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A COEFFICIENT INEQUALITY FOR THE CLASS OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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The aim of this paper is to give a coefficient inequality for the class of analytic functions in the unit disc $D = \{z \mid |z| < 1\}$.

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1. Introduction. Let Ω be the family of functions $\omega(z)$ regular in the disc D and satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in D$.

Next, for arbitrary fixed numbers A and B, $-1 < A \leq 1$, $-1 \leq B < A$, denote by $P(A,B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1.1}$$

regular in D such that $p(z)$ is in $P(A,B)$ if and only if

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)} \tag{1.2}$$

for some function $\omega(z) \in \Omega$ and every $z \in D$. The class $P(A,B)$ was introduced by Janowski [3].

Moreover, let $S^*(A,B,b)$ ($b \neq 0$, complex) denote the family of functions

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{1.3}$$

regular in D and such that $f(z)$ is in $S^*(A,B,b)$ if and only if

$$1 + \frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right) = p(z) \tag{1.4}$$

for some $p(z)$ in $P(A,B)$ and all z in D.

For the aim of this paper we need Jack's lemma [2]. "Let $\omega(z)$ be a regular in the unit disc with $\omega(0) = 0$, then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_1, we can write $z_1\omega'(z_1) = k\omega(z_1)$, where k is real and $k \geq 1$."

2. Coefficient inequality. The main purpose of this paper is to give sharp upper bound of the modulus of the coefficient a_n. Therefore, we need the following lemma.

Lemma 2.1. *The necessary and sufficient condition for* $g(z) = z + a_2 z^2 + \cdots$ *belongs to* $S^*(A,B,b)$ *is*

$$g(z) \in S^*(A,B,b) \Longleftrightarrow g(z) = \begin{cases} z \cdot (1+B\omega(z))^{b(A-B)/B}, & B \neq 0, \\ z \cdot e^{bA\omega(z)}, & B = 0, \end{cases} \tag{2.1}$$

where $\omega(z) \in \Omega$.

Proof. The proof of this lemma is in four steps.

Step 1. Let $B \neq 0$ and

$$g(z) = z \cdot (1+B\omega(z))^{b(A-B)/B}. \tag{2.2}$$

If we take the logarithmic derivative from equality (2.2), we obtain

$$\frac{1}{b}\left(z \cdot \frac{g'(z)}{g(z)} - 1\right) = (A-B)\frac{z \cdot \omega'(z)}{1+B\omega(z)}. \tag{2.3}$$

If we use Jack's lemma [2] in equality (2.3), we get

$$\frac{1}{b}\left(z\frac{g'(z)}{g(z)} - 1\right) = \frac{(A-B)\omega(z)}{1+B\omega(z)}. \tag{2.4}$$

After the simple calculations from (2.4), we see that

$$1 + \frac{1}{b}\left(z \cdot \frac{g'(z)}{g(z)} - 1\right) = \frac{1+A\omega(z)}{1+B\omega(z)}. \tag{2.5}$$

Equality (2.5) shows that $g(z) \in S^*(A,B,b)$.

Step 2. Let $B = 0$ and

$$g(z) = z \cdot e^{bA\omega(z)}. \tag{2.6}$$

Similarly, we obtain

$$1 + \frac{1}{b}\left(z\frac{g'(z)}{g(z)} - 1\right) = \frac{1+A\omega(z)}{1+B\omega(z)} = 1 + A\omega(z). \tag{2.7}$$

This shows that $g(z) \in S^*(A,B,b)$.

Step 3. Let $g(z) \in S^*(A,B,b)$ and $B \neq 0$, then we have

$$1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right)=\frac{1+A\omega(z)}{1+B\omega(z)}. \tag{2.8}$$

Equality (2.8) can be written in the from

$$\frac{g'(z)}{g(z)}=\frac{b(A-B)(\omega(z)/z)}{1+B\omega(z)}+\frac{1}{z}. \tag{2.9}$$

If we use Jack's lemma (2.9), we obtain

$$\frac{g'(z)}{g(z)}=\frac{b(A-B)\omega'(z)}{1+B\omega(z)}+\frac{1}{z}. \tag{2.10}$$

Integrating both sides of equality (2.10), we get

$$g(z)=z\cdot(1+B\omega(z))^{b(A-B)/B}. \tag{2.11}$$

Step 4. Let $g(z) \in S^*(A,B,b)$ and $B = 0$. Similarly, we obtain

$$g(z)=z\cdot e^{bA\omega(z)} \tag{2.12}$$

which ends the proof. □

We note that we choose the branch of $(1+Bw(z))^{b(A-B)/B}$ such that

$$(1+Bw(0))^{b(A-B)/B}=1 \quad \text{at } z=0. \tag{2.13}$$

Theorem 2.2. *If $f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ belongs to $S^*(A,B,b)$, then*

$$\begin{aligned} |a_n| &\le \prod_{k=0}^{n-2}\frac{|b(A-B)+kB|}{k+1} \quad \text{if } B \neq 0,\\ |a_n| &\le \prod_{k=0}^{n-2}\frac{|bA|}{k+1} \quad \text{if } B = 0. \end{aligned} \tag{2.14}$$

These bounds are sharp because the extremal function is

$$f_*(z)=\begin{cases} \dfrac{z}{(1-B\delta z)^{-b(A-B)/B}}, & |\delta|=1,\ \text{if } B\neq 0,\\ ze^{bAz}, & \text{if } B=0. \end{cases} \tag{2.15}$$

Proof. Let $B \neq 0$. If we use the definition of the class $S^*(A,B,b)$, then we write

$$1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)=p(z). \tag{2.16}$$

Equality (2.16) can be written by using the Taylor expansion of $f(z)$ and $p(z)$ in the form

$$\begin{aligned} &z+2a_2z^2+3a_3z^3+\cdots+na_nz^n+\cdots \\ &\quad=(z+a_2z^2+\cdots+a_nz^n+\cdots)(1+bp_1z+bp_2z^2+\cdots+bp_nz^n+\cdots). \end{aligned} \tag{2.17}$$

Evaluating the coefficient of z^n in both sides of (2.17), we get

$$na_n=a_n+bp_1a_{n-1}+bp_2a_{n-2}+\cdots+bp_{n-1}. \tag{2.18}$$

on the other hand,

$$|p_n|\le(A-B). \tag{2.19}$$

Inequality (2.19) was proved by Aouf [1]. If we consider the relations (2.18) and (2.19) together, then we obtain

$$(n-1)|a_n|\le|b||A-B|(1+|a_2|+|a_3|+\cdots+|a_{n-1}|), \tag{2.20}$$

which can be written in the form

$$|a_n|\le\frac{1}{(n-1)}\sum_{k=1}^{n-1}|b||A-B||a_k|,\qquad |a_1|=1. \tag{2.21}$$

To prove (2.14), we will use the induction principle.

Now, we consider inequalities (2.21) and

$$|a_n|\le\prod_{k=0}^{n-2}\frac{|b(A-B)+kB|}{k+1}. \tag{2.22}$$

The right-hand sides of these inequalities are the same because

(i) for $n=2$,

$$\begin{aligned} &|a_n|\le\frac{|b||A-B|}{(n-1)}\sum_{k=1}^{n-1}|a_k|,\qquad |a_1|=1\Longrightarrow|a_2|\le|b||A-B|, \\ &|a_n|\le\prod_{k=0}^{n-2}\frac{|b(A-B)+kB|}{k+1}=|b(A-B)|\Longrightarrow|a_2|\le|b||A-B|; \end{aligned} \tag{2.23}$$

(ii) for $n = 3$,

$$|a_3| \le \frac{|b||A-B|}{(n-1)} \sum_{k=1}^{n-1} |a_k| = \frac{1}{2}|b||A-B|(1+|a_2|)$$

$$\Longrightarrow |a_3| \le \frac{1}{2}|b|^2|A-B|^2 + \frac{1}{2}|b||A-B|,$$

$$|a_3| \le \prod_{k=0}^{n-2} \frac{|b(A-B)+kB|}{k+1} = |b||A-B| \frac{|b(A-B)+B|}{2} \tag{2.24}$$

$$\Longrightarrow |a_3| \le \frac{1}{2}|b||A-B|[|b||A-B|+|B|] \le \frac{1}{2}|b||A-B|[|b||A-B|+1]$$

$$\Longrightarrow |a_3| \le \frac{1}{2}|b|^2|A-B|^2 + \frac{1}{2}|b||A+B|.$$

Suppose that this result is true for $n = p$, then we have

$$\begin{aligned} &|a_n| \le \frac{|b||A-B|}{(n-1)} \sum_{k=1}^{n-1} |a_k|, \\ &|a_1| = 1 \Longrightarrow |a_p| \le \frac{|b||A-B|}{(p-1)}(1+|a_2|+|a_3|+\cdots+|a_{p-1}|), \end{aligned} \tag{2.25}$$

$$\begin{aligned} |a_n| &\le \prod_{k=0}^{n-2} \frac{|b(A-B)+kB|}{k+1} \\ \Longrightarrow |a_p| &\le \prod_{k=0}^{p-2} \frac{|b(A-B)+kB|}{k+1} \\ \Longrightarrow |a_p| &\le \frac{1}{(p-1)!}|b||A-B|(|b||A-B|+1)(|b||A-B|+2) \\ &\cdot (|b||A-B|+3)\cdots(|b||A-B|+(p-2)) \end{aligned} \tag{2.26}$$

from (2.25), (2.26), and induction hypothesis, we have

$$\begin{aligned} &\frac{|b||A-B|}{(p-1)}(1+|a_2|+|a_3|+\cdots+|a_{p-1}|) \\ &\quad = \frac{1}{(p-1)!}|b||A-B|(|b||A-B|+1) \\ &\quad \cdot (|b||A-B|+2)\cdots(|b||A-B|+(p-2)). \end{aligned} \tag{2.27}$$

If we write $x = |b||A-B| > 0$, equality (2.27) can be written in the form.

$$\begin{aligned} &\frac{x}{(p-1)}(1+|a_2|+|a_3|+\cdots+|a_{p-1}|) \\ &\quad = \frac{1}{(p-1)!}x(x+1)(x+2)\cdots(x+(p-2)). \end{aligned} \tag{2.28}$$

After the simple calculation from equality (2.28), we get

$$\begin{aligned}
&\frac{1}{p}(x+(p-1))\frac{1}{(p-1)}(1+|a_2|+|a_3|+\cdots+|a_{p-1}|)\\
&\quad=\frac{1}{p!}(x+1)(x+2)(x+3)\cdots(x+(p-2))(x+(p-1))\\
&\quad\Longrightarrow\frac{1}{p}\left[\frac{x}{p-1}(1+|a_2|+|a_3|+\cdots+|a_{p-1}|)\right]\\
&\qquad+\left[\frac{1}{p}(1+|a_2|+|a_3|+\cdots+|a_{p-1}|)\right]\\
&\quad=\frac{1}{p!}(x+1)(x+2)(x+3)\cdots(x+(p-2))(x+(p-1))\\
&\quad\Longrightarrow\frac{1}{p}|a_p|+\left[\frac{1}{p}(1+|a_2|+|a_3|+\cdots+|a_{p-1}|)\right]\\
&\quad=\frac{1}{p!}(x+1)(x+2)(x+3)\cdots(x+(p-2))(x+(p-1))\\
&\quad\Longrightarrow\frac{x}{p}(1+|a_2|+|a_3|+\cdots+|a_{p-1}|+|a_p|)\\
&\quad=\frac{1}{p!}x(x+1)(x+2)(x+3)\cdots(x+(p-2))(x+(p-1)).
\end{aligned}\tag{2.29}$$

Equality (2.29) shows that the result is valid for $n=p+1$.

Therefore, we have (2.14). □

Corollary 2.3. *The first inequality of (2.14) can be rewritten in the form*

$$\begin{aligned}
|a_n|&\le\prod_{k=0}^{n-2}\frac{|b(A-B)+kb|}{k+1}\\
&=|B(A-B)|\frac{1}{2}|b(A-B)+B|\\
&\quad\cdot\frac{1}{3}|b(A-B)+2B|\cdots\frac{1}{(n-1)}|b(A-B)+(n-2)B|\\
&=\frac{1}{(n-1)!}|b(A-B)|\cdot|b(A-B)+B|\\
&\quad\cdot|b(A-B)+2B|\cdots|b(A-B)+(n-2)B|\\
&\le\frac{1}{(n-1)!}|b(A-B)|(|b(A-B)|+|B|)\\
&\quad\cdot(|b(A-B)|+2|B|)\cdots(|b(A-B)|+(n-2)|B|).
\end{aligned}\tag{2.30}$$

If $A=1$, $B=-1$, and $b=1$, then

$$|a_n|\le\frac{1}{(n-1)!}2\cdot(2+1)\cdot(2+2)\cdots n=\frac{n!}{(n-1)!}=n.\tag{2.31}$$

This is the coefficient inequality for the starlike function which is well known.

Corollary 2.4. *If* $A = 1$*,* $B = -1$*,*

$$|a_n| < \frac{1}{(n-1)!} \prod_{k=0}^{n-2} |2b+k|. \tag{2.32}$$

This inequality was obtained by Aouf [1].

Therefore, by giving the special value to A, B, and b, we obtain the coefficient inequality for the classes $S^*(1,-1,\beta)$, $S^*(1,-1,e^{-i\lambda}\operatorname{Cos}\lambda)$, $S^*(1,-1,(1-\beta)e^{-i\lambda}\operatorname{Cos}\lambda)$, $S^*(1,0,b)$, $S^*(\beta,0,b)$, $S^*(\beta,-\beta,b)$, $S^*(1,(-1+1/M),b)$, and $S^*(1-2\beta,-1,b)$, where $0 \le \beta < 1$, $|\lambda| < \pi/2$, and $M > 1$.

References

[1] M. K. Aouf, *On a class of p-valent starlike functions of order α*, Int. J. Math. Math. Sci. **10** (1987), no. 4, 733–744.

[2] I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. (2) **3** (1971), 469–474.

[3] W. Janowski, *Some extremal problems for certain families of analytic functions. I*, Ann. Polon. Math. **28** (1973), 297–326.

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