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ON SOME PROPERTIES OF ⊕-SUPPLEMENTED MODULES

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A module M is ⊕-supplemented if every submodule of M has a supplement which is a direct summand of M. In this paper, we show that a quotient of a ⊕-supplemented module is not in general ⊕-supplemented. We prove that over a commutative ring R, every finitely generated ⊕-supplemented R-module M having dual Goldie dimension less than or equal to three is a direct sum of local modules. It is also shown that a ring R is semisimple if and only if the class of ⊕-supplemented R-modules coincides with the class of injective R-modules. The structure of ⊕-supplemented modules over a commutative principal ideal ring is completely determined.

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1. Introduction. All rings considered in this paper will be associative with an identity element. Unless otherwise mentioned, all modules will be left unitary modules. Let R be a ring and M an R-module. Let A and P be submodules of M. The submodule P is called a *supplement* of A if it is minimal with respect to the property $A+P=M$. Any $L \leq M$ which is the supplement of an $N \leq M$ will be called a *supplement submodule* of M. If every submodule U of M has a supplement in M, we call M *complemented*. In [25, page 331], Zöschinger shows that over a discrete valuation ring R, every complemented R-module satisfies the following property (P): every submodule has a supplement which is a direct summand. He also remarked in [25, page 333] that every module of the form $M \cong (R/a_1)\times\cdots\times(R/a_n)$, where R is a commutative local ring and a_i $(1 \leq i \leq n)$ are ideals of R, satisfies (P). In [12, page 95], Mohamed and Müller called a module ⊕-supplemented if it satisfies property (P).

On the other hand, let U and V be submodules of a module M. The submodule V is called a complement of U in M if V is maximal with respect to the property $V \cap U = 0$. In [17] Smith and Tercan investigate the following property which they called (C_{11}): every submodule of M has a complement which is a direct summand of M. So, it was natural to introduce a dual notion of (C_{11}) which we called (D_{11}) (see [6, 7]). It turns out that modules satisfying (D_{11}) are exactly the ⊕-supplemented modules. A module M is called a completely ⊕-supplemented (see [5]) (or *satisfies* (D_{11}^{+}) in our terminology, see [6, 7]) if every direct summand of M is ⊕-supplemented.

Our paper is divided into four sections. The purpose of Section 2 is to answer the following natural question: is any factor module of a $\oplus$-supplemented module $\oplus$-supplemented? Some relevant counterexamples are given.

In Section 3 we prove that, over a commutative ring, every finitely generated $\oplus$-supplemented module having dual Goldie dimension less than or equal to three is a direct sum of local modules.

Section 4 describes the structure of $\oplus$-supplemented modules over commutative principal ideal rings.

In the last section we determine the class of rings R with the property that every $\oplus$-supplemented R-module is injective. These turn out to be the class of all left Noetherian V-rings (Proposition 5.3). It is also shown that a ring R is semisimple if and only if the class of $\oplus$-supplemented R-modules coincides with the class of injective R-modules (Proposition 5.5).

For an arbitrary module M, we will denote by $\mathrm{Rad}(M)$ the Jacobson radical of M. The injective hull of M will be denoted by $E(M)$. The annihilator of M will be denoted by $\mathrm{Ann}_R(M)$. A submodule A of M is called *small* in M ($A \ll M$) if $A + B \neq M$ for any proper submodule B of M. A nonzero module H is called *hollow* if every proper submodule is small in H and is called *local* if the sum of all its proper submodules is also a proper submodule. We notice that a local module is just a cyclic hollow module.

2. Quotients of $\oplus$-supplemented modules.

By [23, corollary on page 45], every factor module of a complemented module is complemented. Now, let M be a $\oplus$-supplemented module. In this section we will answer the following natural question: is any factor module of M $\oplus$-supplemented?

First, we mention the following result, which we will use frequently in the sequel.

Proposition 2.1 [6, Proposition 1]. *The following are equivalent for a module M:*

(i) *M is $\oplus$-supplemented;*

(ii) *for any submodule N of M, there exists a direct summand K of M such that $M = N + K$ and $N \cap K$ is small in K.*

A commutative ring R is a valuation ring if it satisfies one of the following three equivalent conditions:

(i) for any two elements a and b, either a divides b or b divides a;

(ii) the ideals of R are linearly ordered by inclusion;

(iii) R is a local ring and every finitely generated ideal is principal.

A module M is called finitely presented if $M \cong F/K$ for some finitely generated free module F and finitely generated submodule K of F. An important result about these modules is that if M is finitely presented and $M \cong F/G$, where F is a finitely generated free module, then G is also finitely generated (see [2]).

Example 2.2. Let R be a commutative local ring which is not a valuation ring and let $n \geq 2$. By [21, Theorem 2], there exists a finitely presented indecomposable module $M = R^{(n)}/K$ which cannot be generated by fewer than n elements. By [6, Corollary 1], $R^{(n)}$ is ⊕-supplemented. However M is not ⊕-supplemented [6, Proposition 2].

The *dual Goldie dimension* of an R-module, denoted by $\text{corank}({}_RM)$, was introduced by Varadarajan in [19]. If $M = 0$, the corank of M is defined as 0. Let $M \neq 0$ and k an integer greater than or equal to one. If there is an epimorphism $f : M \to \prod_{i=1}^{k} N_i$, where each $N_i \neq 0$, we say that the $\text{corank}({}_RM) \geq k$. If $\text{corank}({}_RM) \geq k$ and $\text{corank}({}_RM) \not\geq k+1$, then we define $\text{corank}({}_RM) = k$. If the $\text{corank}({}_RM) \geq k$ for every $k \geq 1$, we say that the $\text{corank}({}_RM) = \infty$. It was shown in [14, 19] that the $\text{corank}({}_RM) < \infty$ if and only if there is an epimorphism $f : M \to \prod_{i=1}^{k} H_i$, where H_i is hollow and $\ker(f)$ is small in M.

As in [20], a module M has the *exchange property* if for any module G, where

$$G = M' \oplus C = \oplus_{i \in I} D_i \tag{2.1}$$

with $M' \cong M$, there are submodules $D_i' \leq D_i$ such that $G = M' \oplus (\oplus_{i \in I} D_i')$.

Before proceeding any further, we consider another example (note that the module considered is decomposable).

Example 2.3. Let R be a commutative local ring which is not a valuation ring. Let a and b be elements of R, neither of them divides the other. By taking a suitable quotient ring, we may assume $(a) \cap (b) = 0$ and $am = bm = 0$, where m is the maximal ideal of R. Let F be a free module with generators x_1, x_2, and x_3. Let K be the submodule generated by $ax_1 - bx_2$ and let $M = F/K$. Thus,

$$M = \frac{Rx_1 \oplus Rx_2 \oplus Rx_3}{R(ax_1 - bx_2)} = (R\overline{x_1} + R\overline{x_2}) \oplus R\overline{x_3}. \tag{2.2}$$

Suppose that M is ⊕-supplemented. There exist submodules H and N of M such that $M = H \oplus N$, $R\overline{x_1} + N = M$, and $R\overline{x_1} \cap N$ is small in N (Proposition 2.1). By the proof of [21, Theorem 2], $R\overline{x_1} + R\overline{x_2}$ is an indecomposable module which cannot be generated by fewer than 2 elements. Thus $\text{corank}(R\overline{x_1} + R\overline{x_2}) = 2$ by [14, Proposition 1.7]. Hence $\text{corank}(M) = 3$. Since $H \cong M/N$ and $M/N \cong R\overline{x_1}/(N \cap R\overline{x_1})$, we get that H is a local direct summand of M and hence $\text{corank}(N) = 2$ (see [14, Corollary 1.9]). Since R is a commutative local ring, $\text{End}_R(R\overline{x_3})$ is a local ring by [4, Theorem 4.1]. Since $R\overline{x_3}$ has the exchange property [20, Proposition 1], there are submodules $H' \leq H$ and $N' \leq N$ such that $M = R\overline{x_3} \oplus H' \oplus N'$. Therefore $R\overline{x_1} + R\overline{x_2} \cong H' \oplus N'$. Thus $H' \oplus N'$ is indecomposable. Hence $N' = 0$ or $H' = 0$. But $\text{corank}(M) = 3$ and $\text{corank}(N) = 2$, so $M = R\overline{x_3} \oplus N$ and $N \cong R\overline{x_1} + R\overline{x_2}$ is indecomposable. Since $\overline{x_1}, \overline{x_2} \in M$, there are $\alpha, \beta \in R$ and $\overline{y_1}, \overline{y_2} \in N$ such that $\overline{x_1} = \alpha\overline{x_3} + \overline{y_1}$ and $\overline{x_2} = \beta\overline{x_3} + \overline{y_2}$. Hence $\overline{x_1} - \alpha\overline{x_3} \in N$ and $\overline{x_2} - \beta\overline{x_3} \in N$. But $M = R\overline{x_3} \oplus [R(\overline{x_1} - \alpha\overline{x_3}) + R(\overline{x_2} - \beta\overline{x_3})]$. Then $N = R(\overline{x_1} - \alpha\overline{x_3}) + R(\overline{x_2} - \beta\overline{x_3})$. Now, $M = R\overline{x_1} + N$ and $\overline{x_3} \in M$, so there exists $\alpha' \in R$ such that $\overline{x_3} - \alpha'\ \overline{x_1} \in N$. Note that $\alpha'\overline{x_1} - \alpha'\alpha\overline{x_3} \in N$ and $(1-\alpha'\alpha)\overline{x_3} \in N \cap R\overline{x_3}$. Thus $(1-\alpha'\alpha)\overline{x_3} = 0$, that is, $(1-\alpha'\alpha)x_3 \in R(ax_1 - bx_2)$. Hence $1-\alpha'\alpha = 0$. So α is invertible and $\alpha^{-1} = \alpha'$. Note that

$$a(\overline{x_1} - \alpha\overline{x_3}) - b(\overline{x_2} - \beta\overline{x_3}) = (b\beta - a\alpha)\overline{x_3}. \tag{2.3}$$

Thus $a(\overline{x_1} - \alpha\overline{x_3}) - b(\overline{x_2} - \beta\overline{x_3}) \neq 0$. Otherwise, $(b\beta - a\alpha)x_3 \in R(ax_1 - bx_2)$, which gives $b\beta = a\alpha$ and then $a = b\beta\alpha'$, which is a contradiction. Since $(b\beta - a\alpha)\overline{x_3} \in N \cap R\overline{x_3}$, then $N \cap R\overline{x_3} \neq 0$, which is a contradiction. It follows that M is not $\oplus$-supplemented. But $Rx_1 \oplus Rx_2 \oplus Rx_3$ is completely $\oplus$-supplemented [6, Corollary 2].

These examples show that a factor module of a $\oplus$-supplemented module is not in general $\oplus$-supplemented.

Proposition 2.5 deals with a special case of factor modules of $\oplus$-supplemented modules. First we prove the following lemma.

Lemma 2.4. *Let M be a nonzero module and let U be a submodule of M such that $f(U) \leq U$ for each $f \in \mathrm{End}_R(M)$. If $M = M_1 \oplus M_2$, then $U = U \cap M_1 \oplus U \cap M_2$.*

Proof. Let $\pi_i : M \to M_i$ $(i = 1,2)$ denote the canonical projections. Let x be an element of U. Then $x = \pi_1(x) + \pi_2(x)$. By hypothesis, $\pi_i(U) \leq U$ for $i = 1,2$. Thus $\pi_i(x) \in U \cap M_i$ for $i = 1,2$. Hence $U \leq U \cap M_1 \oplus U \cap M_2$. It follows that $U = U \cap M_1 \oplus U \cap M_2$. □

Proposition 2.5. *Let M be a nonzero module and let U be a submodule of M such that $f(U) \leq U$ for each $f \in \mathrm{End}_R(M)$. If M is $\oplus$-supplemented, then M/U is $\oplus$-supplemented. If, moreover, U is a direct summand of M, then U is also $\oplus$-supplemented.*

Proof. Suppose that M is $\oplus$-supplemented. Let L be a submodule of M which contains U. There exist submodules N and N' of M such that $M = N \oplus N'$, $M = L + N$, and $L \cap N$ is small in N (Proposition 2.1). By [23, Lemma 1.2(d)], $(N+U)/U$ is a supplement of L/U in M/U. Now apply Lemma 2.4 to get that $U = U \cap N \oplus U \cap N'$. Thus,

$$(N+U) \cap (N'+U) \leq (N+U+N') \cap U + (N+U+U) \cap N'. \tag{2.4}$$

Hence,

$$(N+U) \cap (N'+U) \leq U + (N + U \cap N + U \cap N') \cap N'. \tag{2.5}$$

It follows that $(N+U) \cap (N'+U) \leq U$ and $((N+U)/U) \oplus ((N'+U)/U) = M/U$. Then $(N+U)/U$ is a direct summand of M/U. Consequently, M/U is $\oplus$-supplemented.

Now suppose that U is a direct summand of M. Let V be a submodule of U. Since M is $\oplus$-supplemented, there exist submodules K and K' of M such that

$M = K \oplus K'$, $M = V + K$, and $V \cap K \ll K$ (Proposition 2.1). Thus $U = V + U \cap K$. But $U = U \cap K \oplus U \cap K'$ (Lemma 2.4), hence $U \cap K$ is a direct summand of U. Moreover, $V \cap (U \cap K) = V \cap K$ is small in K. Then, $V \cap (U \cap K)$ is small in $U \cap K$ by [23, Lemma 1.1(b)]. Therefore $U \cap K$ is a supplement of V in U and it is a direct summand of U. Thus U is ⊕-supplemented. □

Corollary 2.6. *Let M be an R-module and $P(M)$ the sum of all its radical submodules. If M is ⊕-supplemented, then $M/P(M)$ is ⊕-supplemented. If, moreover, $P(M)$ is a direct summand of M, then $P(M)$ is also ⊕-supplemented.*

Proof. By Proposition 2.5, it suffices to prove that $f(P(M)) \leq P(M)$ for each $f \in \mathrm{End}_R(M)$. Let N be a radical submodule of M and let f be an endomorphism of M and g its restriction to N. By [1, Proposition 9.14], $g(\mathrm{Rad}(N)) \leq \mathrm{Rad}(f(N))$. But $\mathrm{Rad}(N) = N$ and $f(N) = g(N)$, hence $f(N) \leq \mathrm{Rad}(f(N))$. Thus, $\mathrm{Rad}(f(N)) = f(N)$. This implies that $f(N) \leq P(M)$, and the corollary is proved. □

We recall that a module M is called semi-Artinian if every nonzero quotient module of M has nonzero socle. For a module $_RM$, we define

$$\mathrm{Sa}(M) = \sum_{\substack{U \leq M \\ U \text{ semi-Artinian}}} U. \tag{2.6}$$

By [18, Chapter VIII, Section 2, Corollary 2.2], if R is a left Noetherian ring and $_RM$ a semi-Artinian left R-module, then M is the sum of its submodules of finite length.

If R is a commutative Noetherian ring and M is an R-module, then $\mathrm{Sa}(M) = L(M)$, the sum of all Artinian submodules of M.

Corollary 2.7. *Let M be a ⊕-supplemented R-module. Then $M/\mathrm{Sa}(M)$ is ⊕-supplemented. If, moreover, $\mathrm{Sa}(M)$ is a direct summand of M, then $\mathrm{Sa}(M)$ is also ⊕-supplemented.*

Proof. By Proposition 2.5, it suffices to prove that $f(\mathrm{Sa}(M)) \leq \mathrm{Sa}(M)$ for each $f \in \mathrm{End}_R(M)$. Let U be a semi-Artinian submodule of M and let f be an endomorphism of M and g its restriction to U. Thus $U/\mathrm{Ker}(g) \cong g(U)$. Hence $f(U) \cong U/\mathrm{Ker}(g)$. But it is easy to check that $U/\mathrm{Ker}(g)$ is a semi-Artinian module. Therefore, $f(U)$ is semi-Artinian. □

Remark 2.8. Let M be a ⊕-supplemented module. It is clear that $M/\mathrm{Rad}(M)$ and $M/\mathrm{Soc}(M)$ are also ⊕-supplemented (see Proposition 2.5 and [1, Propositions 9.14 and 9.8]).

3. Some properties of finitely generated ⊕-supplemented modules. A module M is called *supplemented* if for any two submodules A and B with $A + B = M$, B contains a supplement of A.

The proof of the next result is taken from [6, Lemma 2], but is given for the sake of completeness.

Lemma 3.1. *Let M be a $\oplus$-supplemented R-module. If M contains a maximal submodule, then M contains a local direct summand.*

Proof. Let L be a maximal submodule of M. Since M is $\oplus$-supplemented, there exists a direct summand K of M such that K is a supplement of L in M. Then for any proper submodule X of K, X is contained in L since L is a maximal submodule and $L+X$ is a proper submodule of M by minimality of K. Hence $X \le L\cap K$ and X is small in K by [12, Lemma 4.5]. Thus K is a hollow module, and the lemma is proved. □

Proposition 3.2. *If M is a $\oplus$-supplemented module such that* $\operatorname{Rad}(M)$ *is small in M, then M can be written as an irredundant sum of local direct summands of M.*

Proof. Since $\operatorname{Rad}(M)$ is small in M, M contains a maximal submodule and hence M contains a local direct summand by Lemma 3.1. Let N be the sum of all local direct summands of M. If N is a proper submodule of M, then there exists a maximal submodule L of M such that $N \le L$ (see [8, Proposition 9 and Theorem 8]). Let P be a direct summand of M such that P is a supplement of L in M. Note that P is a local module (see the proof of Lemma 3.1) and hence it is contained in N, so $M = L+P \le L+N = L$. This is a contradiction. Hence we have $N = M$. Now let $M = \sum_{i\in I} L_i$ where each L_i is a local direct summand of M. Then,

$$\frac{M}{\operatorname{Rad}(M)} = \sum_{i\in I}\left[\frac{L_i+\operatorname{Rad}(M)}{\operatorname{Rad}(M)}\right] \tag{3.1}$$

and each

$$\frac{L_i+\operatorname{Rad}(M)}{\operatorname{Rad}(M)} \cong \frac{L_i}{L_i\cap \operatorname{Rad}(M)} \tag{3.2}$$

is simple by [23, Lemma 1.1(c)]. Hence

$$\frac{M}{\operatorname{Rad}(M)} = \bigoplus_{k\in K}\left[\frac{L_k+\operatorname{Rad}(M)}{\operatorname{Rad}(M)}\right] \tag{3.3}$$

for some subset $K \subseteq I$. Thus $M = \sum_{k\in K} L_k$ since $\operatorname{Rad}(M)$ is small in M. Clearly, the sum $\sum_{k\in K} L_k$ is irredundant. □

Corollary 3.3. *Let R be a commutative ring and M a finitely generated R-module. If M is $\oplus$-supplemented, then $M = H_1+H_2+\cdots+H_n$, where each H_i is a local direct summand of M and $n = \operatorname{corank}(M)$.*

Proof. By Proposition 3.2, $M = H_1 + H_2 + \cdots + H_n$, where each H_i is a local direct summand of M and the sum $\sum_{i=1}^{n} H_i$ is irredundant. By [16, Corollary 4.6], M is supplemented. Therefore $n = \operatorname{corank}(M)$ by [14, Proposition 1.7] and [19, Lemma 2.36 and Theorem 2.39]. □

Remark 3.4. (i) The module $M = (R\overline{x_1} + R\overline{x_2}) \oplus R\overline{x_3}$ in Example 2.3 is not ⊕-supplemented. On the other hand, M can be written as follows: $M = (R\overline{x_1} + R\overline{x_2}) \oplus R(\overline{x_1} - \overline{x_3})$; $M = (R\overline{x_1} + R\overline{x_2}) \oplus R(\overline{x_2} - \overline{x_3})$; and $M = R(\overline{x_1} - \overline{x_3}) + R(\overline{x_2} - \overline{x_3}) + R\overline{x_3}$. Therefore M is an irredundant sum of local direct summands of M. However, M is not ⊕-supplemented.

(ii) In the same example, we have that $K = R\overline{x_1} + R\overline{x_2}$ is an indecomposable direct summand of

$$M = \frac{Rx_1 \oplus Rx_2 \oplus Rx_3}{R(ax_1 - bx_2)} = (R\overline{x_1} + R\overline{x_2}) \oplus R\overline{x_3}. \tag{3.4}$$

Then K is not an irredundant sum of local direct summands. This example shows that, in general, a direct summand of a module which is written as an irredundant sum of local direct summands does not have the same property.

Proposition 3.5. *Let M be a finitely generated ⊕-supplemented module such that $k = \operatorname{corank}(M) \le 2$. Then M is a direct sum of local modules.*

Proof. It is clear that if $k = 1$, then M is a local module. Now suppose that $k = 2$. Since M is ⊕-supplemented, M contains a local direct summand H (Lemma 3.1). Let K be a submodule of M such that $M = H \oplus K$. By [14, Corollary 1.9], we have $\operatorname{corank}(K) = 1$ and hence K is a local module (see [19, Proposition 1.11]). Thus M is a direct sum of local modules, as required. □

Our next objective is to prove that over a commutative ring, if M is a finitely generated ⊕-supplemented module with $\operatorname{corank}(M) = 3$, then M is a direct sum of local modules. We first prove the following generalization of [11, Lemma 2.3].

Lemma 3.6. *Let $L_1, L_2, \ldots, L_n$ be indecomposable direct summands of a module M such that $\operatorname{End}_R(L_i)$ is a local ring for each i ($1 \le i \le n$). If $L_i \ncong L_j$ for all $i \ne j$, then $\sum_{i=1}^{n} L_i$ is direct and is a direct summand of M.*

Proof. We use induction over n. Assume that $L_1 + L_2 + \cdots + L_{n-1}$ is a direct sum and is a direct summand of M and let $L = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$. There exists a submodule N of M such that $M = L \oplus N$. By [20, Proposition 1], L_n has the exchange property. Thus, $M = L_n \oplus L' \oplus N'$ for some submodules L' and N' of M with $L' \le L$ and $N' \le N$. Let N'' and L'' be two submodules of M such that $N = N' \oplus N''$ and $L = L' \oplus L''$. Hence $M = L' \oplus N' \oplus L'' \oplus N''$. Therefore, $L_n \cong L'' \oplus N''$. This implies that $L'' = 0$ or $N'' = 0$. Hence $L' = L$ or $N' = N$. Suppose that $N' = N$. Thus $L_n \oplus L' \cong L$. By the Krull-Schmidt-Azumaya theorem, every indecomposable direct summand of L is isomorphic to one of the L_i, $1 \le i \le n-1$. It follows that L_n is isomorphic to one of the L_i, $1 \le i \le n-1$, which is a contradiction. Therefore $L' = L$ and $M = L_n \oplus L \oplus N'$, that is, $M = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1} \oplus L_n \oplus N'$, and the lemma is proved. □

Corollary 3.7. *Suppose that R is commutative or left Noetherian. Let L_1, $L_2, \dots, L_n$ be hollow local direct summands of a module M. If $L_i \ncong L_j$ for all $i \ne j$, then $\sum_{i=1}^{n} L_i$ is direct and is a direct summand of M.*

Proof. This is a consequence of [4, Theorems 4.1 and 4.2] and Lemma 3.6. □

Proposition 3.8. *Suppose that R is a commutative ring. Let M be a finitely generated $\oplus$-supplemented module such that all the hollow direct summands of M are isomorphic. Then M is a direct sum of hollow local modules.*

Proof. By Proposition 3.2, we can write $M = H_1 + H_2 + \cdots + H_n$ as an irredundant sum of hollow local direct summands. By hypothesis, $H_1 \cong H_2 \cong \cdots \cong H_n$. Thus,

$$\operatorname{Ann}_R(H_1) = \operatorname{Ann}_R(H_2) = \cdots = \operatorname{Ann}_R(H_n). \tag{3.5}$$

Hence,

$$\operatorname{Ann}_R(M) = \bigcap_{i=1}^{n} \operatorname{Ann}_R(H_i) = \operatorname{Ann}_R(H_i) \quad \text{for each } i \ (1 \le i \le n). \tag{3.6}$$

Therefore all hollow local direct summands of M are isomorphic to R/I, where $I = \operatorname{Ann}_R(M)$. Let H be a local submodule of M such that H is not small in M. Since M is $\oplus$-supplemented, there exist submodules N and N' of M such that $H + N = M$, $N' \oplus N = M$, and $H \cap N$ is small in N (Proposition 2.1). It follows that $N' \cong M/N \cong H/(H \cap N)$. Hence, N' is a local module. This implies that $\operatorname{Ann}_R(N') = I$ and $\operatorname{Ann}_R(H/(H \cap N)) = I$. Thus, the set $\{r \in R \mid rx \in N\} = I$, where $H = Rx$. Let $y \in H \cap N$. There exists $\alpha \in R$ with $y = \alpha x$. So $\alpha \in I$ and hence $y = 0$ since $I \subseteq \operatorname{Ann}_R(H)$. Therefore $H \cap N = 0$ and $M = H \oplus N$. It follows that every nonsmall local submodule of M is a direct summand of M. Note that $\operatorname{corank}(M) < \infty$ (Corollary 3.3). Applying [23, corollary on page 45] and [8, Proposition 9], we get that M is a direct sum of local modules. □

Corollary 3.9. *Let R be a commutative ring and M a finitely generated $\oplus$-supplemented module with* $\operatorname{corank}(M) = 3$. *Then M is a direct sum of local modules.*

Proof. Let F_0 be an irredundant set of representatives of the local direct summands of M (F_0 is not empty by Lemma 3.1). By Corollary 3.7, $\operatorname{Card}(F_0) \le 3$. If $\operatorname{Card}(F_0) = 3$, then M is a direct sum of local modules (Corollary 3.7). If $\operatorname{Card}(F_0) = 2$ and $F_0 = \{L_1, L_2\}$, then there exists a submodule L_3 of M such that

$M = L_1 \oplus L_2 \oplus L_3$ (Corollary 3.7). But $\operatorname{corank}(M) = 3$. Therefore $\operatorname{corank}(L_3) = 1$ (see [14, Corollary 1.9]) and hence L_3 is a local module. If $\operatorname{Card}(F_0) = 1$, then M is a direct sum of local modules by Proposition 3.8. □

Remark 3.10. (i) If M is a finitely generated ⊕-supplemented module with $\operatorname{corank}(M) \le 2$, then M is completely ⊕-supplemented (see [6, Proposition 6] and Proposition 3.5).

(ii) If R is a commutative ring and M a finitely generated ⊕-supplemented module with $\operatorname{corank}(M) = 3$, then M is completely ⊕-supplemented (see [6, Corollary 6] and Corollary 3.9).

4. ⊕-supplemented modules over commutative principal ideal rings. In this section, the structure of ⊕-supplemented modules over a principal ideal ring is completely determined.

Let R be a commutative Noetherian ring. Let Ω be the set of all maximal ideals of R. As in [24, page 53], if $m \in \Omega$ and M is an R-module, we denote the m-local component of M by $K_m(M) = \{x \in M \mid x = 0$ *or* the only maximal ideal over $\operatorname{Ann}_R(x)$ is $m\}$. We call M m-local if $K_m(M) = M$ or, equivalently, if m is the only maximal ideal over each $p \in \operatorname{Ass}(M)$. In this case, m is an R_m-module by the following operation: $(r/s)x := rx'$ with $x = sx'$ ($r \in R$, $s \in R\backslash m$). The submodules of M over R and over R_m are identical.

For $K(M) = \{x \in M \mid Rx$ is complemented$\}$, we always have a decomposition $K(M) = \oplus_{m\in\Omega} K_m(M)$ and for a complemented module M, we have $M = K(M)$ [24, Theorems 2.3 and 2.5].

A principal ideal ring is called *special* if it has only one prime ideal $p \ne R$ and p is nilpotent [22, page 245].

Theorem 4.1. *Let R be a commutative local principal ideal ring (not necessarily a domain) with maximal ideal m.*

(i) *If m is nilpotent, then every R-module is ⊕-supplemented.*

(ii) *If m is not nilpotent, then R is a domain and ${}_RM$ is a ⊕-supplemented R-module if and only if $M \cong R^a \oplus Q^b \oplus (Q/R)^c \oplus B(1,\dots,n)$, where Q is the quotient field of R and $B(1,\dots,n)$ denotes the direct sum of arbitrarily many copies of $R/m,\dots,R/m^n$, for some positive integer n.*

Proof. (i) Suppose that m is nilpotent. By [1, Theorem 15.20], R is an Artinian principal ideal ring. Thus, every R-module is ⊕-supplemented by [7, Theorem 1.1].

(ii) Suppose that m is not nilpotent. Then R is not a special principal ideal ring. By [22, Chapter IV, Section 15, Theorem 33], R is a principal ideal domain and the result follows from [12, Proposition A.7]. □

The proof of the following result can be found in [7, Proposition 2.1].

Proposition 4.2. *Let R be a commutative Noetherian ring and M an R-module. The following assertions are equivalent:*

(i) *M is $\oplus$-supplemented;*
(ii) *$M = K(M)$ and $K_m(M)$ is $\oplus$-supplemented for all $m \in \Omega$.*

Corollary 4.3. *Let R be a commutative principal ideal ring (not necessarily a domain) and M an R-module. The following conditions are equivalent:*
(i) *M is $\oplus$-supplemented;*
(ii) (1) *the ring R/p is local for all $p \in \mathrm{Ass}(M)$;*
(2) *if $m \in \Omega$ such that mR_m is not nilpotent, then $K_m(M) \cong R_m^a \oplus Q(R_m)^b \oplus [Q(R_m)/R_m]^c \oplus B_m(1,\dots,n_m)$ (in Mod-R_m), where $Q(R_m)$ is the quotient field of R_m and $B_m(1,\dots,n_m)$ denotes the direct sum of arbitrarily many copies of $R_m/mR_m,\dots,R_m/(mR_m)^{n_m}$, for some positive integer n_m.*

Proof. See Proposition 4.2, [13, Proposition 2.2(b)], and Theorem 4.1. □

Proposition 4.4 (see [7, Corollary 2.2])**.** *Let R be a commutative Noetherian ring and M an R-module. The following assertions are equivalent:*
(i) *M is completely $\oplus$-supplemented;*
(ii) *$M = K(M)$ and $K_m(M)$ is completely $\oplus$-supplemented for all $m \in \Omega$.*

Corollary 4.5. *Let R be a commutative principal ideal ring (not necessarily a domain) and M an R-module. Then M is $\oplus$-supplemented if and only if M is completely $\oplus$-supplemented.*

Proof. By Proposition 4.4 and the proof of Theorem 4.1, it suffices to prove the result for an R-module M over a local principal ideal domain R with maximal ideal $m \neq 0$. If M is $\oplus$-supplemented, then $M \cong R^a \oplus Q^b \oplus (Q/R)^c \oplus B(1,\dots,n)$, where Q is the quotient field of R and $B(1,\dots,n)$ denotes the direct sum of arbitrarily many copies of $R/m,\dots,R/m^n$ (Theorem 4.1). By [7, Theorem 2.1], $Q^b \oplus (Q/R)^c$ and $R^a \oplus B(1,\dots,n)$ both are $\oplus$-supplemented. By [6, Corollary 2], $R^a \oplus B(1,\dots,n)$ is completely $\oplus$-supplemented. Now consider the module $Q^b \oplus (Q/R)^c$. Since Q and Q/R are injective, $\mathrm{End}_R(Q)$ and $\mathrm{End}_R(Q/R)$ are local rings (see [1, Lemma 25.4]). By [1, Corollary 12.7] and [12, Proposition A.7], $Q^b \oplus (Q/R)^c$ is completely $\oplus$-supplemented. Hence $Q^b \oplus (Q/R)^c \oplus R^a \oplus B(1,\dots,n)$ is completely $\oplus$-supplemented (see [7, Corollary 2.1]). □

5. Some rings whose modules are $\oplus$-supplemented. A ring R is called a *left V-ring* if every simple left R-module is injective. The ring R is called an *SSI*-ring if every semisimple left R-module is injective.

Lemma 5.1. *Let M be a module with $\mathrm{Rad}(M) = 0$. Then M is $\oplus$-supplemented if and only if M is semisimple.*

Proof. This is clear by [19, Proposition 3.3]. □

Corollary 5.2. *Let R be a left V-ring and M an R-module. Then M is $\oplus$-supplemented if and only if M is semisimple.*

Proof. By [3, page 236, Theorem (Villamayor)], for every left R-module, $\mathrm{Rad}(M)=0$. Therefore, every ⊕-supplemented R-module is semisimple (Lemma 5.1). □

Proposition 5.3. *Let R be a ring. The following statements are equivalent:*
(i) *every ⊕-supplemented R-module is injective;*
(ii) *R is a left Noetherian V-ring.*

Proof. (i)⇒(ii). Since every semisimple R-module is ⊕-supplemented, every semisimple R-module is injective. Thus R is an SSI-ring. By [3, Proposition 1], R is a left Noetherian V-ring.

(ii)⇒(i). Let M be a ⊕-supplemented R-module. Since R is a left V-ring, M is semisimple (Corollary 5.2). Thus M is an injective R-module (see [3, Proposition 1]). □

Corollary 5.4. *Let R be a commutative ring. The following are equivalent:*
(i) *every ⊕-supplemented R-module is injective;*
(ii) *R is semisimple.*

Proof. (i)⇒(ii). It is a consequence of Proposition 5.3 and [3, page 236, Proposition 1 and its first corollary].

(ii)⇒(i) This application is obvious. □

Proposition 5.5. *The following assertions are equivalent for a ring R:*
(i) *for every R-module M, M is ⊕-supplemented if and only if M is injective;*
(ii) *R is semisimple.*

Proof. (i)⇒(ii). Suppose that R satisfies the stated condition. By Proposition 5.3, R is a left Noetherian V-ring. Now, let M be an injective R-module. Then M is ⊕-supplemented and, since R is a V-ring, M is semisimple (Corollary 5.2). Therefore R is a semisimple ring.

(ii)⇒(i). It is easy to show that every R-module is ⊕-supplemented and every R-module is injective. □

Remark 5.6. If R is a commutative local Noetherian ring having an injective hollow radical R-module H, then the R-module $M = H^{(\mathbb{N})}$ is injective. However M is not ⊕-supplemented (see [7, Remark 2.1(3)]). For example, if R is a local Dedekind domain with quotient field K, then $K^{(\mathbb{N})}$ is an injective R-module which is not ⊕-supplemented.

Our next objective is to determine the class of commutative Noetherian rings R with the property that every injective R-module is ⊕-supplemented. First we prove the following lemma.

Lemma 5.7. *Let R be a quasi-Frobenius ring (not necessarily commutative). Then every injective R-module is ⊕-supplemented.*

Proof. By [10, Theorem 15.9], every injective R-module is projective. Since R is left perfect, every projective R-module is $\oplus$-supplemented (see [6, Proposition 13]) and the result is proved. □

Proposition 5.8. *For a commutative Noetherian ring R, the following statements are equivalent:*

(i) *every injective R-module is $\oplus$-supplemented;*

(ii) *R is Artinian and $E(R/m)$ is a local R-module for each maximal ideal m of R;*

(iii) *R is Artinian and R/I_m is a quasi-Frobenius ring for each maximal ideal m of R, where $I_m = \text{Ann}_R(E(R/m))$.*

Proof. (i)⇒(ii). By [15, page 53, corollary of Theorem 2.32] and [10, Corollary 3.86], it suffices to prove that $E(R/p)$ is a finitely generated R-module for each prime ideal p of R. Since $E(R/p)$ is indecomposable (see [15, page 53, corollary of Theorem 2.32]) and $E(R/p)$ is $\oplus$-supplemented, $E(R/p)$ is hollow [6, Proposition 2]. By Remark 5.6, $E(R/p)$ is not radical. Thus, $E(R/p)$ is a local R-module.

(ii)⇒(iii). Let m be a maximal ideal of R. Since $E(R/m)$ is a local R-module, $E(R/m) \cong R/I_m$ where $I_m = \text{Ann}_R(E(R/m))$. Thus, R/I_m is an injective R-module. By [9, Theorem 203], R/I_m is an injective (R/I_m)-module, that is, the ring R/I_m is self-injective. Since R/I_m is an Artinian ring, R/I_m is a quasi-Frobenius ring, and the result is proved.

(iii)⇒(i). Let M be an injective R-module. By [15, Theorem 4.5], we can write $M = \oplus_{i\in I} E(R/m_i)$ where the m_i are maximal ideals of R. Now, $E(R/m_i)$ is an (R/I_{m_i})-module and the (R/I_{m_i})-submodules of $E(R/m_i)$ are the same as the R-submodules of $E(R/m_i)$, therefore ${}_R(E(R/m_i))$ is $\oplus$-supplemented (see Lemma 5.7 and [9, Theorem 203]). By [6, Proposition 2], $E(R/m_i)$ $(i \in I)$ is a hollow R-module. By [1, Corollary 15.21], $\text{Rad}(E(R/m_i))$ is small in $E(R/m_i)$. Thus, $E(R/m_i)$ $(i \in I)$ is a local R-module. It follows by [1, Corollary 15.21] and [6, Corollary 2] that M is $\oplus$-supplemented. □

Proposition 5.9. *Let p be a prime ideal of a commutative Noetherian ring R such that $E(R/p)$ is hollow. Then there is a maximal ideal m of R such that*

(i) *m is the only maximal ideal over p;*

(ii) *$E(R/p)$ has the structure of an R_m-module;*

(iii) *the submodules of $E(R/p)$ over R and over R_m are identical.*

Moreover, as an R_m-module, $E(R/p)$ is isomorphic to an injective envelope of $R_m/S^{-1}p$ where $S = R\setminus m$.

Proof. Suppose that $E(R/p)$ is hollow. Since [13, Proposition 1.1] gives that $E(R/p)$ is m-local for some $m \in \Omega$, m is the only maximal ideal over p, $E(R/p)$ has the structure of an R_m-module, and the R_m-submodules of $E(R/p)$ are exactly the R-submodules of $E(R/p)$. It remains to show the last assertion. By [15, Proposition 5.5], $E(R/p)$ is injective as an R_m-module. Now,

$E(R/p)$ is indecomposable as an R-module and its R_m-submodules are also R-submodules so that $E(R/p)$ is also indecomposable as an R_m-module. Since $\mathrm{Ass}_R(E(R/p)) = \{p\}$, there is an element $x \in E(R/p)$ such that $\mathrm{Ann}_R(x) = p$. But it is easy to check that $\mathrm{Ann}_{R_m}(x) = S^{-1}p$ with $S = R\backslash m$ and $S^{-1}p$ is a prime ideal of R_m. Then $E(R/p)$ is isomorphic to an injective envelope of $R_m/S^{-1}p$ by [15, page 53, Corollary of Theorem 2.32]. □

Proposition 5.10. *Let p be a prime ideal of a commutative Noetherian ring R. Then the following are equivalent:*
(i) *$E(R/p)$ is hollow local;*
(ii) *p is maximal and R_p is a quasi-Frobenius ring.*

Proof. (i)⇒(ii). Suppose that $E(R/p)$ is hollow local. By Proposition 5.9, $E(R/p)$ is m-local for some maximal ideal m of R and as an R_m-module, $E(R_m/S^{-1}p)$ is hollow local, where $S = R\backslash m$. Since R_m is Noetherian local, R_m is Artinian by [9, Theorem 207]. Hence $S^{-1}p$ is a maximal ideal of R_m. Thus $S^{-1}p = S^{-1}m$. Therefore $p = m$ is maximal. Moreover, by [15, page 47, Corollary 2], $\mathrm{Ann}_{R_m}(E(R_m/S^{-1}m)) = 0$. Then $E(R_m/S^{-1}m) \cong R_m$. So R_m is self-injective. Therefore R_m is a quasi-Frobenius ring.

(ii)⇒(i). Suppose that p is maximal and R_p is a quasi-Frobenius ring. Put $E = E(R/p)$. By [15, Proposition 4.23], $E(R/p) = \sum_{n=1}^{\infty} \mathrm{Ann}_E(p^n)$. Then E is p-local. Thus E is an R_p-module and the submodules of E over R and over R_p are identical. The proof of Proposition 5.9 shows that, as an R_p-module, E is isomorphic to $E(R_p/pR_p)$, where pR_p denotes the unique maximal ideal of R_p. On the other hand, since R_p is a self-injective Artinian local ring, $E(R_p/pR_p)$, as an R_p-module, is isomorphic to R_p (see [10, Theorem 15.27]). Hence $E(R_p/pR_p)$ is a local R_p-module. Consequently, E is a local R-module. □

Lemma 5.11. *Let R be a commutative ring. If R is Noetherian and R_m is quasi-Frobenius for every maximal ideal m of R, then R is quasi-Frobenius.*

Proof. Let m be a maximal ideal of R. Since R_m is quasi-Frobenius, then R_m is Artinian and so mR_m, the maximal ideal of R_m, is a minimal prime ideal. Therefore m is a minimal prime ideal of R. The ring R is Noetherian and every prime ideal is maximal, hence R is Artinian. Let $R = R_1 \times \cdots \times R_t$ where each R_i is Artinian and local. Since each R_i is a localization of R, then R_i is quasi-Frobenius for each $i = 1,\dots,t$. It is not difficult to see that a finite product of rings is quasi-Frobenius if and only if each factor is quasi-Frobenius (see [10, Theorem 15.27]). Hence $R = R_1 \times \cdots \times R_t$ is quasi-Frobenius. □

Theorem 5.12. *For a commutative Noetherian ring R, the following statements are equivalent:*
(i) *every injective R-module is ⊕-supplemented;*
(ii) *R_m is quasi-Frobenius for each maximal ideal m of R;*
(iii) *R is quasi-Frobenius.*

Proof. (i)⇒(ii). It is a consequence of Propositions 5.8 and 5.10.
(ii)⇒(iii). It is clear by Lemma 5.11.
(iii)⇒(i). See Lemma 5.7. □

Proposition 5.13. *For a V-ring, the following statements are equivalent:*
(i) *R is semisimple;*
(ii) *every R-module is $\oplus$-supplemented.*

Proof. (i)⇒(ii). It is obvious.
(ii)⇒(i). Suppose that every R-module is $\oplus$-supplemented. By Corollary 5.2, every R-module is semisimple. Thus R is semisimple, as required. □

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