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# SOME VERSIONS OF ANDERSON'S AND MAHER'S INEQUALITIES I

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We prove the orthogonality (in the sense of Birkhoff) of the range and the kernel of an important class of elementary operators with respect to the Schatten $p$-class.

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**1. Introduction.** Let $H$ be a separable infinite-dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded operators on $H$ into itself. Given $A,B \in B(H)$, we define the generalized derivation $\delta_{A,B} : B(H) \mapsto B(H)$ by $\delta_{A,B}(X) = AX - XB$ and the elementary operator derivation $\Delta_{A,B} : B(H) \mapsto B(H)$ by $\Delta_{A,B}(X) = AXB - X$. Denote $\delta_{A,A} = \delta_A$ and $\Delta_{A,A} = \Delta_A$.

In [1, Theorem 1.7], Anderson shows that if $A$ is normal and commutes with $T$, then, for all $X \in B(H)$,

$$\left\|T + \delta_A(X)\right\| \geq \|T\|. \tag{1.1}$$

It is shown in [10] that if the pair $(A,B)$ has the Fuglede-Putnam property (in particular, if $A$ and $B$ are normal operators) and $AT = TB$, then, for all $X \in B(H)$,

$$\left\|T + \delta_{A,B}(X)\right\| \geq \|T\|. \tag{1.2}$$

Duggal [4] showed that the above inequality (1.2) is also true when $\delta_{A,B}$ is replaced by $\Delta_{A,B}$. The related inequality (1.1) was obtained by the author [11] showing that if the pair $(A,B)$ has the Fuglede-Putnam property $(\mathrm{FP})_{C_p}$, then

$$\left\|T + \delta_{A,B}(X)\right\|_p \geq \|T\|_p \tag{1.3}$$

for all $X \in B(H)$, where $C_p$ is the von Neumann-Schatten class, $1 \leq p < \infty$, and $\|\cdot\|_p$ is its norm for all $X \in B(H)$ and for all $T \in C_p \cap \ker \delta_{A,B}$. In all of the above results, $A$ was not arbitrary. In fact, certain normality-like assumptions have been imposed on $A$. A characterization of $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(\delta_A|C_p)$ (the range of $\delta_A|C_p$) for a general operator $A$, has been carried out by Kittaneh [7], showing that if $T$ has the polar decomposition $T = U|T|$, then

$$\left\|T+\delta_A(X)\right\|_p \geq \|T\|_p \tag{1.4}$$

for all $X \in C_p$ $(1 < p < \infty)$ if and only if $|T|^{p-1}U^* \in \ker\delta_A$. By a simple modification in the proof of the above inequality, we can prove that this inequality is also true in the general case, that is, if $T$ has the polar decomposition $T = U|T|$, then $\|T+\delta_{A,B}(X)\|_p \geq \|T\|_p$ for all $X \in C_p$ $(1 < p < \infty)$ if and only if $|T|^{p-1}U^* \in \ker\delta_{B,A}$. In Sections 1, 2, 3, and 4, we prove these results in the case where we consider $E_{A,B}$ instead of $\delta_{A,B}$, which leads us to prove that if $T \in C_p$ and $\ker E_{A,B} \subseteq \ker E^*_{A,B}$, then

$$\left\|T+E_{A,B}(X)\right\|_p \geq \|T\|_p \tag{1.5}$$

for all $X \in C_p$ $(1 < p < \infty)$ if and only if $T \in \ker E_{A,B}$. In Sections 5 and 6, we minimize the map $\|S+E_{A,B}(X)\|_p$ and we classify its critical points.

**2. Preliminaries.** Let $T \in B(H)$ be compact and let $s_1(X) \geq s_2(X) \geq \cdots \geq 0$ denote the singular values of $T$, that is, the eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in their decreasing order. The operator $T$ is said to belong to the Schatten $p$-class $C_p$ if

$$\|T\|_p = \left[\sum_{i=1}^{\infty} s_j(T)^p\right]^{1/p} = \left[\operatorname{tr}(T)^p\right]^{1/p}, \quad 1 \leq p < \infty, \tag{2.1}$$

where tr denotes the trace functional. Hence, $C_1$ is the trace class, $C_2$ is the Hilbert-Schmidt class, and $C_\infty$ is the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\| \tag{2.2}$$

denoting the usual operator norm. For the general theory of the Schatten $p$-classes, the reader is referred to [8, 13].

Recall that the norm $\|\cdot\|$ of the $B$-space $V$ is said to be Gateaux differentiable at nonzero elements $x \in V$ if

$$\lim_{t\to 0,\, t\in\mathbb{R}} \frac{\|x+ty\|-\|x\|}{t} = \Re D_x(y) \tag{2.3}$$

for all $y \in V$. Here $\mathbb{R}$ denotes the set of reals, $\Re$ denotes the real part, and $D_x$ is the unique support functional (in the dual space $V^*$) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$. The Gateaux differentiability of the norm at $x$ implies that $x$ is a smooth point of the sphere of radius $\|x\|$.

It is well known (see [8] and the references therein) that, for $1 < p < \infty$, $C_p$ is a uniformly convex Banach space. Therefore, every nonzero $T \in C_p$ is a smooth point and, in this case, the support functional of $T$ is given by

$$D_T(X) = \operatorname{tr}\left[\frac{|T|^{p-1}UX^*}{\|T\|_p^{p-1}}\right] \tag{2.4}$$

for all $X \in C_p$, where $T = U|T|$ is the polar decomposition of $T$.

**Definition 2.1.** Let $E$ be a complex Banach space. We define the orthogonality in $E$. We say that $b \in E$ is orthogonal to $a \in E$ if, for all complex $\lambda$, there holds

$$\|a + \lambda b\| \geq \|a\|. \tag{2.5}$$

This definition has a natural geometric interpretation, namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, that is, if and only if this complex line is a tangent one. Note that if $b$ is orthogonal to $a$, then $a$ needs not be orthogonal to $b$. If $E$ is a Hilbert space, then from (2.5), it follows that $\langle a, b\rangle = 0$, that is, orthogonality in the usual sense.

## 3. Main results.

In this section, we characterize $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(\Delta_{A,B}|C_p)$ (the range of $\Delta_{A,B}|C_p$) for a general pair of operators $A$, $B$.

**Lemma 3.1** [7]. *Let $u$ and $v$ be two elements of a Banach space $V$ with norm $\|\cdot\|$. If $u$ is a smooth point, then $D_u(v) = 0$ if and only if*

$$\|u + zv\| \geq \|u\| \tag{3.1}$$

*for all $z \in \mathbb{C}$ (the complex numbers).*

**Theorem 3.2.** *Let $A, B \in B(H)$ and $T \in C_p$ $(1 < p < \infty)$. Then*

$$\left\|T + \Delta_{A,B}(X)\right\|_p \geq \|T\|_p \tag{3.2}$$

*for all $X \in B(H)$ with $\Delta_{A,B}(X) \in C_p$ if and only if $\operatorname{tr}(|T|^{p-1}U^*\Delta_{A,B}(X)) = 0$ for all such $X$.*

**Proof.** The theorem is an immediate consequence of equality (2.4) and Lemma 3.1. □

**Theorem 3.3.** *Let $A, B \in B(H)$ and $T \in C_p$ $(1 < p < \infty)$. Then*

$$\left\|T + \Delta_{A,B}(X)\right\|_p \geq \|T\|_p \tag{3.3}$$

*for all $X \in C_p$ if and only if $\tilde{T} = |T|^{p-1}U^* \in \ker \Delta_{B,A}$.*

**Proof.** By virtue of Theorem 3.2, it is sufficient to show that $\operatorname{tr}(\widetilde{T}\Delta_{A,B}(X)) = 0$ for all $X \in C_p$ if and only if $\widetilde{T} \in \ker\Delta_{B,A}$.

Choose $X$ to be the rank-one operator $f \otimes g$ for some arbitrary elements $f$ and $g$ in $H$; then $\operatorname{tr}(\widetilde{T}(AXB - X)) = \operatorname{tr}((B\widetilde{T}A - \widetilde{T})X) = 0$ implies that $\langle \Delta_{B,A}(\widetilde{T})f, g\rangle = 0 \Leftrightarrow \widetilde{T} \in \ker\Delta_{B,A}$. Conversely, assume that $\widetilde{T} \in \ker\Delta_{B,A}$, that is, $B\widetilde{T}A = \widetilde{T}$.

Since $\widetilde{T}X$ and $\widetilde{T}\Delta_{B,A}$ are trace classes for all $X \in C_p$, we get

$$\begin{aligned} \operatorname{tr}(\widetilde{T}(AXB - X)) &= \operatorname{tr}(\widetilde{T}AXB - \widetilde{T}X) \\ &= \operatorname{tr}(XB\widetilde{T}A - X\widetilde{T}) \\ &= \operatorname{tr}(X\Delta_{B,A}(\widetilde{T})) = 0. \end{aligned} \tag{3.4}$$

□

**Lemma 3.4.** *Let $A, B \in B(H)$ and $S \in B(H)$ such that $\ker\Delta_{A,B} \subseteq \ker\Delta_{A^*,B^*}$. If $AU|S|^{p-1}B = U|S|^{p-1}$, where $p > 1$ and $S = U|S|$ is the polar decomposition of $S$, then $AU|S|B = U|S|$.*

**Proof.** If $T = |S|^{p-1}$, then

$$AUTB = UT. \tag{3.5}$$

We prove that

$$AUT^nB = UT^n. \tag{3.6}$$

If $ATB = T = A^*TB^*$, then $BT^*T = BT^*ATB = T^*TB$, and thus $B|T| = |T|B$ and $BT^2 = T^2B$. Since $B$ commutes with the positive operator $T^2$, then $B$ commutes with its square roots, that is,

$$BT = TB. \tag{3.7}$$

By (3.5) and (3.7) we obtain (3.6). Let $f(t)$ be the map defined on $\sigma(T) \subset R^+$ by

$$f(t) = t^{1/(p-1)}, \quad 1 < p < \infty. \tag{3.8}$$

Since $f$ is the uniform limit of a sequence $(P_i)$ of polynomials without constant term (since $f(0) = 0$), it follows from (3.3) that $AUP_i(T)B = UP_i(T)$. Therefore, $AUT^{1/(p-1)}B = UT^{1/(p-1)}$. □

**Theorem 3.5.** *Let $A$ and $B$ be operators in $B(H)$ such that $\ker\Delta_{A,B} \subseteq \ker\Delta_{A^*,B^*}$. Then $T \in \ker\Delta_{A,B} \cap C_p$ if and only if*

$$\left\|S + \Delta_{A,B}(X)\right\|_p \geq \|S\|_p \tag{3.9}$$

*for all $X \in C_p$.*

**Proof.** If $S \in \ker \Delta_{A,B}$, then, by applying [11, Theorem 3.4], it follows that

$$\left\|S+\Delta_{A,B}(X)\right\|_p \geq \|S\|_p \tag{3.10}$$

for all $X \in C_p$. Conversely, if

$$\left\|S+\Delta_{A,B}(X)\right\|_p \geq \|S\|_p \tag{3.11}$$

for all $X \in C_p$, then, from Theorem 3.3, $A|S|^{p-1}U^*B = |S|^{p-1}U^*$. Since $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$, $A^*|S|^{p-1}U^*B^* = |S|^{p-1}U^*$. By taking adjoints, we get $AU|S|^{p-1}B = U|S|^{p-1}$. From Lemma 3.4, it follows that $AU|S|B = U|S|$. That is, $S \in \ker \Delta_{A,B}$. □

**Theorem 3.6.** *Let $A,B \in B(H)$. If*
(1) *$A,B \in \mathcal{L}(H)$ such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in \mathcal{H}$,*
(2) *$A$ is invertible and $B$ is such that $\|A^{-1}\|\|B\| \leq 1$,*
(3) *$A = B$ is a cyclic subnormal operator,*
*then, $T \in \ker \Delta_{A,B} \cap C_p$ if and only if*

$$\left\|S+\Delta_{A,B}(X)\right\|_p \geq \|S\|_p \tag{3.12}$$

*for all $X \in C_p$.*

**Proof.** The result of Tong [14, Lemma 1] guarantees that the above condition implies that for all $T \in \ker(\delta_{A,B}|\mathcal{K}(\mathcal{H}))$, $\overline{R(T)}$ reduces $A$, $\ker(T)^\perp$ reduces $B$, and $A|_{\overline{R(T)}}$ and $B|_{\ker(T)^\perp}$ are unitary operators. Take $\mathcal{H}_1 = \mathcal{H} = \overline{\operatorname{ran} S} \oplus \overline{\operatorname{ran} S}^\perp$ and $\mathcal{H}_2 = \mathcal{H} = \ker S \oplus \ker S^\perp$. According to the decomposition of $\mathcal{H}$ and for $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$, $A_2 : \mathcal{H}_2 \to \mathcal{H}_2$, and $S : \mathcal{H}_2 \to \mathcal{H}_1$, we can write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B^* = \begin{pmatrix} B_1^* & 0 \\ 0 & B_2^* \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.13}$$

From $ASB = S$, it follows that $A_1SB_1 = S$, and since $A_1$ and $B_1$ are unitary operators, we obtain $A_1^*SB_1^* = S$, and the result holds by the above theorem.

The above inequality holds in particular if $A = B$ is isometric; in other words, $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$.

(2) In this case, it suffices to take $A_1 = \|B\|^{-1}A$ and $B_1 = \|B\|^{-1}B$, then $\|A_1x\| \geq \|x\| \geq \|B_1x\|$, and the result holds by (1) for all $x \in \mathcal{H}$.

(3) Since $T$ commutes with $A$, it follows that $T$ is subnormal [15]. But any compact subnormal operator is normal; hence, $T$ is normal. By applying Fuglede-Putnam theorem, we get that $ATA = T$ implies $A^*TA^* = T$. □

## 4. The case where $n > 1$.

Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be $n$-tuples of operators in $B(H)$. In this section, we characterize $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(E_{A,B}|C_p)$ (the range of $E_{A,B}|C_p$) for a general pair of operators $A$ and $B$.

By the same argument used in the proofs of Theorems 3.2 and 3.3, we prove the following theorems.

**Theorem 4.1.** *Let $A=(A_1,A_2,\dots,A_n)$ and $B=(B_1,B_2,\dots,B_n)$ be $n$-tuples of operators in $B(H)$ and $T\in C_p$ $(1<p<\infty)$. Then*

$$\left\|T+E_{A,B}(X)\right\|_p \geq \|T\|_p \tag{4.1}$$

*for all $X\in B(H)$ with $E_{A,B}(X)\in C_p$ if and only if $\operatorname{tr}(|T|^{p-1}U^*E_{A,B}(X))=0$ for all such $X$.*

**Theorem 4.2.** *Let $A=(A_1,A_2,\dots,A_n)$ and $B=(B_1,B_2,\dots,B_n)$ be $n$-tuples of operators in $B(H)$ and $T\in C_p$ $(1<p<\infty)$. Then*

$$\left\|T+E_{A,B}(X)\right\|_p \geq \|T\|_p \tag{4.2}$$

*for all $X\in C_p$ if and only if $\tilde{T}=|T|^{p-1}U^*\in\ker E_{A,B}$.*

**Lemma 4.3.** *Let $C=(C_1,C_2,\dots,C_n)$ be $n$-tuple of operators in $B(H)$ such that $\sum_{i=1}^n C_iC_i^*\leq 1$, $\sum_{i=1}^n C_i^*C_i\leq 1$, and $\ker E_C\subseteq\ker E_{C^*}$. If*

$$\sum_{i=1}^n C_iU|S|^{p-1}C_i=U|S|^{p-1}, \tag{4.3}$$

*where $p>1$ and $S=U|S|$ is the polar decomposition of $S$, then*

$$\sum_{i=1}^n C_iU|S|C_i=U|S|. \tag{4.4}$$

**Proof.** If $T=|S|^{p-1}$, then

$$\sum_{i=1}^n C_iUTC_i=UT. \tag{4.5}$$

We prove that

$$\sum_{i=1}^n C_iUT^nC_i=UT^n. \tag{4.6}$$

It is known that if $\sum_{i=1}^n C_iC_i^*\leq 1$, $\sum_{i=1}^n C_i^*C_i\leq 1$, and $\ker E_c\subseteq\ker E_c^*$, then the eigenspaces corresponding to distinct nonzero eigenvalues of the compact positive operator $|S|^2$ reduce each $C_i$ (see [3, Theorem 8], [14, Lemma 2.3]). In particular, $|S|$ commutes with $C_i$ for all $1\leq i\leq n$. This implies also that $|S|^{p-1}=T$ commutes with each $C_i$ for all $1\leq i\leq n$. Hence $C_i|T|=|T|C_i$ and $C_iT^2=T^2C_i$.

Since $C_i$ commutes with the positive operator $T^2$, then $C_i$ commutes with its square roots, that is,

$$C_iT = TC_i. \tag{4.7}$$

By the same arguments used in the proof of Lemma 3.4, the proof of this lemma can be completed. □

**Theorem 4.4.** *Let $C = (C_1, C_2, \dots, C_n)$ be $n$-tuple of operators in $B(H)$ such that $\sum_{i=1}^n C_iC_i^* \le 1$, $\sum_{i=1}^n C_i^*C_i \le 1$, and $\ker E_C \subseteq \ker E_{C^*}$. Then $S \in \ker E_C \cap C_p$ $(1 < p < \infty)$ if and only if*

$$\|S + E_C(X)\|_p \ge \|S\|_p \tag{4.8}$$

*for all $X \in C_p$.*

**Proof.** If $S \in \ker E_C$, then, from [14, Theorem 2.4], it follows that $\|S + E_C(X)\|_p \ge \|S\|_p$ for all $X \in C_p$. Conversely, if $\|S + E_C(X)\|_p \ge \|S\|_p$ for all $X \in C_p$, then, from Theorem 4.2, $\sum_{i=1}^n C_i|S|^{p-1}U^*C_i = |S|^{p-1}U^*$. Since $\ker E_C \subseteq \ker E_{C^*}$, $\sum_{i=1}^n C_i^*|S|^{p-1}U^*C_i^* = |S|^{p-1}U^*$. Taking adjoints, we get $\sum_{i=1}^n C_iU \times |S|^{p-1}C_i = U|S|^{p-1}$, and from Lemma 4.3, it follows that $\sum_{i=1}^n C_iU|S|C_i = U|S|$, that is, $S \in \ker E_C$. □

**Theorem 4.5.** *Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ be $n$-tuples of operators in $B(H)$ such that $\sum_{i=1}^n A_iA_i^* \le 1$, $\sum_{i=1}^n A_i^*A_i \le 1$, $\sum_{i=1}^n B_iB_i^* \le 1$, $\sum_{i=1}^n B_i^*B_i \le 1$, and $\ker E_{A,B} \subseteq \ker E_{A^*,B^*}$.*

*Then $T \in \ker E_{A,B} \cap C_p$ if and only if*

$$\|S + E_{A,B}(X)\|_p \ge \|S\|_p \tag{4.9}$$

*for all $X \in C_p$.*

**Proof.** It suffices to take the Hilbert space $H \oplus H$ and the operators

$$C_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \tag{4.10}$$

and apply Theorem 4.4. □

**5. Remarks.** (1) It is known (see [8] and the references therein) that the smooth points of $K(H)$ are those compact operators that attain their norm at a unique (up to multiplication by a constant of modulus one) unit vector. It has been shown in [8] that a nonzero $T \in B(H)$ is a smooth point if and only if $T$ attains its norm at a unique (up to multiplication by a constant of modulus one) unit vector $e \in H$ and $\|T\|_e \le \|T\|$, where $\|T\|_e$ is the essential norm of $T$, that is, the norm of $\pi(T)$, where $\pi$ is the quotient map of $B(H)$ onto $B(H)/K(H)$. In this case,

$$D_T(X) = \operatorname{tr}\left[\frac{(e\otimes Te)}{\|T\|}X\right] = \left\langle Xe, \frac{Te}{\|T\|}\right\rangle \tag{5.1}$$

for all $X \in B(H)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product on $H$ and $e\otimes Te$ is the rank-one operators defined by $(e\otimes Te)f = \langle f, Te\rangle e$ for all $f \in H$.

Hence, for the usual operator norm, Theorems 3.2, 3.3, 4.1, and 4.2 can be combined in the following formulation. Let $A, B \in B(H)$ and $T \in B(H)$ be a smooth point. If $\tilde{T} = e\otimes Te$, then the following statements are equivalent:

(i) $\|T + E_{A,B}(X)\| \geq \|T\|$ for all $X \in B(H)$,

(ii) $\operatorname{tr}(T^{\sim}E_{A,B}(X)) = 0$ for all $X \in B(H)$,

(iii) $\tilde{T} \in \ker E_{A,B}$.

(2) It is still possible to give a characterization similar to this given in the usual operator norm for the norm $\|\cdot\|_\infty$. However, in this case, we have to assume that $T$ is a smooth point, that is, the given norm is Gateaux differentiable at $T$ and $\tilde{T} = e\otimes Te$, where $e$ is the unique (up to multiplication by a constant of modulus one) unit vector at which $T$ attains its norm.

(3) It is well known that the Hilbert-Schmidt class $C_2$ is a Hilbert space under the inner product $\langle Y, Z\rangle = \operatorname{tr} Z^*Y$.

We remark here that, for the Hilbert Schmidt norm $\|\cdot\|_2$, the orthogonality results in Theorems 3.3, 3.5, 4.1, and 4.2 are to be understood in the usual Hilbert-space sense. Note in the case $\tilde{T} = |T|U^* = T^*$ that

$$\|T + E_{A,B}(X)\|_2^2 = \|E_{A,B}(X)\|_2^2 + \|T\|_2^2 \tag{5.2}$$

for all $X \in C_2$ if and only if $T^* \in \ker E_{A,B}$.

(4) Theorem 4.4 does not hold in the case $0 < p \leq 1$ because the functional calculus argument involving the function $t \mapsto t^{1/(p-1)}$, where $0 \leq t < \infty$, is only valid for $1 < p < \infty$. We ask if there is another proof where this theorem still holds in the case $0 < p < 1$. For the case $p = 1$, this theorem still holds see [12, Theorem 2.3].

**6. On minimizing $\|T - (AXB - X)\|_p^p$.** Maher [9, Theorem 3.2] showed that, if $A$ is normal, $AT = TA$, $1 \leq p < \infty$, and $S \in \ker\delta_{A,B} \cap C_p$; then the map $F_p$ defined by $F_p(X) = \|S - (AX - XA)\|_p^p$ has a global minimizer at $V$ if, and for $1 < p < \infty$ only if, $AV - VA = 0$. In other words, we have

$$\|S - (AX - XA)\|_p^p \geq \|T\|_p^p \tag{6.1}$$

if, and for $1 < p < \infty$ only if, $AV - VA = 0$. In [10] we generalized Maher's result, showing that if the pair $(A, B)$ has the property $(\mathrm{FP})_{C_p}$, that is, ($AT = TB$, where $T \in C_p$ implies $A^*T = TB^*$), $1 \leq p < \infty$ and $S \in \ker\delta_{A,B} \cap C_p$, then the map $F_p$

defined by $F_p(X) = \|S - (AX - XB)\|_p^p$ has a global minimizer at $V$ if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In other words, we have

$$\|S - (AX - XB)\|_p^p \geq \|T\|_p^p \tag{6.2}$$

if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In this paper, we obtain an inequality similar to (6.1), where the operator $AX - XB$ is replaced by the operator $\Delta_{A,B}(X) = AXB - X$ (in the case $n = 1$). We prove that if $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$ and $T \in \ker \Delta_{A,B} \cap C_p$, then the map $F_p$ defined by $F_p(X) = \|T - (AXB - X)\|_p^p$ has a global minimizer at $V$ if, and for $1 < p < \infty$ only if, $AVB - V = 0$. In other words, we have

$$\|T - (AXB - X)\|_p^p \geq \|T\|_p^p \tag{6.3}$$

if, and for $1 < p < \infty$ only if, $AVB - V = 0$. Additionally, we show that if $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$ and $T \in \ker \Delta_{A,B} \cap C_p$, $1 < p < \infty$, then the map $F_p$ has a critical point at $W$ if and only if $AWB - W = 0$, that is, if $\mathcal{D}_W F_p$ is the Frechet derivative at $W$ of $F_p$, the set

$$\{W \in \mathcal{B}(H) : \mathcal{D}_W F_p = 0\} \tag{6.4}$$

coincides with $\ker \Delta_{A,B}$ (the kernel of $\Delta_{A,B}$).

**Theorem 6.1** [2]. *If $1 < p < \infty$, then the map*

$$F_p : C_p \longmapsto \mathbb{R}^+ \tag{6.5}$$

*defined by $X \mapsto \|X\|_p^p$ is differentiable at every $X \in C_p$ with derivative $\mathcal{D}_X F_p$ given by*

$$\mathcal{D}_X F_p(T) = p \operatorname{Re} \operatorname{tr} \left( |X|^{p-1} U^* T \right), \tag{6.6}$$

*where $\operatorname{tr}$ denotes trace, $\operatorname{Re} z$ is the real part of a complex number $z$, and $X = U|X|$ is the polar decomposition of $X$. If $\dim \mathcal{H} < \infty$, then the same result holds for $0 < p \leq 1$ at every invertible $X$.*

**Theorem 6.2** [6]. *If $\mathcal{U}$ is a convex set of $C_p$ with $1 < p < \infty$, then the map $X \mapsto \|X\|_p^p$, where $X \in \mathcal{U}$, has at most a global minimizer.*

**Definition 6.3.** Let $\mathcal{U}(A,B) = \{X \in B(H) : AXB - X \in C_p\}$ and let $F_p : \mathcal{U} \mapsto \mathbb{R}^+$ be the map defined by $F_p(X) = \|T - (AXB - X)\|_p^p$, where $T \in \ker \Delta_{A,B} \cap C_p$ $(1 \leq p < \infty)$.

By a simple modification in the proof of Lemma 4.3, we can proof the following lemma.

**Lemma 6.4.** *Let $A, B \in B(H)$ and $S \in B(H)$ such that $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$. If $A|S|^{p-1}U^*B = |S|^{p-1}U^*$, where $p > 1$ and $S = U|S|$ is the polar decomposition of $S$, then $A|S|U^*B = |S|U^*$.*

**Theorem 6.5.** *Let $A,B \in B(H)$. If $\ker\Delta_{A,B} \subseteq \ker\Delta_{A^*,B^*}$ and $T \in \ker\Delta_{A,B} \cap C_p$, then, for $1 \le p < \infty$, the map $F_p$ has a global minimizer at $W$ if, and for $1 < p < \infty$ only if, $AWB - W = 0$.*

**Proof.** If $\ker\Delta_{A,B} \subseteq \ker\Delta_{A^*,B^*}$, then it follows from Theorem 3.5 that $\|T-(AXB-X)\|_p^p \ge \|T\|_p^p$, that is, $F_p(X) \ge F_p(W)$. Conversely, if $F_p$ has a minimum, then

$$||T-(AWB-WB)||_p^p = \|S\|_p^p. \tag{6.7}$$

Since $\mathcal{U}$ is convex, then the set $\mathcal{V} = \{T-(AXB-X);\ X \in \mathcal{U}\}$ is also convex. Thus Theorem 6.2 implies that $S-(AWB-W) = S$. □

**Theorem 6.6.** *Let $A,B \in B(H)$. If $\ker\Delta_{A,B} \subseteq \ker\Delta_{A^*,B^*}$ and $S \in \ker\Delta_{A,B} \cap C_p$, then, for $1 < p < \infty$, the map $F_p$ has a critical point at $W$ if and only if $AWB - W = 0$.*

**Proof.** Let $W,S \in \mathcal{U}$ and let $\phi$ and $\varphi$ be two maps defined, respectively, by $\phi : X \mapsto S-(AXB-X)$ and $\varphi : X \mapsto \|X\|_p^p$.

Since the Frechet derivative of $F_p$ is given by

$$\mathfrak{D}_W F_p(T) = \lim_{h\to 0} \frac{F_p(W+hT)-F_p(W)}{h}, \tag{6.8}$$

it follows that $\mathfrak{D}_W F_p(T) = [\mathfrak{D}_{S-(AWB-W)}](ATB-T)$. If $W$ is a critical point of $F_p$, then $\mathfrak{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying Theorem 6.1, we get

$$\begin{aligned}\mathfrak{D}_W F_p(T) &= p\operatorname{Re}\operatorname{tr}\left[\,|S-(AWB-W)|^{p-1}W^*(ATB-T)\right]\\ &= p\operatorname{Re}\operatorname{tr}\left[Y(ATB-T)\right] = 0,\end{aligned} \tag{6.9}$$

where $S-(AWB-W) = W|S-(AWB-W)|$ is the polar decomposition of the operator $S-(AWB-W)$, and $Y = |S-(AWB-W)|^{p-1}W^*$.

An easy calculation shows that $AYB - Y = 0$, that is,

$$A\,|S-(AWB-W)|^{p-1}W^*B = |S-(AWB-W)|^{p-1}W^*. \tag{6.10}$$

It follows from Lemma 6.4 that

$$A\,|S-(AWB-W)|\,W^*B = |S-(AWB-W)|\,W^*. \tag{6.11}$$

By taking adjoints and since $\ker\Delta_{A,B} \subseteq \ker\Delta_{A^*,B^*}$, we get $A(T-(AWB-W))B = (T-(AWB-W))$. Then $A(AWB-W)B = (AWB-W)$.

Hence $AWB-W \in R(\Delta_{A,B}) \cap \ker\Delta_{A,B}$. It is easy to see that (arguing as in the proof of Theorem 3.5) if $A,B \in \mathcal{B}(H)$, $\ker\Delta_{A,B} \subseteq \ker\Delta_{A^*,B^*}$, and $T \in \ker\Delta_{A,B}$, where $T \in \mathcal{B}(H)$, then

$$\|T - (AXB - X)\| \geq \|T\| \tag{6.12}$$

holds for all $X \in \mathcal{B}(H)$ and for all $T \in \ker \Delta_{A,B}$. Hence $AWB - W = 0$.

Conversely, if $AWB = W$, then $W$ is a minimum, and since $F_p$ is differentiable, then $W$ is a critical point. □

**Theorem 6.7.** *Let $A, B \in \mathcal{B}(H)$ such that $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$, $S \in \ker \Delta_{A,B} \cap C_p$ $(0 < p \leq 1)$, $\dim \mathcal{H} < \infty$, and $S - (AWB - W)$ is invertible. Then $F_p$ has a critical point at $W$ if $AWB - W = 0$.*

**Proof.** Suppose that $\dim \mathcal{H} < \infty$. If $AWB - W = 0$, then $S$ is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$. If we take

$$Y = |S|^{p-1} U^* \tag{6.13}$$

with $S = U|S|$ the polar decomposition and since $ASB = S$ implies $BS^*A = S^*$, then $AS^*S = AS^*BSA = S^*SA$, and this implies that $|S|^2 A = A|S|^2$ and $|S|A = A|S|$.

Since $BS^*A = S^*$, that is, $A|S|U^*B = |S|U^*$, $|S|(AU^*B - U^*) = 0$, and since $A|S|^{p-1} = |S|^{p-1}A$, then

$$AYB - Y = A|S|^{p-1}U^*B - |S|^{p-1}U^* = |S|^{p-1}(AU^*B - U^*) \tag{6.14}$$

so that $AYB - Y = 0$ and $\operatorname{tr}[(AYB - Y)T] = 0$ for all $T \in B(H)$. Since $S = S - (AWB - W)$, then

$$\begin{aligned} 0 &= \operatorname{tr}[YATB - YAT] = \operatorname{tr}[Y(ATB - T)] \\ &= p \operatorname{Re}\operatorname{tr}[Y(ATB - AT)] = p \operatorname{Re}\operatorname{tr}[|S|^{p-1}U^*(ATB - T)] \\ &= (\mathcal{D}_T \phi)(ATB - T) = (\mathcal{D}_W F_p)(T). \end{aligned} \tag{6.15}$$

□

**Remark 6.8.** (1) In Theorem 6.6, the implication "$W$ is a critical point $\Rightarrow$ $AWB - WB = 0$" does not hold in the case $0 < p \leq 1$ because the functional calculus argument involving the function $t \mapsto t^{1/(p-1)}$, where $0 \leq t < \infty$, is only valid for $1 < p < \infty$.

(2) Theorems 3.5, 6.5, 6.6, and 6.7 hold in particular if $A$ and $B$ are contractions. Indeed, it is known [4] that if $A$ and $B$ are contractions and $\Delta_{A,B}(S) = 0$, where $S \in C_p$, then $\Delta_{A^*,B^*}(S) = \delta_{A^*,B}(S) = \delta_{A,B^*}(S) = 0$.

(3) The set

$$\mathcal{S} = \{X : AXB - X \in C_p\} \tag{6.16}$$

contains $C_p$ for if $X \in C_p$, then $X \in \mathcal{S}$ and, for example, $I \in \mathcal{S}$ but $I \notin C_p$. If $A \in C_p$, the conclusions of Theorems 6.5, 6.6, and 6.7 hold for all $X \in B(H)$.

**7. On minimizing** $\|T-(\sum_{i=1}^{n}A_iXB_i-X)\|_p^p$**.** Let $A=(A_1,A_2,\dots,A_n)$ and $B=(B_1,B_2,\dots,B_n)$ be $n$-tuples of operators in $B(H)$. We define the elementary operator $E_{A,B}:B(H)\mapsto B(H)$ by $E_{A,B}(X)=\sum_{i=1}^{n}A_iXB_i-X$.

Denote $E_{A,A}=E_A$. In this section, we prove that if $\sum_{i=1}^{n}A_iA_i^*\le 1$, $\sum_{i=1}^{n}A_i^*A_i \le 1$, $\sum_{i=1}^{n}B_iB_i^*\le 1$, $\sum_{i=1}^{n}B_i^*B_i\le 1$, $\ker E_{A,B}\subseteq \ker E_{A^*,B^*}$, and $T\in\ker\Delta_{A,B}\cap C_p$, then the map $F_p$ defined by $F_p(X)=\|T-E_{A,B}(X)\|_p^p$ has a global minimizer at $V$ if, and for $1<p<\infty$ only if, $\sum_{i=1}^{n}A_iVB_i-V=0$. In other words, we have

$$\left\|T-E_{A,B}(X)\right\|_p^p\ge\|T\|_p^p \tag{7.1}$$

if, and for $1<p<\infty$ only if, $\sum_{i=1}^{n}A_iVB_i-V=$. Additionally, we show that if $\ker E_{A,B}\subseteq E_{A^*,B^*}$ and $T\in\ker E_{A,B}\cap C_p$ $(1<p<\infty)$, then the map $F_p$ has a critical point at $W$ if and only if $\sum_{i=1}^{n}A_iWB_i-W=0$, that is, if $D_WF_p$ is the Frechet derivative of $F_p$ at $W$, the set

$$\{W\in L(H):D_WF_p=0\} \tag{7.2}$$

coincides with $\ker E_{A,B}$ (the kernel of $E_{A,B}$).

**Definition 7.1.** Let $\mathcal{U}(A,B)=\{X\in B(H):(\sum_{i=1}^{n}C_iXC_i-X)\in C_p\}$ and let $F_p:\mathcal{U}\mapsto R^+$ be the map defined by $F_p(X)=\|T-(\sum_{i=1}^{n}C_iXC_i-X)\|_p^p$, where $T\in\ker E_C\cap C_p$ $(1\le p<\infty)$.

**Lemma 7.2.** *Let $C=(C_1,C_2,\dots,C_n)$ be $n$-tuple of operators in $B(H)$ such that $\sum_{i=1}^{n}C_iC_i^*\le 1$, $\sum_{i=1}^{n}C_i^*C_i\le 1$, and $\ker E_c\subseteq\ker E_c^*$. If $\sum_{i=1}^{n}C_i|S|^{p-1}U^*C_i=|S|^{p-1}U^*$, where $p>1$ and $S=U|S|$ is the polar decomposition of $S$, then $\sum_{i=1}^{n}C_i|S|U^*C_i=|S|U^*$.*

**Proof.** By the same arguments as in the proof of Lemma 4.3, the proof can be completed. □

**Theorem 7.3.** *Let $C=(C_1,C_2,\dots,C_n)$ be $n$-tuple of operators in $B(H)$. If $\sum_{i=1}^{n}C_iC_i^*\le 1$, $\sum_{i=1}^{n}C_i^*C_i\le 1$, $\ker E_C\subseteq\ker E_{C^*}$, and $T\in\ker\Delta_{A,B}\cap C_p$, then, for $1\le p<\infty$, the map $F_p$ has a global minimizer at $W$ if, and for $1<p<\infty$ only if, $\sum_{i=1}^{n}C_iWC_i-W=0$.*

**Proof.** If $\sum_{i=1}^{n}C_iC_i^*\le 1$, $\sum_{i=1}^{n}C_i^*C_i\le 1$, and $\ker E_c\subseteq\ker E_c^*$, it follows from Theorem 4.4 that

$$\left\|T-\left(\sum_{i=1}^{n}C_iXC_i-X\right)\right\|_p^p\ge\|T\|_p^p, \tag{7.3}$$

that is, $F_p(X)\ge F_p(W)$. Conversely, if $F_p$ has a minimum, then

$$\left\|T-\left(\sum_{i=1}^{n}C_iWC_i-W\right)\right\|_p^p=\|T\|_p^p. \tag{7.4}$$

Since $\mathcal{U}$ is convex, then the set

$$\mathcal{V} = \left\{T - \left(\sum_{i=1}^{n} C_i X C_i - X\right);\ X \in \mathcal{U}\right\} \tag{7.5}$$

is also convex. Thus Theorem 6.2 implies that $T - (\sum_{i=1}^{n} C_i W C_i - W) = T$. □

**Theorem 7.4.** *Let $C = (C_1, C_2, \dots, C_n)$ be $n$-tuple of operators in $B(H)$. If $\sum_{i=1}^{n} C_i C_i^* \le 1$, $\sum_{i=1}^{n} C_i^* C_i \le 1$, $\ker E_c \subseteq \ker E_c^*$, and $T \in \ker E_C \cap C_p$, then, for $1 \le p < \infty$, the map $F_p$ has a critical point at $W$ if, and for $1 < p < \infty$ only if,*

$$\sum_{i=1}^{n} C_i W C_i - W = 0. \tag{7.6}$$

**Proof.** Let $W, S \in U$ and let $\phi$ and $\varphi$ be two maps defined, respectively, by

$$\phi : X \longmapsto S - \left(\sum_{i=1}^{n} C_i X C_i - X\right), \qquad \varphi : X \longmapsto \|X\|_p^p. \tag{7.7}$$

Since the Frechet derivative of $F_p$ is given by

$$\mathfrak{D}_W F_p(T) = \lim_{h \to 0} \frac{F_p(W + hT) - F_p(W)}{h}, \tag{7.8}$$

it follows that

$$\mathfrak{D}_W F_p(T) = \left[\mathfrak{D}_{S - (\sum_{i=1}^{n} C_i W C_i - W)}\right]\left(\sum_{i=1}^{n} C_i T C_i - T\right). \tag{7.9}$$

If $W$ is a critical point of $F_p$, then $\mathfrak{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying Theorem 6.1, we get

$$\begin{aligned} \mathfrak{D}_W F_p(T) &= p \operatorname{Re} \operatorname{tr}\left[\left|S - \left(\sum_{i=1}^{n} C_i W C_i - W\right)\right|^{p-1} W^* \left(\sum_{i=1}^{n} C_i T C_i - T\right)\right] \\ &= p \operatorname{Re} \operatorname{tr}\left[Y\left(\sum_{i=1}^{n} C_i T C_i - T\right)\right] = 0, \end{aligned} \tag{7.10}$$

where

$$S - \left(\sum_{i=1}^{n} C_i W C_i - W\right) = W\left|S - \left(\sum_{i=1}^{n} C_i W C_i - W\right)\right| \tag{7.11}$$

is the polar decomposition of the operator $S - (\sum_{i=1}^{n} C_i W C_i - W)$, and

$$Y = \left|S - \left(\sum_{i=1}^{n} C_i W C_i - W\right)\right|^{p-1} W^*. \tag{7.12}$$

An easy calculation shows that

$$\left(\sum_{i=1}^{n} C_i Y C_i - Y\right) = 0, \tag{7.13}$$

that is,

$$\sum_{i=1}^{n} C_i \left| S - \left(\sum_{i=1}^{n} C_i W C_i - W\right)\right|^{p-1} W^* C_i = |S - (AWB - W)|^{p-1} W^*. \tag{7.14}$$

It follows from Lemma 7.2 that

$$\sum_{i=1}^{n} C_i \left| S - \left(\sum_{i=1}^{n} C_i W C_i - W\right)\right| W^* C_i = \left| S - \left(\sum_{i=1}^{n} C_i W C_i - W\right)\right| W^*. \tag{7.15}$$

By taking adjoints, and since $\ker E_C \subseteq \ker E_{C^*}$, we get

$$\sum_{i=1}^{n} C_i \left(T - \left(\sum_{i=1}^{n} C_i W C_i - W\right)\right) C_i = \left(T - \left(\sum_{i=1}^{n} C_i W C_i - W\right)\right), \tag{7.16}$$

and then

$$\sum_{i=1}^{n} C_i \left[\left(\sum_{i=1}^{n} C_i W C_i - W\right)\right] C_i = \left(\sum_{i=1}^{n} C_i W C_i - W\right). \tag{7.17}$$

Hence

$$\sum_{i=1}^{n} C_i W C_i - W \in R(E_C) \cap \ker E_{C^*}. \tag{7.18}$$

It is easy to see that (arguing as in the proof of [14, Proposition 4.3]) if $C = (C_1, C_2, \dots, C_n)$ is $n$-tuple of operator in $B(H)$ such that

$$\sum_{i=1}^{n} C_i C_i^* \le 1, \qquad \sum_{i=1}^{n} C_i^* C_i \le 1, \tag{7.19}$$

$\ker E_c \subseteq \ker E_c^*$, and $T \in \ker \Delta_C$, where $T \in B(H)$, then

$$\|T - \Delta_C X\| \ge \|T\| \tag{7.20}$$

holds for all $X \in B(H)$ and for all $T \in \ker E_c$. Hence $\sum_{i=1}^{n} C_i W C_i - W = 0$.

Conversely, if $\sum_{i=1}^{n} C_i W C_i = W$, then $W$ is minimum, and since $F_p$ is differentiable, $W$ is a critical point. □

**Theorem 7.5.** *Let $C = (C_1, C_2, \dots, C_n)$ be $n$-tuple of operators in $B(H)$. If*

$$\sum_{i=1}^{n} C_i C_i^* \le 1, \qquad \sum_{i=1}^{n} C_i^* C_i \le 1, \tag{7.21}$$

*such that* $\ker E_c \subseteq \ker E_c^*$*,* $S \in \ker E_C \cap C_p$ $(0 < p \leq 1)$*,* $\dim H < \infty$*, and* $S - (\sum_{i=1}^n C_i W C_i - W)$ *is invertible, then* $F_p$ *has a critical point at* $W$ *if* $\sum_{i=1}^n C_i W C_i - W = 0$.

**Proof.** Suppose that $\dim H < \infty$. If $\sum_{i=1}^n C_i W C_i - W = 0$, then $S$ is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$ taking $Y = |S|^{p-1}U^*$, where $S = U|S|$ is the polar decomposition of $S$.

It is known that if

$$\sum_{i=1}^n C_i C_i^* \leq 1, \qquad \sum_{i=1}^n C_i^* C_i \leq 1, \qquad \ker E_c \subseteq \ker E_c^*, \tag{7.22}$$

the eigenspaces corresponding to distinct nonzero eigenvalues of the compact positive operator $|S|^2$ reduce each $C_i$ (see [5, Theorem 8] and [14, Lemma 2.3]). In particular, $|S|$ commutes with $C_i$ for all $1 \leq i \leq n$. Hence

$$C_i|S| = |S|C_i. \tag{7.23}$$

Since $\sum_{i=1}^n C_i S^* C_i = S^*$, that is,

$$\sum_{i=1}^n C_i |S| U^* C_i = |S| U^*, \tag{7.24}$$

then

$$|S|\left(\sum_{i=1}^n C_i U^* C_i - U^*\right) = 0, \tag{7.25}$$

and since

$$A|S|^{p-1} = |S|^{p-1}A, \tag{7.26}$$

then

$$\sum_{i=1}^n C_i Y C_i - Y = \sum_{i=1}^n C_i |S|^{p-1} U^* C_i - |S|^{p-1} U^* = |S|^{p-1}\left(\sum_{i=1}^n C_i U^* C_i - U^*\right) \tag{7.27}$$

so that $\sum_{i=1}^n C_i Y C_i - Y = 0$ and $\operatorname{tr}[(\sum_{i=1}^n C_i Y C_i - Y)T] = 0$ for all $T \in B(H)$. Since

$$S = S - \left(\sum_{i=1}^n C_i W C_i - W\right), \tag{7.28}$$

then

$$\begin{aligned}0 &= \operatorname{tr}\left[Y\sum_{i=1}^{n}C_iTC_i - YT\right] = \operatorname{tr}\left[Y\left(\sum_{i=1}^{n}C_iTC_i - T\right)\right] \\ &= p\operatorname{Re}\operatorname{tr}\left[Y\left(\sum_{i=1}^{n}C_iTC_i - T\right)\right] = p\operatorname{Re}\operatorname{tr}\left[|S|^{p-1}U^*\left(\sum_{i=1}^{n}C_iTC_i - T\right)\right] \\ &= (\mathfrak{D}_T\phi)\left(\sum_{i=1}^{n}C_iTC_i - T\right) = (\mathfrak{D}_W F_p)(T).\end{aligned} \tag{7.29}$$

□

**Theorem 7.6.** *Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be $n$-tuples of operators in $B(H)$ such that*

$$\sum_{i=1}^{n} A_iA_i^* \le 1, \qquad \sum_{i=1}^{n} A_i^*A_i \le 1, \qquad \sum_{i=1}^{n} B_iB_i^* \le 1, \qquad \sum_{i=1}^{n} B_i^*B_i \le 1. \tag{7.30}$$

*If*

$$\ker E_{A,B} \subseteq \ker E_{A^*,B^*} \tag{7.31}$$

*and $T \in \ker E_{A,B} \cap C_p$, then for $1 \le p < \infty$,*

(i) *the map $F_p$ has a global minimizer at $W$ if, and for $1 < p < \infty$ only if,*

$$\sum_{i=1}^{n} A_iWB_i - W = 0; \tag{7.32}$$

(ii) *the map $F_p$ has a critical point at $W$ if, and for $1 < p < \infty$ only if,*

$$\sum_{i=1}^{n} A_iWB_i - W = 0; \tag{7.33}$$

(iii) *for $0 < p \le 1$, $\dim H < \infty$, and $S - (\sum_{i=1}^{n} C_iWC_i - W)$ invertible, $F_p$ has a critical point at $W$ if*

$$\sum_{i=1}^{n} A_iWB_i - W = 0. \tag{7.34}$$

**Proof.** It suffices to take the Hilbert space $H \oplus H$ and operators (4.10) and apply Theorems 7.3, 7.4, and 7.5. □

**Remark 7.7.** (1) In Theorem 7.4, the implication "$W$ is a critical point $\Rightarrow$ $\sum_{i=1}^{n} A_iWB_i - W = 0$" does not hold in the case $0 < p \le 1$ because the functional calculus argument involving the function $t \mapsto t^{1/(p-1)}$, where $0 \le t < \infty$, is only valid for $1 < p < \infty$.

(2) The set $S = \{X : AXB - X \in C_p\}$ contains $C_p$. If $X \in C_p$, then $X \in S$ and, for example, $I \in S$ but $I \notin C_p$. If $A \in C_p$, the conclusion of Theorem 7.6 holds for all $X \in B(H)$.

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