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A RENORMING OF ℓ_2, RARE BUT WITH THE FIXED-POINT PROPERTY

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We give an example of a renorming of ℓ_2 with the fixed-point property (FPP) for nonexpansive mappings, but which seems to fall out of the scope of all the commonly known sufficient conditions for FPP.

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1. Introduction. Let C be a subset of a Banach space $(X, \|\cdot\|)$. A mapping $T: C \to X$ is called nonexpansive whenever $\|T(x)-T(y)\| \le \|x-y\|$ for all $x, y \in C$. The space $(X, \|\cdot\|)$ has the *fixed-point property* (FPP) if every nonexpansive self-mapping of each nonempty bounded closed convex subset C of X has a fixed point. If the same property holds for every weakly compact convex subset of X we say that $(X, \|\cdot\|)$ has the *weak fixed-point property* WFPP.

It has been known from the outset of the study of this property that it depends strongly on "nice" geometrical properties of the space. For instance, a celebrated result due to Kirk [21] establishes that those Banach spaces with *normal structure* (NS) have the WFPP.

Recently, it has been shown that a subspace of $L^1([0,1])$ has WFPP if and only if it is reflexive (see [4]). This result is a partial answer to a longtime open major question in metric fixed-point theory: does every reflexive Banach space have FPP? (See [22].) A particular case of this question is: does every superreflexive Banach space have FPP? Although superreflexive spaces have the FPP for isometries [24], the question for general nonexpansive mappings remains still unsolved.

A complication in the general study of FPP is the obvious fact that fixed-point properties are not invariant under renormings. In fact, it is not known whether there exists a nontrivial class of isomorphic Banach spaces such that every member of this class has the FPP. (A trivial class with this property is the Banach spaces isomorphic to a Schur space.)

Since Hilbert spaces have the FPP, this raises the natural question of whether every Banach space which is isomorphic to a Hilbert space does indeed enjoy the FPP. This problem is essentially of separable nature (see [12, page 35]). Thus, without loss of generality, we can merely ask: *does every (equivalent) renorming of ℓ_2 have the FPP?*

The aim of this paper is to overview the state of the art on this question and to give an example of a renorming of ℓ_2 which seems to fall out of the scope of all the commonly known sufficient conditions for FPP.

2. The state of the art. In this section, we give a survey on the main fixed-point theorems which can be applied in order to check the FPP for renormings of ℓ_2, the classical space

$$\ell_2 := \left\{ (x(n)) : \|x\|_2 := \left(\sum_{n=1}^{\infty} |x(n)|^2 \right)^{1/2} < \infty \right\} \tag{2.1}$$

endowed with its usual Euclidean norm.

2.1. Stability. There are results ensuring the FPP for those Banach spaces $(\ell_2, \|\cdot\|)$ which are sufficiently close (in the sense of Banach-Mazur) to $(\ell_2, \|\cdot\|_2)$. Such results are known as stability ones. Perhaps the deepest stability result is the following one.

Theorem 2.1 (Lin [25]). *If the Banach-Mazur distance from X to $(\ell_2, \|\cdot\|_2)$ is less than $\sqrt{(5+\sqrt{13})/2}$, then X has the FPP.*

This behavior of spreading the FPP to spaces close to $(\ell_2, \|\cdot\|_2)$ still holds when $(\ell_2, \|\cdot\|_2)$ is replaced by some other spaces. For example, we consider for $\beta > 1$ the spaces $E_\beta := (\ell_2, |\cdot|_\beta)$, where

$$|x|_\beta := \max\{\|x\|_2, \beta\|x\|_\infty\}. \tag{2.2}$$

It is known that if $\|\cdot\|$ is a norm on ℓ_2 such that, for every $v \in \ell_2$,

$$|v|_\beta \le \|v\| \le b|v|_\beta \tag{2.3}$$

with $b < \sqrt{2}$, then $(\ell_2, \|\cdot\|)$ has the FPP.

For more information about the stability of FPP, see [8].

2.2. Normal structure and its generalizations. There are many sufficient conditions to determine whether the FPP holds for Banach spaces isomorphic to ℓ_2. However, the most relevant in metric fixed-point theory is the NS. Recall that a Banach space $(X, \|\cdot\|)$ is said to have NS if for each nontrivial bounded convex subset C of X there exists a *nondiametral* point $p \in C$, that is, a point $p \in C$ such that

$$\sup\{\|p - x\| : x \in C\} < \operatorname{diam}(C). \tag{2.4}$$

This property was introduced by Brodskii and Milman and has been significant in the development of fixed-point theory (see [12, 13]). Of course, reflexive spaces with NS have FPP [21].

In fact, the result of Kirk on NS was the starting point of considerable research aimed at finding other geometrical properties of Banach spaces which imply NS; for instance, uniform convexity and uniform smoothness. However, in some way, it is very unusual for spaces to fail to have NS (see [12, page 65]). Nevertheless, one can find renormings of ℓ_2 with NS, which lack uniform convexity and several of its generalizations (see [10, 28]).

Karlovitz [18, 19] first noted that even in reflexive spaces NS is not essential for FPP. In particular, E_β fails to have NS for $\beta \geq \sqrt{2}$.

The concept of *asymptotic normal structure* (ANS) was introduced by Baillon and Schöneberg in 1981 [2]. A Banach space $(X, \|\cdot\|)$ has ANS if each nonempty, bounded, closed, and convex subset C of X which contains more than one point has the following property: if a sequence (x_n) in C satisfies $\|x_n - x_{n+1}\| \to 0$, there exists a point $x \in C$ such that

$$\liminf \|x - x_n\| < \operatorname{diam}(C). \tag{2.5}$$

Clearly, if $(X, \|\cdot\|)$ has NS, then it has ANS, but the converse is not true. In fact, for $\beta \in [\sqrt{2}, 2)$, the spaces E_β have ANS but not NS. In the same work [2] the following result was shown.

Theorem 2.2 (Baillon and Schöneberg [2]). *Every reflexive Banach space with ANS has the FPP.*

2.3. Orthogonal convexity. Some well-known sufficient conditions for the WFPP are independent of the NS. One of them is the so-called orthogonal convexity, a property of geometric nature which among other things implies the *Banach-Saks* property (see [23]). It is known that every uniformly convex Banach space is orthogonally convex. Other examples include E_β, ℓ_1, c_0, c, and the James space J.

Recall that a Banach space $(X, \|\cdot\|)$ is *orthogonally convex* (OC) if for every weakly null sequence (x_n) with

$$D[(x_n)] = \limsup_n \left(\limsup_m \|x_n - x_m\| \right) > 0, \tag{2.6}$$

there exists $\beta > 0$ such that $A_\beta[(x_n)] < D[(x_n)]$, where

$$\begin{gathered} A_\beta[(x_n)] = \limsup_n \left(\limsup_m |M_\beta(x_n, x_m)| \right), \\ M_\beta(x, y) = \left\{ z \in X : \max\{\|z - x\|, \|z - y\|\} \leq \frac{1}{2}(1+\beta)\|x - y\| \right\} \end{gathered} \tag{2.7}$$

for any x and y in X, and $|A| = \sup\{\|x\| : x \in A\}$ for any subset A of X. A generalization of the property of OC can be found in [3].

We consider the following renorming of ℓ_2. For $x \in \ell_2$,

$$|||x||| := \max\left\{ \frac{1}{3}\|x\|_2, \sup_{n \geq 2} |x(1) + x(n) + x(n+1)| \right\}. \tag{2.8}$$

This norm was introduced by van Dulst [29]. The space $(\ell_2, |||\cdot|||)$ lacks ANS although it is OC (see [16]). In this paper, the following result was given.

Theorem 2.3. *Every OC Banach space has the WFPP.*

Hence, reflexive OC Banach spaces have FPP. On the other hand, there are renormings of ℓ_2 with NS which fail to be OC.

2.4. Conditions depending on a (Schauder) basis. A Schauder basis (e_n) of $(X, \|\cdot\|)$ is called *unconditional* if for every choice of signs $\theta = (\theta_n)$, one has that $\sum_{n=1}^{\infty} \theta_n a_n e_n$ is convergent provided $\sum_{n=1}^{\infty} a_n e_n$ is also convergent. In this case, the number

$$\mu := \sup\left\{\left\|\sum_{n=1}^{\infty} \theta_n a_n e_n\right\| : \left\|\sum_{n=1}^{\infty} a_n e_n\right\| = 1,\ \theta_n = \pm 1\right\} \tag{2.9}$$

is called the *unconditional constant* of (e_n).

If (e_n) is an unconditional basis for X, then it is well known that for each nonempty $F \subset \mathbb{N}$, the projection

$$P_F\left(\sum_{n=1}^{\infty} a_n e_n\right) := \sum_{n\in F} a_n e_n \tag{2.10}$$

is well defined and the constant $c := \sup\{\|P_F\|\}$ is finite. It is easy to verify that $1 \le c \le \mu \le 2c$. The basis (e_n) is called *strongly monotonous* whenever $c = 1$. The following theorems are two sufficient conditions for WFPP and in terms of these constants are well known.

Theorem 2.4 (Khamsi [20]). *Let $(X, \|\cdot\|)$ be a reflexive Banach space with an unconditional basis such that*

$$c(\mu + 2) < 4. \tag{2.11}$$

Then $(X, \|\cdot\|)$ has the FPP.

Theorem 2.5 (Lin [24]). *Let $(X, \|\cdot\|)$ be a superreflexive Banach space with $c = 1$. Then $(X, \|\cdot\|)$ has the FPP.*

In order to be able to apply Theorems 2.4 and 2.5 to a specific space, one must note that they are strongly dependent on a previous choice of a basis. For renormings of ℓ_2, we have the standard basis (e_n) as a quite natural option.

We observe that E_β falls into the scope of Theorems 2.4 and 2.5 since c and μ are equal to 1 for the standard basis in this space.

On the other hand, one has, for the van Dulst renorming,

$$|||(-1,1,1,0,\ldots)||| = 1, \qquad |||(1,1,1,0,\ldots)||| = 3. \tag{2.12}$$

Hence, $\mu \geq 3$. Moreover, for $F = \{2,3\} \subset \mathbb{N}$, $\||P_F(-1,1,1,0,\ldots)|\| = 2$ which implies that $c \geq \|P_F\| \geq 2$. Therefore, with respect to the canonical basis, $c(2+\mu) \geq 10$. It is not clear whether these constants can be improved with respect to another basis.

2.5. Near uniform smoothness. Recall that $(X,\|\cdot\|)$ is weakly nearly uniformly smooth (WNUS) if for some $\varepsilon > 0$, there exists $\mu > 0$ such that if $0 < t < \mu$ and (x_n) is a basic sequence in B_X, then there exists $k > 1$ so that $\|x_1 + tx_k\| \leq 1 + \varepsilon t$. A coefficient closely related to this property was defined by García-Falset [6] as

$$R(X) := \sup\left\{\liminf_{n\to\infty} \|x_n + x\| : x, x_n \in B_X \ (n = 1,2,\ldots),\ x_n \rightharpoonup 0\right\}. \tag{2.13}$$

(Here, B_X is the unit ball of $(X,\|\cdot\|)$ and the symbol $\rightharpoonup$ stands for the weak convergence.) It turns out that $(X,\|\cdot\|)$ is WNUS if and only if X is reflexive and $R(X) < 2$. Moreover, one has the following result.

Theorem 2.6 (García-Falset [7]). *If $R(X) < 2$, then $(X,\|\cdot\|)$ has WFPP.*

The relationship between the orthogonal convexity and the condition $R(X) < 2$ is not clear. Other coefficients have been defined with the aim of improving the above theorem (see [1]).

2.6. Uniformly noncreasy Banach spaces. An interesting sufficient condition for FPP was recently given by Prus [27]. Given two functionals $x^*, y^* \in S_{X^*}$ and a scalar $\delta \in [0,1]$, we let

$$\begin{aligned} S(x^*,\delta) &:= \{x \in B_X : x^*(x) \geq 1-\delta\}, \\ S(x^*,y^*,\delta) &:= S(x^*,\delta) \cap S(y^*,\delta). \end{aligned} \tag{2.14}$$

The following definitions were also given in [27]: we say that the unit sphere S_X of a Banach space X has a *crease* if there are $x^*, y^* \in S_{X^*}$, $x^* \neq y^*$, such that

$$\operatorname{diam}(S(x^*,y^*,0)) > 0. \tag{2.15}$$

A Banach space X is called *noncreasy* if S_X does not have a crease.

We say that a Banach space X is *uniformly noncreasy* (UNC) if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x^*, y^* \in S_{X^*}$ with $\|x^* - y^*\| \geq \varepsilon$, we have

$$\operatorname{diam} S(x^*,y^*,\delta) \leq \varepsilon. \tag{2.16}$$

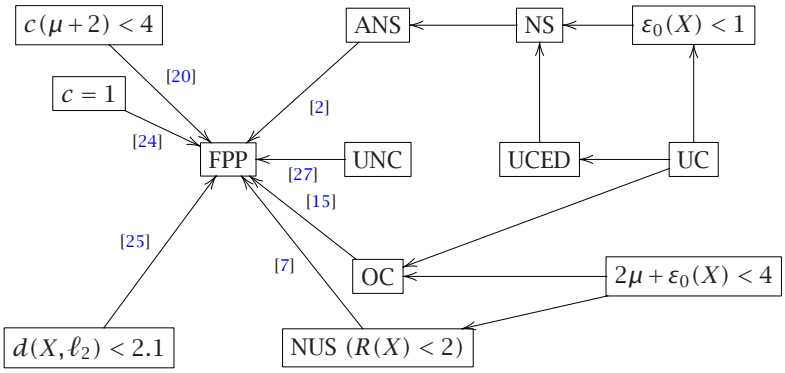
Examples of UNC Banach spaces are the uniformly convex and uniformly smooth Banach spaces. The following theorem is the main result of [27].

Theorem 2.7 (Prus [27]). *UNC Banach spaces have WFPP.*

In particular, the space $E_{\sqrt{2}}$ is UNC but does not have NS. There are renormings of ℓ_2 for which the corresponding Banach space is UNC but fails to be

both OC and NUS. A class of Banach spaces more general than UNC Banach spaces has been given recently in [9].

2.7. Summary. The arrows in the following diagram are used for implications. Nondefined symbols or abbreviations are standard in Banach space geometry. Numbered arrows are sufficient conditions for FPP. No (oriented) arrow between two cells means that effective example or separation is well known.



(2.17)

3. Example. For $x \in \ell_2$, define

$$\begin{aligned} M(x) &:= \sup\{|x(i)|+|x(j)| : 1 \le i < j\}, \\ S(x) &:= \sup\{|x(1)+x(n)+x(n+1)+x(n+2)| : n \ge 2\}. \end{aligned} \tag{3.1}$$

We consider the Banach space $E = (\ell_2, \|\cdot\|)$ whose norm $\|\cdot\|$ is defined by

$$\|x\| := \max\left\{\frac{1}{3}\|x\|_2, M(x), S(x)\right\}. \tag{3.2}$$

Our aim here is to show that E lacks all geometrical sufficient conditions for FPP, listed above, although it still verifies the FPP.

It is easy to see that for every $x \in \ell_2$,

$$\frac{1}{3}\|x\|_2 \le \|x\| \le 2\|x\|_2, \tag{3.3}$$

and since both inequalities are sharp, Lin's stability theorem (Theorem 2.1) cannot be applied in this case.

3.1. The space E is not uniformly nonsquare. Given a Banach space X, the *modulus of convexity* of X is the function $\delta_X : [0,2] \to [0,1]$ defined as $\delta_X(\varepsilon) = \inf\{1-(1/2)\|x+y\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon\}$, and the *characteristic of convexity* of X is the constant $\varepsilon_0(X) := \sup\{\varepsilon \in [0,2] : \delta(\varepsilon) = 0\}$. Recall that X is *uniformly convex* if $\varepsilon_0(X) = 0$ and that X is *uniformly nonsquare* if $\varepsilon_0(X) < 2$. It is well known that $\varepsilon_0(X) < 1$ implies that $(X, \|\cdot\|)$ is reflexive and has NS

[11], so it has the FPP. However, for uniformly nonsquare Banach spaces, the FPP has been proved only under additional assumptions on the space (see [5] or [17]).

Proposition 3.1. *The characteristic of convexity of E is 2.*

Let (e_n) be the standard Schauder basis of ℓ_2. As $\|e_1 - e_2\| = 2$ and $\|e_i\| = 1$ for every positive integer i,

$$\delta_E(2) \le 1 - \frac{1}{2}\|e_1 + e_2\| = 0. \tag{3.4}$$

3.2. The space E does not have ANS

Proposition 3.2. *The space $(\ell_2, \|\cdot\|)$ does not have ANS.*

Proof. Consider the set C defined as

$$C = \left\{ (x(n)) : \|x\|_2 \le 1,\ \|x\|_\infty \le \frac{1}{2},\ x(n) \ge 0\ (n \ge 1),\ x(2k-1) = 0\ (k=1,2,\dots) \right\}. \tag{3.5}$$

We claim that C is a closed, bounded, and convex set with $\operatorname{diam}_{\|\cdot\|}(C) = 1$, which contains a diametral sequence (x_n) with $x_{n+1} - x_n \to 0$.

To prove that C is closed and convex is easy. To see that $\operatorname{diam}_{\|\cdot\|}(C) = 1$, observe that for all $x, y \in C$,

$$\frac{1}{3}\|x - y\|_2 = \frac{1}{3}\left[\|x\|_2^2 + \|y\|_2^2 - 2\langle x, y\rangle\right]^{1/2} \le \frac{1}{3}\left[\|x\|_2^2 + \|y\|_2^2\right]^{1/2} \le 1. \tag{3.6}$$

Moreover, for positive integers $i \ne j$, we have $x(i), y(i), x(j), y(j) \in [0, 1/2]$, which implies that

$$|x(i) - y(i)| + |x(j) - y(j)| \le 1. \tag{3.7}$$

Thus, $M(x - y) \le 1$.

That $S(x - y) \le 1$ follows from $x(1) = y(1) = 0$ and the fact that in the following inequality one of the three summands on the right-hand side is zero:

$$\begin{aligned} &|x(n) - y(n) + x(n+1) - y(n+1) + x(n+2) - y(n+2)| \\ &\quad \le |x(n) - y(n)| + |x(n+1) - y(n+1)| \\ &\qquad + |x(n+2) - y(n+2)|. \end{aligned} \tag{3.8}$$

These previous calculations show that $\operatorname{diam}_{\|\cdot\|}(C) \le 1$. To see the equality consider the sequence (v_n) in C, defined by $v_n := (1/2)(e_{4n} + e_{4n+2})$, and notice that $\|v_n - v_{n+1}\| = 1$.

We consider the sequence (x_n) in C defined by

$$\begin{aligned} x_n &:= \frac{(2k+1)^2-n}{4k+1}v_k + v_{k+1} \quad \text{if } (2k)^2 < n \le (2k+1)^2, \\ x_n &:= v_{k+1} + \frac{n-(2k+1)^2}{4k+3}v_k \quad \text{if } (2k+1)^2 < n \le (2k+2)^2. \end{aligned} \tag{3.9}$$

It is straightforward (but tedious) to check that $x_{n+1}-x_n \to 0$.

To see that (x_n) is a diametral sequence, observe that $v_k = (1/2)(e_{4k}+e_{4k+2})$ and $v_{k+1} = (1/2)(e_{4k+4}+e_{4k+6})$ have disjoint support, and then we can write

$$x_n - x = \left(\dots, \frac{1}{2} - x(4k+4), \frac{1}{2} - x(4k+6), \dots\right), \tag{3.10}$$

where either $(2k)^2 < n \le (2k+1)^2$ or $(2k+1)^2 < n \le (2k+2)^2$. Consequently,

$$\begin{aligned} 1 \ge \|x_n - x\| \ge M(x_n - x) &\ge \left|\frac{1}{2} - x(4k+4)\right| + \left|\frac{1}{2} - x(4k+6)\right| \\ &= 1 - x(4k+4) - x(4k+6), \end{aligned} \tag{3.11}$$

and since $x \in \ell_2$, this shows that

$$\lim_{n\to\infty} \|x_n - x\| = 1 = \operatorname{diam}(C). \tag{3.12}$$

□

3.3. The space E is not OC

Proposition 3.3. *The space $(\ell_2, \|\cdot\|)$ is not OC.*

Proof. Consider the sequence (e_n) of unitary vectors in ℓ_2. For $n, m \in \mathbb{N}$, $n \ne m$, we have

$$\|e_n + e_m\| = \|e_n - e_m\| = 2, \tag{3.13}$$

from where $D[(e_n)] = 2$. Hence, for $z = e_n + e_m$, we have $\|z\| = 2$ and

$$\|z - e_n\| = \|z - e_m\| = \frac{1}{2}\|e_n - e_m\|. \tag{3.14}$$

Therefore, $z \in M_\beta(e_m, e_n)$ for every $\beta > 0$ and $m \ne n$, which implies that

$$\begin{aligned} |M_\beta(e_m, e_n)| &\ge 2, \\ A_\beta[(e_n)] = \limsup_n \left(\limsup_m |M_\beta(e_n, e_m)|\right) &\ge 2 = D[(e_n)]. \end{aligned} \tag{3.15}$$

As this holds for any positive β, it follows that $(\ell_2, \|\cdot\|)$ is not OC. □

3.4. The space E is not WNUS

Proposition 3.4. *The space E is not WNUS, that is, $R((\ell_2, \|\cdot\|)) = 2$.*

Proof. As $e_n \in B_{(\ell_2,\|\cdot\|)}$ $(n = 1,2,\dots)$ and $e_n \rightharpoonup 0_{\ell_2}$, then

$$R(X) \geq \liminf_{n\to\infty} \|e_n + e_1\| = 2. \tag{3.16}$$

□

3.5. The space E has creases

Proposition 3.5. *The space $E = (\ell_2, \|\cdot\|)$ is not UNC.*

Proof. Let $f,g \in E^*$ be the functionals given by

$$f(x) = x(3) + x(5), \qquad g(x) = x(5) + x(7). \tag{3.17}$$

If we take

$$\begin{aligned} v &:= \left(0,0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\dots\right) \in S_X, \\ w &:= \left(0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\dots\right) \in S_X, \end{aligned} \tag{3.18}$$

then

$$1 = |f(v)| \leq \|f\|, \qquad 1 = |g(v)| \leq \|g\|. \tag{3.19}$$

On the other hand, for every $x \in E$, we have

$$\begin{aligned} |f(x)| &= |x(3) + x(5)| \leq M(x) \leq \|x\|, \\ |g(x)| &= |x(5) + x(7)| \leq M(x) \leq \|x\|, \end{aligned} \tag{3.20}$$

and therefore, $\|f\| = \|g\| = 1$. Since $(f-g)(x) = x(3) - x(7)$, obtain

$$\begin{aligned} 1 &= (f-g)\left(0,0,\frac{1}{2},0,0,0,\frac{-1}{2},0,\dots\right) \leq \|f-g\|, \\ |(f-g)(x)| &\leq |x(3)| + |x(7)| \leq M(x) \leq \|x\|, \end{aligned} \tag{3.21}$$

from where $\|f-g\| = 1$.

Moreover, $f(v) = f(w) = g(v) = g(w) = 1$, and hence, for every $\delta \in [0,1]$,

$$v, w \in S(f,g,\delta), \tag{3.22}$$

which implies that $\operatorname{diam}(S(f,g,\delta)) \geq \|v-w\| = 1$. In summary, given $\eta > 0$ small enough, there exists $\varepsilon = 1-\eta$ such that for all $\delta \in [0,1]$, there exist $f,g \in S_X$ with $\|f-g\| \geq \varepsilon$ and $\operatorname{diam}(S(f,g,\delta)) > \varepsilon$. This contradicts the definition of being UNC. □

3.6. Unconditional constants for the space E. With respect to the standard basis (e_n), we can point out that

$$\left\| -\frac{1}{2}e_1 + \frac{1}{2}e_2 - \frac{1}{2}e_3 + \frac{1}{2}e_4 \right\| = 1, \tag{3.23}$$

but

$$\left\|\frac{1}{2}e_1+\frac{1}{2}e_2+\frac{1}{2}e_3+\frac{1}{2}e_4\right\| = 2. \tag{3.24}$$

Hence,

$$\mu := \sup\left\{\left\|\sum_{n=1}^{\infty}\theta_n a_n e_n\right\| : \left\|\sum_{n=1}^{\infty} a_n e_n\right\| = 1,\ \theta_n = \pm 1\right\} \geq 2. \tag{3.25}$$

On the other hand, $\|-(1/2)e_1+(1/2)e_2+(1/2)e_3+(1/2)e_4\| = 1$ and

$$\left\|P_{\{2,3,4\}}\left(-\frac{1}{2}e_1+\frac{1}{2}e_2+\frac{1}{2}e_3+\frac{1}{2}e_4\right)\right\| = \frac{3}{2}. \tag{3.26}$$

Hence, $c \geq \|P_{\{2,3,4\}}\| \geq 3/2$.

Therefore, $c(\mu+2) \geq 6$, so Khamsi theorem does not apply in this case. Nevertheless, an obvious question arises.

Question 3.6. Is there an unconditional Schauder basis (w_n) on E for which $c(\mu+2) < 4$?

3.7. The space E enjoys the FPP. The proof of this result uses the fact that "pathological" sequences exist in those Banach spaces lacking the FPP. A classical result in this direction is the well-known Goebel-Karlovitz lemma (see [12]) whose statement is as follows: assume that $T : K \to K$ is a nonexpansive mapping, where K is a nonempty, bounded, closed, and convex subset of a Banach space X. If K is minimal for T (i.e., minimal for the properties of being nonempty, bounded, closed, convex, and T-invariant) and (x_n) is an approximate fixed-point sequence (AFPS) for T (which means that $x_n - T(x_n) \to 0$), then (x_n) is diametral in K, that is, $\|x_n - x\| \to \operatorname{diam}(K)$ for any $x \in K$. If $\operatorname{diam}(K) > 0$ (which is the case when T has no fixed points in K), this behavior seems unusual. And the more pathology is strange, the larger the class of Banach spaces for which such pathology fails (and then have the FPP).

In this context, a significant impulse to the theory was given by Maurey [26] who proved that between two AFPSs there is always another AFPS. This result was subsequently refined by Lin [24] (and many others) who obtained more sophisticated pathologies, in most cases via ultrapower techniques. The result needed for our purposes is simpler than those and reads as follows.

Lemma 3.7. *Suppose that $T : K \to K$ is nonexpansive with K nonempty, closed, bounded, and convex and let $d = \operatorname{diam}(K)$. If (x_n^i), $i = 1,2,3$, are AFPS for T, then there exists (z_n), an AFPS for T, such that $\limsup \|z_n - x_n^i\| \leq (2/3)d$, $i = 1,2,3$.*

Proof. Since (x_n^i), $i = 1,2,3$, are AFPS for T, we may choose a sequence (α_n) of real numbers in $(0,1)$ such that $\alpha_n \to 0$ and

$$\frac{1}{\alpha_n}\max\left\{\|T(x_n^i)-x_n^i\| : i = 1,2,3\right\} \xrightarrow{n\to\infty} 0. \tag{3.27}$$

By the Banach contraction principle, there exists a sequence (z_n) in K such that

$$z_n = (1-\alpha_n)T(z_n) + \alpha_n \frac{x_n^1 + x_n^2 + x_n^3}{3}. \tag{3.28}$$

Observe that (z_n) is an AFPS for T because $\|z_n - T(z_n)\| \le \alpha_n d$ and $\alpha_n d \to 0$.

Now, use the nonexpansiveness of T to obtain, for (i,j,k), any permutation of $(1,2,3)$,

$$\begin{aligned}\|z_n - x_n^i\| \le (1-\alpha_n)\|z_n - x_n^i\| + (1-\alpha_n)\|T(x_n^i) - x_n^i\| \\ + \alpha_n \frac{\|x_n^i - x_n^j\| + \|x_n^i - x_n^k\|}{3}.\end{aligned} \tag{3.29}$$

Finally, grouping the terms containing $\|z_n - x_n^i\|$, obtain that

$$\|z_n - x_n^i\| \le \frac{1-\alpha_n}{\alpha_n}\|T(x_n^i) - x_n^i\| + \frac{2}{3}d, \tag{3.30}$$

from where the result follows because of the choice of $\{\alpha_n\}$. □

Theorem 3.8. *The space E has the FPP.*

Proof. To prove that E has the FPP we will argue by contradiction: suppose that E lacks the FPP and let $T : K \to K$ be a fixed-point free nonexpansive mapping defined on the nonempty, convex, and weakly compact set K. We assume that K is minimal for T (the existence of such a minimal set is guaranteed by Zorn's lemma) and let (x_n) be an AFPS for T. Since K is weakly compact, (x_n) has a subsequence which is weakly convergent, say to x_0, and since any subsequence of (x_n) is an AFPS for T, we may assume that (x_n) is itself weakly convergent. To simplify the formulae and arguments we may also assume, by translating and dilating K if necessary, that $0 \in K$, $d = 1$, and that (x_n) is weakly null.

Using that (x_n) is a weakly null sequence in ℓ_2, there exists a subsequence of it, still denoted by (x_n), to which a sequence (u_n) in ℓ_2 can be associated with the properties of that $\|u_n - x_n\| \to 0$, the set $\operatorname{supp}(u_n) = \{i : u_n(i) \neq 0\}$ is finite, $\min\operatorname{supp}(u_1) > 1$, and also $\max\operatorname{supp}(u_n) + 4 < \min\operatorname{supp}(u_{n+1})$.

Notice that $\lim_n \langle x_n, x_{n+1}\rangle = 0$. To deduce it use that u_n and u_{n+1} have disjoint supports and the following inequality:

$$\begin{aligned} |\langle x_n, x_{n+1}\rangle| &= |\langle x_n, x_{n+1}\rangle - \langle u_n, u_{n+1}\rangle| \\ &\le |\langle x_n, x_{n+1}\rangle - \langle x_n, u_{n+1}\rangle + \langle x_n, u_{n+1}\rangle - \langle u_n, u_{n+1}\rangle| \\ &\le \|x_n\|\|x_{n+1} - u_{n+1}\| + \|x_n - u_n\|\|u_{n+1}\|. \end{aligned} \tag{3.31}$$

Consider the AFPSs for T given by $x_n^1 = x_{3n}$, $x_n^2 = x_{3n+1}$, and $x_n^3 = x_{3n+2}$, and also denote by u_n^i the corresponding subsequences of (u_n). And apply Lemma 3.7 to obtain an AFPS for T, (z_n), such that

$$\limsup_n \|z_n - x_n^i\| \le \frac{2}{3}, \quad i = 1,2,3. \tag{3.32}$$

Since K is minimal for T and $0 \in K$, the Goebel-Karlovitz lemma ensures that $\lim \|z_n\| = 1$. Our next goal will be to obtain a contradiction with this. To do so, recall that $\|z_n\| = \max\{(1/3)\|z_n\|_2, M(z_n), S(z_n)\}$, and we proceed to estimate each quantity.

Start with $\|z_n\|_2$. Since $\langle x_n^1, x_n^2\rangle \to 0$, the easy-to-check equality

$$\|z_n\|_2^2 = \|z_n - x_n^1\|_2^2 + \|z_n - x_n^2\|_2^2 - \|x_n^1 + x_n^2 - z_n\|_2^2 - 2\Re\langle x_n^1, x_n^2\rangle \tag{3.33}$$

yields

$$\limsup \|z_n\|_2^2 \le \limsup \|z_n - x_n^1\|_2^2 + \limsup \|z_n - x_n^2\|_2^2 \le 8. \tag{3.34}$$

So, $\limsup (1/3)\|z_n\|_2 \le \sqrt{8}/3$.

We see now that $\limsup M(z_n) \le 2/3$. For this, take two positive integers i and j at random. Assume that $i < j$. Since the supports of u_n^1, u_n^2, and u_n^3 are pairwise disjoint, there is one of them, say $u_n^{k(i,j,n)}$, whose support contains neither i nor j. Then,

$$\begin{aligned} |z_n(i)| + |z_n(j)| &= \left|z_n(i) - u_n^{k(i,j,n)}(i)\right| + \left|z_n(j) - u_n^{k(i,j,n)}(j)\right| \\ &\le \left\|z_n - u_n^{k(i,j,n)}\right\| \\ &\le \left\|z_n - x_n^{k(i,j,n)}\right\| + \left\|x_n^{k(i,j,n)} - u_n^{k(i,j,n)}\right\| \\ &\le \max\{\|z_n - x_n^s\| : s = 1,2,3\} \\ &\quad + \max\{\|x_n^s - u_n^s\| : s = 1,2,3\}. \end{aligned} \tag{3.35}$$

From this, obtain that

$$M(z_n) \le \max\{\|z_n - x_n^s\| : s = 1,2,3\} + \max\{\|x_n^s - u_n^s\| : s = 1,2,3\}, \tag{3.36}$$

and then, bearing in mind that $\|x_n^k - u_n^k\| \xrightarrow{n\to\infty} 0$ and $\limsup_n \|z_n - x_n^k\| \le 2/3$ for $k \in \{1,2,3\}$, one derives that

$$\limsup_n M(z_n) \le \limsup_n \left(\max\{\|z_n - x_n^s\| : s = 1,2,3\}\right) \le \frac{2}{3}. \tag{3.37}$$

Finally, we see that

$$\limsup_n S(z_n) = \limsup_n \left(\sup_{r\ge 2} |z_n(1) + z_n(r) + z_n(r+1) + z_n(r+2)|\right) \le \frac{2}{3}. \tag{3.38}$$

Consider any positive integer r and choose $k(r) \in \{1,2,3\}$ such that $\operatorname{supp}(u_n^{k(r)})$ does not contain any of r, $r+1$, $r+2$ (this is possible because $\max\operatorname{supp}(u_j) + 4 < \min\operatorname{supp}(u_{j+1})$ for any j). This, together with the fact that $u_n^k(1) = 0$, gives

$$\begin{aligned}
&|z_n(1) + z_n(r) + z_n(r+1) + z_n(r+2)| \\
&\quad = |(z_n - u_n^{k(r)})(1) + (z_n - u_n^{k(r)})(r) + (z_n - u_n^{k(r)})(r+1) \\
&\qquad + (z_n - u_n^{k(r)})(r+2)| \\
&\quad \le S(z_n - u_n^{k(r)}) \le \left\|z_n - u_n^{k(r)}\right\| \le \left\|z_n - x_n^{k(r)}\right\| + \left\|x_n^{k(r)} - u_n^{k(r)}\right\| \\
&\quad \le \max\{\|z_n - x_n^s\| : s = 1,2,3\} + \max\{\|x_n^s - u_n^s\| : s = 1,2,3\}.
\end{aligned} \tag{3.39}$$

Hence,

$$S(z_n) \le \max\{\|z_n - x_n^s\| : s = 1,2,3\} + \max\{\|x_n^s - u_n^s\| : s = 1,2,3\} \tag{3.40}$$

and then $\limsup S(z_n) \le 2/3$.

From all these estimates, obtain

$$\begin{aligned}
\limsup \|z_n\| &= \limsup \max\left\{\frac{1}{3}\|z_n\|_2, M(z_n), S(z_n)\right\} \\
&\le \max\left\{\frac{\sqrt{8}}{3}, \frac{2}{3}\right\} < 1,
\end{aligned} \tag{3.41}$$

which is the desired contradiction with the Goebel-Karlovitz lemma. □

4. Final remark. Indeed, it is very unusual for spaces to fail to have NS (see [12, page 65]). Then, it seems even more unusual to fail to have both asymptotic normal structure and orthogonal convexity, near uniform smoothness and uniform noncreasyness. But our example is not quite sophisticated and fails all of these and other sufficient conditions for the FPP. One might think that the list of such known sufficient conditions for the FPP is still not large enough. Then one should seek for wider geometrical sufficient conditions.

On the other hand, we quote the following intriguing sentence from [14]:

> "let us finish with the metamathematical statement not quite clear but in our opinion in some sense true: for any sufficient condition for FPP there exists a set having FPP, which does not satisfy it."

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