\lim_{x\to\infty} \left(\cos\left(x + \frac{\pi}{4}\right) - \sum_{i=k}^{\infty} (-1)^i \frac{x^{2i-0.5}}{(2i-0.5)!} \right) \right] = 0. \end{aligned} \tag{3.38}$$

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