IJMMS 2003:59, 3769–3776
PII. S0161171203112070
http://ijmms.hindawi.com

GENERALIZATIONS OF BERNOULLI NUMBERS AND POLYNOMIALS

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Received 3 December 2001

The concepts of Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$, and the generalized Bernoulli numbers $B_n(a,b)$ are generalized to the one $B_n(x;a,b,c)$ which is called the generalized Bernoulli polynomials depending on three positive real parameters. Numerous properties of these polynomials and some relationships between B_n, $B_n(x)$, $B_n(a,b)$, and $B_n(x;a,b,c)$ are established.

2000 Mathematics Subject Classification: 11B68, 33E20.

1. Introduction. It is well known that Bernoulli's numbers and polynomials play important roles in mathematics. They are main objects in the theory of special functions [5]. Their definitions can be given as follows.

Definition 1.1. The numbers B_n, $0 \le n \le \infty$, are called Bernoulli numbers if

$$\phi(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n, \quad |t| < 2\pi. \tag{1.1}$$

Definition 1.2. The functions $B_n(x)$, $0 \le n \le \infty$, are called Bernoulli polynomials if they satisfy

$$\phi(x;t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \quad |t| < 2\pi,\ x \in \mathbb{R}. \tag{1.2}$$

The usual definition of higher-order Bernoulli polynomials is

$$\frac{t^\sigma e^{ut}}{(e^t - 1)^\sigma} = \sum_{n=0}^{\infty} \frac{B_n^\sigma(u)}{n!} t^n, \quad |t| < 2\pi. \tag{1.3}$$

In [2, 4] the second and third authors generalized the concept of Bernoulli numbers as follows.

Definition 1.3. Let $a,b>0$ and $a\neq b$. The generalized Bernoulli numbers $B_n(a,b)$ are defined by

$$\phi(t;a,b)=\frac{t}{b^t-a^t}=\sum_{n=0}^{\infty}\frac{B_n(a,b)}{n!}t^n,\quad |t|<\frac{2\pi}{|\ln b-\ln a|}. \tag{1.4}$$

Among other things, some basic properties and relationships between B_n, $B_n(x)$, and $B_n(a,b)$ were also studied in [2, 4] initially and originally.

In this note, we first give definitions of the generalized Bernoulli polynomials, which generalize the concepts stated above, and then research their basic properties and relationships with Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$, and the generalized Bernoulli numbers $B_n(a,b)$.

2. Definitions and properties of generalized Bernoulli polynomials.

It is easy to see that the following definition is a natural and essential generalization of the concepts of Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$, and the generalized Bernoulli numbers $B_n(a,b)$.

Definition 2.1. Let $a,b,c>0$ and $a\neq b$. The generalized Bernoulli polynomials $B_n(x;a,b,c)$ for nonnegative integer n are defined by

$$\phi(x;t;a,b,c)=\frac{tc^{xt}}{b^t-a^t}=\sum_{n=0}^{\infty}\frac{B_n(x;a,b,c)}{n!}t^n,\quad |t|<\frac{2\pi}{|\ln b-\ln a|},\ x\in\mathbb{R}. \tag{2.1}$$

The generalized Bernoulli polynomials $B_n(x;a,b,c)$ have the following properties which are stated as theorems below.

Theorem 2.2. *Let $a,b,c>0$ and $a\neq b$. For $x\in\mathbb{R}$ and $n\geq 0$,*

$$\begin{gathered}B_n(x;1,e,e)=B_n(x),\qquad B_n(0;a,b,c)=B_n(a,b),\\ B_n(0;1,e,e)=B_n,\qquad B_n(x;a,b,1)=B_n(a,b),\qquad B_n(x;1,e,1)=B_n,\end{gathered} \tag{2.2}$$

$$B_n(x;a,b,c)=\sum_{k=0}^{n}\binom{n}{k}[\ln c]^{n-k}B_k(a,b)x^{x-k}, \tag{2.3}$$

$$B_n(x;a,b,c)=\sum_{k=0}^{n}\binom{n}{k}[\ln c]^{n-k}[\ln b-\ln a]^{k-1}B_k\left(\frac{\ln a}{\ln a-\ln b}\right)x^{x-k}, \tag{2.4}$$

$$B_n(x;a,b,c)=\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{n}{k}\binom{k}{j}[\ln c]^{n-k}[\ln a]^{k-j}\left[\ln\frac{b}{a}\right]^{j-1}B_jx^{x-k}. \tag{2.5}$$

Proof. Applying Definition 1.3 to the term $t/(b^t - a^t)$ and expanding the exponential function c^{xt} at $t = 0$ yields

$$\begin{aligned} \frac{tc^{xt}}{b^t - a^t} &= \left(\sum_{k=0}^{\infty} \frac{B_k(a,b)}{k!} t^k\right)\left(\sum_{i=0}^{\infty} \frac{x^i (\ln c)^i}{i!} t^i\right) \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{(\ln c)^{k-i}}{i!(k-i)!} B_i(a,b) x^{k-i} t^k \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} B_k(a,b) x^{n-k}\right) \frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

Combining (2.6) and (2.1) and equating their coefficients of t^n produces formula (2.3).

The following two formulae were provided in [2, 4]:

$$B_n(a,b) = (\ln b - \ln a)^{n-1} B_n\left(\frac{\ln a}{\ln a - \ln b}\right), \tag{2.7}$$

$$B_n(a,b) = \sum_{i=0}^{n} (-1)^{n-i} (\ln b - \ln a)^{i-1} (\ln a)^{n-i} \binom{n}{i} B_i. \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.3) leads to (2.4) and (2.5).

The formulae in (2.2) are obvious. □

Now we give some results about derivatives and integrals of the generalized Bernoulli polynomials $B_n(x;a,b,c)$ as follows.

Theorem 2.3. *Let $a,b,c > 0$, $a \neq b$, $n \geq 0$, and $x \in \mathbb{R}$. For any nonnegative integer ℓ and real numbers α and β,*

$$\frac{\partial^\ell B_n(x;a,b,c)}{\partial x^\ell} = \frac{n!}{(n-\ell)!} (\ln c)^\ell B_{n-\ell}(x;a,b,c), \tag{2.9}$$

$$\int_\alpha^\beta B_n(t;a,b,c)\,dt = \frac{1}{(n+1)\ln c} \left[B_{n+1}(\beta;a,b,c) - B_{n+1}(\alpha;a,b,c)\right], \tag{2.10}$$

where $B_0(x;a,b,c) = 1/(\ln b - \ln a)$.

Proof. Formula (2.9) follows from standard arguments and induction.

Integrating on both sides of (2.9) with respect to variable x for $\ell = 1$ gives formula (2.10). □

Theorem 2.4. *Let $a,b,c > 0$, $a \neq b$, $n \geq 0$, and $x \in \mathbb{R}$. Then*

$$B_n(x+1;a,b,c) = \sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} B_k(x;a,b,c), \tag{2.11}$$

$$B_n(x+1;a,b,c) = B_n\left(x; \frac{a}{c}, \frac{b}{c}, c\right), \tag{2.12}$$

and, for $m \geq 2$,

$$\begin{aligned} B_m(x+1;a,b,c) &= B_m(x;a,b,c) + m(\ln c)^{m-1}x^{m-1} \\ &\quad + \sum_{k=0}^{m-1}\binom{m}{k}\left[(\ln a)^{m-k} - (\ln b)^{m-k} + (\ln c)^{m-k}\right]B_k(x;a,b,c). \end{aligned} \tag{2.13}$$

Proof. By the definition of the generalized Bernoulli polynomials, we have

$$\frac{tc^{(x+1)t}}{b^t - a^t} = \sum_{n=0}^{\infty}\frac{B_n(x+1;a,b,c)}{n!}t^n, \tag{2.14}$$

$$\begin{aligned} \frac{tc^{(x+1)t}}{b^t - a^t} &= \frac{tc^{xt}}{b^t - a^t}\cdot c^t \\ &= \left(\sum_{n=0}^{\infty}\frac{B_n(x;a,b,c)}{n!}t^n\right)\left(\sum_{k=0}^{\infty}\frac{(\ln c)^k}{k!}t^k\right) \\ &= \sum_{n=0}^{\infty}\frac{\sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}B_k(x;a,b,c)}{n!}t^n. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) and equating their coefficients of t^n leads to formula (2.11).

Similarly, since

$$\frac{tc^{(x+1)t}}{b^t - a^t} = \frac{tc^{xt}}{(b/c)^t - (a/c)^t} = \sum_{n=0}^{\infty}\frac{B_n(x;a/c,b/c,c)}{n!}t^n, \tag{2.16}$$

equating the coefficients of t^n in (2.14) and (2.16) leads to formula (2.12).

Straightforward computation gives

$$\begin{aligned} \frac{tc^{(x+1)t}}{b^t - a^t} &= tc^{xt} + \frac{tc^{xt}(a^t - b^t + c^t)}{b^t - a^t} \\ &= \sum_{n=0}^{\infty}\frac{(\ln c)^n x^n}{n!}t^{n+1} \\ &\quad + \left(\sum_{n=0}^{\infty}\frac{B_n(x;a,b,c)}{n!}t^n\right)\left(\sum_{\ell=0}^{\infty}\frac{\left[(\ln a)^{\ell} - (\ln b)^{\ell} + (\ln c)^{\ell}\right]}{\ell!}t^{\ell}\right) \\ &= \sum_{n=0}^{\infty}\frac{(\ln c)^n x^n}{n!}t^{n+1} \\ &\quad + \sum_{n=0}^{\infty}\left[\sum_{\ell=0}^{n}\binom{n}{\ell}\left[(\ln a)^{n-\ell} - (\ln b)^{n-\ell} + (\ln c)^{n-\ell}\right]B_{\ell}(x;a,b,c)\right]\frac{t^n}{n!} \end{aligned}$$

$$
\begin{aligned}
&= B_0(x;a,b,c) + \left[1 + B_1(x;a,b,c) + B_0(x;a,b,c)(\ln a - \ln b + \ln c)\right] t \\
&\quad + \sum_{n=2}^{\infty} \left[n(\ln c)^{n-1} x^{n-1} + B_n(x;a,b,c) \right] \frac{t^n}{n!} \\
&\quad + \sum_{n=2}^{\infty} \left\{ \sum_{\ell=0}^{n-1} \binom{n}{\ell} \left[(\ln a)^{n-\ell} - (\ln b)^{n-\ell} + (\ln c)^{n-\ell} \right] B_\ell(x;a,b,c) \right\} \frac{t^n}{n!}.
\end{aligned}
\tag{2.17}
$$

Equating (2.1) and (2.17) yields (2.13). □

Corollary 2.5. *For $n \geq 1$, $b > 0$, and $x \in \mathbb{R}$,*

$$
B_n(x+1;1,b,b) = B_n(x;1,b,b) + n(\ln b)^{n-1} x^{n-1}. \tag{2.18}
$$

Remark 2.6. *Taking $b = e$ in (2.18), the following well-known result is deduced:*

$$
B_n(x+1) = B_n(x) + n x^{n-1}, \quad n \geq 1. \tag{2.19}
$$

Similarly, from (2.9), it follows that

$$
B_i'(t) = i B_{i-1}(t), \qquad B_0(t) = 1. \tag{2.20}
$$

Actually, the Bernoulli polynomials $B_i(t)$, $i \in \mathbb{N}$, are uniquely determined by formulae (2.19) and (2.20), see [1, identities 23.1.5 and 23.1.6] *or* [5].

Theorem 2.7. *Let $a,b,c > 0$, $a \neq b$, $n \geq 0$, and $x \in \mathbb{R}$. Then*

$$
\begin{aligned}
B_n(1-x;a,b,c) &= (-1)^n B_n\left(x; \frac{c}{b}, \frac{c}{a}, c\right) \\
&= B_n\left(-x; \frac{a}{c}, \frac{b}{c}, \frac{1}{c}\right),
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
B_n(x+y;a,b,c) &= \sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} B_k(x;a,b,c) y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} B_k(y;a,b,c) x^{n-k}.
\end{aligned}
\tag{2.22}
$$

Proof. From Definition 2.1, it follows that

$$
\frac{t c^{(1-x)t}}{b^t - a^t} = \sum_{n=0}^{\infty} \frac{B_n(1-x;a,b,c)}{n!} t^n. \tag{2.23}
$$

Meanwhile, we have

$$\begin{aligned}\frac{tc^{(1-x)t}}{b^t-a^t}&=\frac{tc^{-xt}}{(b/c)^t-(a/c)^t}=\sum_{n=0}^{\infty}\frac{B_n(-x;a/c,b/c,c)}{n!}t^n,\\ \frac{tc^{(1-x)t}}{b^t-a^t}&=\frac{-tc^{x(-t)}}{(c/a)^{-t}-(c/b)^{-t}}=\sum_{n=0}^{\infty}(-1)^n\frac{B_n(x;c/b,c/a,c)}{n!}t^n.\end{aligned}\tag{2.24}$$

Therefore, formula (2.21) follows from equating series expansions in (2.23) and (2.24).

Similarly, we have

$$\begin{aligned}\frac{tc^{(x+y)t}}{b^t-a^t}&=\sum_{n=0}^{\infty}\frac{B_n(x+y;a,b,c)}{n!}t^n,\\ \frac{tc^{(x+y)t}}{b^t-a^t}&=\frac{tc^{xt}}{b^t-a^t}\cdot c^{yt}\\ &=\left(\sum_{n=0}^{\infty}\frac{B_n(x;a,b,c)}{n!}t^n\right)\left(\sum_{i=0}^{\infty}\frac{y^i(\ln c)^i}{i!}t^i\right)\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}y^{n-k}(\ln c)^{n-k}B_k(x;a,b,c)\right)\frac{t^n}{n!},\\ \frac{tc^{(x+y)t}}{b^t-a^t}&=\frac{tc^{yt}}{b^t-a^t}\cdot c^{xt}\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(\ln c)^{n-k}B_k(y;a,b,c)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.25}$$

Hence, formula (2.22) follows from equating series expansions in (2.25). The proof is complete. □

Theorem 2.8. *Let m and n be natural numbers. Then, for any positive number b, the following identity holds:*

$$\begin{aligned}\sum_{j=1}^{m}j^n&=\frac{1}{(n+1)(\ln b)^n}\left[B_{n+1}(m+1;1,b,b)-B_{n+1}(0;1,b,b)\right]\\ &=\frac{1}{(n+1)(\ln b)^n}\left[B_{n+1}(m+1;1,b,b)-B_{n+1}(1;1,b,b)\right].\end{aligned}\tag{2.26}$$

Proof. Rewriting formula (2.18) yields

$$x^{n-1}=\frac{1}{n(\ln b)^{n-1}}\left[B_n(x+1;1,b,b)-B_n(x;1,b,b)\right],\tag{2.27}$$

which implies

$$j^n = \frac{1}{(n+1)(\ln b)^n}\left[B_{n+1}(j+1;1,b,b) - B_{n+1}(j;1,b,b)\right]. \tag{2.28}$$

Summing up on both sides of (2.28) from 0 to m or from 1 to m with respect to j easily leads to formula (2.26). □

Remark 2.9. The calculation of values of $\sum_{j=1}^{m} j^n$ is an interesting problem that has been investigated in many works, see, for example, [3].

Remark 2.10. It follows from the identities (2.3) and (2.7), combined with [1, identity 23.1.7], that

$$B_n(x;a,b,c) = (\ln b - \ln a)^{n-1} B_n\left(\frac{\ln a - x\ln c}{\ln a - \ln b}\right). \tag{2.29}$$

Remark 2.11. At last, it is pointed out that the Bernoulli and Euler numbers and the Bernoulli and Euler polynomials can be further generalized to more general results in this manner. These conclusions will be published in some subsequent papers.

Acknowledgments. This note was finalized during the third author's visit to the RGMIA with grants from the Victoria University and Jiaozuo Institute of Technology. The authors would like to express many thanks to the anonymous referee for many valuable comments and suggestions. The first three authors were supported in part by NNSF of China, Grant 10001016, SF for the Prominent Youth of Henan Province, Grant 0112000200, SF of Henan Innovation Talents at Universities, NSF of Henan Province, Grant 004051800, SF for Pure Research of Natural Science of the Education Department of Henan Province, Grant 1999110004, Doctor Fund of Jiaozuo Institute of Technology, China. The fourth author was partially supported by a grant of the Faculty Research Council of the University of Texas-Pan American.

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