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ATOMICAL GROTHENDIECK CATEGORIES

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Motivated by the study of Gabriel dimension of a Grothendieck category, we introduce the concept of atomical Grothendieck category, which has only two localizing subcategories, and we give a classification of this type of Grothendieck categories.

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1. Introduction. Given a Grothendieck category $\mathcal{A}$, we can associate with it the lattice of all localizing categories of $\mathcal{A}$ and denote it by $\text{Tors}(\mathcal{A})$. We will show (Theorem 3.3) that if $\mathcal{A}$ has Gabriel dimension, then the lattice $\text{Tors}(\mathcal{A})$ is semi-Artinian. Moreover, the Gabriel dimension of $\mathcal{A}$ is exactly the Loewy length of this lattice. Example 3.4 proves that the converse statement does not hold. (Therefore, the properties of the lattice $\text{Tors}(\mathcal{A})$ do not determine the properties of the category $\mathcal{A}$.) These facts suggest introducing the concept of atomical Grothendieck category. Precisely, $\mathcal{A}$ will be called atomical if the lattice $\text{Tors}(\mathcal{A})$ has only two elements, that is, $\mathcal{A}$ has only two localizing subcategories, namely, $\{0\}$ and $\mathcal{A}$. The classification of atomical Grothendieck categories is given in Section 4.

2. Preliminaries. Throughout this paper, $\mathcal{A}$ will denote a Grothendieck category, that is, an abelian category with a generator, such that colimits exist and direct limits are exact.

It is well known that in a Grothendieck category each object X has an injective hull, denoted in the sequel by $E(X)$.

If $\mathcal{A}$ is a category, then by a subcategory $\mathcal{B}$ of $\mathcal{A}$ we will always mean a full subcategory of $\mathcal{A}$.

A subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *closed* (or hereditary pretorsion class) if it is closed with respect to kernels, cokernels, and direct sums.

By $\sigma[X]$ we denote the full subcategory of $\mathcal{A}$ whose objects are subobjects of X-generated objects. These objects are said to be subgenerated by X, and X is a subgenerator of $\sigma[X]$. This is the smallest closed full subcategory of $\mathcal{A}$ containing X.

By definition, the objects of $\sigma[X]$ form a closed subcategory in $\mathcal{A}$. On the other hand, every closed subcategory $\mathcal{T}$ in $\mathcal{A}$ is of the form $\sigma[X]$ for some object X; for example, for X equals the direct sum of all (nonisomorphic) cyclic objects in $\mathcal{T}$.

Following Goldman [2], a functor $\tau : \mathcal{A} \to \mathcal{A}$ is called a kernel functor if

(1) it is a subfunctor of the identity functor, that is, $\tau(M) \subseteq M$ and $f : M \to M'$ implies $f(\tau(M)) \subseteq \tau(M')$;

(2) $N \subseteq M$ implies $\tau(N) = N \cap \tau(M)$.

The trivial kernel functors 0 and ∞ are defined by setting $0(X) = 0$, and $\infty(X) = X$, for every object $X \in \mathcal{A}$.

For any kernel functor τ, X is called a τ-torsion module if $\tau(X) = X$, and a τ-torsion-free module if $\tau(X) = 0$. The collection of $\mathcal{T}_\tau$ of all the τ-torsion module is a closed subcategory of $\mathcal{A}$. Conversely, for any closed subcategory $\mathcal{C}$, there exists a unique kernel functor τ such that $\mathcal{C} = \mathcal{T}_\tau$.

Lemma 2.1. *Let G be a generator of $\mathcal{A}$ and $\mathcal{C}$ a closed subcategory. Then*

$$\mathcal{C} = \sigma[\oplus\{G/X \mid G/X \in \mathcal{C}\}]. \tag{2.1}$$

Corollary 2.2. *The closed subcategories (and hence, the kernel functors) form a set.*

Proposition 2.3. *The set of all closed subcategories of $\mathcal{A}$ is a complete lattice. For a family $\{X_\lambda\}_\Lambda$ of objects of $\mathcal{A}$,*

$$\begin{aligned} \bigvee_\Lambda \sigma[X_\lambda] &= \sigma[\oplus_\Lambda X_\lambda], \\ \bigwedge_\Lambda \sigma[X_\lambda] &= \bigcap_\Lambda \sigma[X_\lambda]. \end{aligned} \tag{2.2}$$

Remark 2.4. (1) (cf. [4]). For a coalgebra C, the lattice of all closed subcategories of the category of comodules over C is anti-isomorphic to the lattice of subcoalgebras of C.

(2) The Serre subcategories of $\mathcal{A}$ (i.e., the subcategories $\mathcal{S}$ of $\mathcal{A}$ satisfying that for any exact sequence from $\mathcal{A}$,

$$0 \longrightarrow X'' \longrightarrow X \longrightarrow X' \longrightarrow 0, \tag{2.3}$$

where X is in $\mathcal{S}$ if and only if X' and X'' are in $\mathcal{S}$) do not form a set. For example, we consider the Grothendieck category $\mathcal{V}$ of vector spaces over a division ring. For any cardinal α, the subcategory of all vector spaces of dimension less than or equal to α is a Serre subcategory. Thus, the Serre subcategories of $\mathcal{V}$ are not a set.

We now recall the notion of semi-Artinian lattice. Let L be an upper continuous and modular lattice. An atom of L is a nonzero element $a \in L$ such that whenever $b \in L$ and $b < a$, then $b = 0$, that is, the interval $[0,a]$ has exactly two elements, 0 and a. If $a, b \in L$ and $x < y$, then the interval $[x,y]$ is simple if y is an atom in the sublattice $[x,y]$ of L. The lattice is called semiatomic if 1 is a joint of atoms, and L is called *semi-Artinian* if for every $x \in L$, $x \neq 1$, the sublattice $[x,1]$ of L contains an atom.

The *(ascending) Loewy series* of L,

$$s_0(L) < s_1(L) < \cdots < s_{\lambda(L)}(L), \tag{2.4}$$

is defined recursively as follows: $s_0(L) = 0$, $s_1(L)$ is the socle $\mathrm{Soc}(L)$ of L (i.e., the join of all atoms of L), and if the elements $s_\beta(L)$ of L have been defined for all ordinals $\beta < \alpha$, then $s_\alpha(L) = \bigvee_{\beta<\alpha} s_\beta(L)$ if α is a limit ordinal and $s_\alpha = \mathrm{Soc}([s_\gamma(L), 1])$ if $\alpha = \gamma + 1$.

The *Loewy length* $\lambda(L)$ of L is the least ordinal such that $s_\lambda(L) = s_{\lambda+1}(L)$.

3. Gabriel dimension and localizing subcategories. A subcategory $\mathcal{T} \subseteq \mathcal{A}$ is a localizing subcategory if it is closed under subobjects, quotient objects, extensions, and coproducts. If $\mathcal{B} \subseteq \mathcal{A}$ is an arbitrary subcategory, we denote by $\mathcal{T}(\mathcal{B})$ the smallest localizing subcategory containing $\mathcal{B}$.

Examples 3.1. (i) An object $A \in \mathcal{A}$ is *singular* if there exists a short exact sequence

$$0 \longrightarrow A'' \longrightarrow A' \longrightarrow A \longrightarrow 0, \tag{3.1}$$

where the monomorphism is essential.

In any Grothendieck category, we can always consider the *Goldie localizing subcategory*, denoted by $\mathcal{G}$, as the smallest localizing subcategory containing the singular objects.

(ii) We can associate to any injective object $E \in \mathcal{A}$ a localizing subcategory

$$\mathcal{T}_E = \{A \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(A, E) = 0\}. \tag{3.2}$$

This localizing subcategory is said to be cogenerated by E.

(iii) For a projective object $P \in \mathcal{A}$, we can define

$$\mathcal{T}_P = \{A \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(P, A) = 0\}. \tag{3.3}$$

It is clear that $\mathcal{T}_P$ is a localizing subcategory closed under direct product.

(iv) If S is a simple object in $\mathcal{A}$, we denote by $\mathcal{A}_S$ the smallest localizing subcategory containing S. In fact,

$$\mathcal{A}_S = \{M \in \mathcal{A} \mid N \subset M,\ M/N \text{ contains a simple object isomorphic to } S\}. \tag{3.4}$$

The objects in this localizing subcategory are called S-primary.

Let $\mathcal{T}$ be a localizing subcategory. The corresponding torsion functor or idempotent kernel functor is denoted by

$$t_{\mathcal{T}} : \mathcal{A} \longrightarrow \mathcal{T}. \tag{3.5}$$

This functor assigns to an object $A \in \mathcal{A}$ the maximal subobject $t_{\mathcal{T}}(A) \subseteq A$ in $\mathcal{T}$. An object $X \in \mathcal{A}$ is $\mathcal{T}$-torsion-free (resp., $\mathcal{T}$-torsion) if $t_{\mathcal{T}}(X) = 0$ (resp.,

$t_{\mathcal{T}}(X) = X$). Let $H^1 t_{\mathcal{T}}$ denote the first higher derived functor of the left exact functor $t_{\mathcal{T}}$. A $\mathcal{T}$-torsion-free object $E \in \mathcal{A}$ is $\mathcal{T}$-closed if $H^1 t_{\mathcal{T}} = 0$.

If $\mathcal{T}$ is a localizing subcategory of $\mathcal{A}$, we can consider the quotient category $\mathcal{A}/\mathcal{T}$. We denote by $T_{\mathcal{T}} : \mathcal{A} \to \mathcal{A}/\mathcal{T}$ the canonical functor and by $S_{\mathcal{T}} : \mathcal{A}/\mathcal{T} \to \mathcal{A}$ the right adjoint functor of $T_{\mathcal{T}}$.

It is well known that the category $\mathcal{A}/\mathcal{T}$ is equivalent to the full subcategory of $\mathcal{A}$ of $\mathcal{T}$-closed objects.

It is well known that $\mathcal{A}$ has a set of localizing subcategories $\operatorname{Tors}(\mathcal{A})$. Given a family of localizing subcategories $(\mathcal{C}_i)_{i\in I}$, we define the meet by $\bigwedge_{i\in I} \mathcal{C}_i = \bigcap_{i\in I} \mathcal{C}_i$, and the join by $\bigvee_{i\in I} \mathcal{C}_i$, as the smallest localizing subcategory containing the union of the $\mathcal{C}_i$. Notice that $\operatorname{Tors}(\mathcal{A})$ is not a sublattice of the lattice of all closed subcategories of $\mathcal{A}$. It is also known that this set is a frame (i.e., it is a complete lattice L such that $a \wedge (\bigvee X) = \bigvee\{a \wedge x \mid x \in X\}$ for each element a and subset X of L). Frames are also known as local lattices, complete Heyting algebras, or complete Brouwerian lattices. The lattice of closed subcategories is not a frame in general.

We need the following preliminary result.

Proposition 3.2. *Let $\mathcal{A}$ be a Grothendieck category and let $\mathcal{C} \subseteq \mathcal{A}$ be a localizing subcategory. There exists a bijective correspondence between the localizing subcategories of $\mathcal{A}/\mathcal{C}$ and the localizing subcategories $\mathcal{B}$ of $\mathcal{A}$ containing $\mathcal{C}$. Moreover,* $\operatorname{Tors}(\mathcal{A}/\mathcal{C})$ *is a subframe of* $\operatorname{Tors}(\mathcal{A})$

Proof. Let $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ be the canonical functor. Consider $\mathcal{B}$, a localizing subcategory of $\mathcal{A}$ containing $\mathcal{C}$, then $T(\mathcal{B}) = \{Z \in \mathcal{A}/\mathcal{C} \mid Z \cong T(X),\ X \in \mathcal{B}\}$ is a localizing subcategory of $\mathcal{A}/\mathcal{C}$. In fact, it is clear that $T(\mathcal{B})$ is closed under subobjects, quotients, and direct sums. It remains to show that $T(\mathcal{B})$ is closed under extensions. First, we observe that $T(\mathcal{B}) = \{Z \in \mathcal{A}/\mathcal{C} \mid S(Z) \in \mathcal{B}\}$. To see this, consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow X \longrightarrow ST(X) \cong S(Z) \longrightarrow \operatorname{Coker} f \longrightarrow 0, \tag{3.6}$$

where $\operatorname{Ker} f$, $\operatorname{Coker} f \in \mathcal{C}$. Therefore, $\operatorname{Ker} f$, $\operatorname{Coker} f \in \mathcal{B}$, and $X \in \mathcal{B}$ if and only if $S(Z) \in \mathcal{B}$. Now if

$$0 \longrightarrow Z' \longrightarrow Z \longrightarrow Z'' \longrightarrow 0 \tag{3.7}$$

is an exact sequence in $\mathcal{A}/\mathcal{C}$, with $Z', Z'' \in T(\mathcal{B})$, we apply the functor S to obtain

$$0 \longrightarrow S(Z') \longrightarrow S(Z) \longrightarrow S(Z''). \tag{3.8}$$

Thus, $S(Z) \in \mathcal{B}$ and $Z \in T(\mathcal{B})$.

Let $\mathcal{D}$ be a localizing subcategory in $\mathcal{A}/\mathcal{C}$. we define $T^{-1}\mathcal{D} = \{X \in \mathcal{A} \mid T(X) \in \mathcal{D}\}$. Since T is an exact functor which commutes with direct sums, then $T^{-1}(\mathcal{D})$ is a localizing subcategory which contains $\mathcal{C}$. It is not difficult to see that these two operations establish a bijection between the localizing subcategories of $\mathcal{A}/\mathcal{C}$ and the localizing subcategories $\mathcal{B}$ of $\mathcal{A}$ containing $\mathcal{C}$. $\square$

We now recall the notion of Gabriel dimension of a Grothendieck category $\mathcal{A}$. For any ordinal α, we will denote by $\mathcal{C}_\alpha$ the localizing subcategory defined in the following way: $\mathcal{C}_0$ is the zero subcategory; $\mathcal{C}_1$ is the smallest localizing subcategory containing all simple objects; if $\alpha = \beta + 1$, an object X of $\mathcal{A}$ will be contained in $\mathcal{C}_\alpha$ if and only if $T_\beta(X) \in Ob(\mathcal{A}/\mathcal{C}_\beta)_1$, where $T_\beta : \mathcal{A} \to \mathcal{A}/\mathcal{C}_\beta$ is the canonical functor; and if α is a limit ordinal, then $\mathcal{C}_\alpha$ is the localizing subcategory generated by all localizing subcategories $\mathcal{C}_\beta$, with $\beta \le \alpha$.

It is clear that if $\alpha \le \alpha'$, then $\mathcal{C}_\alpha \subseteq \mathcal{C}_{\alpha'}$. Hence, there exists an ordinal τ such that $\mathcal{C}_\tau = \mathcal{C}_\alpha$ for any ordinal $\alpha \ge \tau$. We define $\mathcal{C}_\tau = \cup_\alpha C_\alpha$.

The set of localizing subcategories $\mathcal{C}_\alpha$ is called the *Gabriel filtration* of $\mathcal{A}$. We say that an object X of $\mathcal{A}$ has *Gabriel dimension* if X is in $\mathcal{C}_\tau$. Then the smallest ordinal α verifying X in $\mathcal{C}_\alpha$ is called the Gabriel dimension of X.

We say that $\mathcal{A}$ has Gabriel dimension if $\mathcal{A} = \mathcal{C}_\tau$ or, equivalently, any object of $\mathcal{A}$ has Gabriel dimension. We are now ready for the main result of this section.

Theorem 3.3. *Let $\mathcal{A}$ be Grothendieck category. If $\mathcal{A}$ has Gabriel dimension α, then* $\mathrm{Tors}(\mathcal{A})$ *is a semi-Artinian lattice with Loewy length α.*

Proof. We will show this result by transfinite induction. If $G\text{-}\dim\mathcal{A} = 1$, then $\mathcal{A} = \mathcal{C}_1$, the localizing subcategory generated by the simple objects of $\mathcal{A}$. Hence, $\mathcal{A} = \bigvee \mathcal{C}_S$.

Now, we assume that the result is true for any Grothendieck category of Gabriel dimension $\beta < \alpha$. If $\alpha = \gamma + 1$ is not a limit ordinal, then any object $X \in \mathcal{A}$ belongs to $\mathcal{C}_\alpha$ or, equivalently, $T_\gamma(X) \in (\mathcal{A}/\mathcal{C}_\gamma)_1$. Now, $\mathcal{C}_\gamma = s_\gamma(\mathrm{Tors}(\mathcal{C}_\gamma))$. If $X \in \mathcal{A}$ satisfies that $T_\gamma(X)$ is a simple object in $\mathcal{A}/\mathcal{C}_\gamma$, then $(\mathcal{A}/\mathcal{C}_\gamma)_{T_\gamma(X)}$ is an atom in $\mathrm{Tors}(\mathcal{A}/\mathcal{C}_\gamma)$. By Proposition 3.2, $T_\gamma^{-1}((\mathcal{A}/\mathcal{C}_\gamma)_{T_\gamma(X)})$ is an atom in $[\mathcal{C}_\gamma, \mathcal{A}]$. We will see that $\mathcal{A} = \bigvee T_\gamma^{-1}(\mathcal{A}/\mathcal{C}_\gamma)_{T_\gamma(X)}$. Let $A \in \mathcal{A}$ and consider $A \to A' \to 0$, with $A' \ne 0$. Applying the functor T_γ, we obtain $T_\gamma(A) \to T_\gamma(A') \to 0$. If $T(A') = 0$, the proof is finished; otherwise $T_\gamma(A')$ contains a simple object $T_\gamma(X)$. Therefore, we have $K \to X \to A'$ and A' contains X/K which is in $T_\gamma^{-1}(\mathcal{A}/\mathcal{C}_\gamma)_{T_\gamma(X)}$. If α is a limit ordinal $\mathcal{A} = \bigcup_{\beta<\alpha} \mathcal{C}_\beta$, then $\mathcal{A} = \bigvee_{\beta<\alpha} \mathcal{C}_\beta$. $\square$

The next example shows that the converse of Theorem 3.3 is not true.

Example 3.4. Let R be a commutative nondiscrete valuation domain of Krull dimension 1, with maximal ideal M. Then

(i) $M^2 = M$,

(ii) if $x \in M$, then $\bigcap_{n \ge 0} Rx^n = 0$,

(iii) $\mathrm{Tors}(R\text{-Mod})$ has four elements:

$$\{0\} \subseteq (R\text{-Mod})_{R/M} \subseteq \mathcal{T} \subseteq R\text{-Mod}, \tag{3.9}$$

where $\mathcal{T}$ is the usual torsion theory in a domain and $(R\text{-Mod})_{R/M}$ is a semisimple category,

(iv) the quotient category R-Mod$/(R$-Mod$)_{R/M}$ has no simple objects,
(v) R-Mod has no Gabriel dimension.

Proof. (i) Take $x \in M$. Since the valuation is not discrete, we can find an element $y \in M$ such that $v(y^2) = 2v(y) < v(x)$. Hence, $x \in (y^2) \subseteq M^2$ and $M = M^2$.

(ii) Let $\mathfrak{Q} = \bigcap_{n\geq 0} Rx^n$. We claim that $\mathfrak{Q}$ is a prime ideal. Let $a, b \in R$ with $a \notin \mathfrak{Q}$ and $b \notin \mathfrak{Q}$. Hence, there exist n and m such that $a \notin Rx^n$ and $b \notin Rx^m$. Thus, $Rx^n \subset Ra$ and $Rx^m \subset Rb$. Then $Rx^{n+m} \subset Rx^n b \subset Rab$ and $ab \notin \mathfrak{Q}$. Therefore, $\mathfrak{Q} = 0$.

(iii) Let $\mathscr{C}$ be a localizing subcategory properly containing $(R$-Mod$)_{R/M}$ and let I be a nonzero ideal. We take $J \subsetneq M$ with $R/J \in \mathscr{C}$. Thus, there exist $x \in M \setminus J$. By (ii), $\bigcap_{n\geq 0} Rx^n = 0$, and it follows that $Rx^n \subseteq I$ for some n and $R/I \in \mathscr{C}$.

(iv) Any simple object of R-Mod$/(R$-Mod$)_{R/M}$ is given by an $(R$-Mod$)_{R/M}$-critical ideal, but this kind of ideals is prime. This prime is 0. So the cocritical module is isomorphic to R. Therefore, R/I is semisimple for every nonzero ideal I of R—a contradiction.

(v) The proof follows from (iv). □

4. Atomical Grothendieck categories.

We have proved in Theorem 3.3 that if a Grothendieck category has Gabriel dimension, then the lattice of localizing subcategories is semi-Artinian. Example 3.4 shows that the converse is not true. This fact suggests the study of Grothendieck categories $\mathscr{A}$ with the property that the category $\mathscr{A}$ is an atom in the lattice $\mathrm{Tors}(\mathscr{A})$, that is, $\mathscr{A}$ has only two localizing subcategories $\{0\}$ and $\mathscr{A}$.

Definition 4.1. A Grothendieck category $\mathscr{A}$ is called *atomical* if it has only two localizing subcategories, namely, $\{0\}$ and $\mathscr{A}$.

A maximal localizing category $\mathscr{T}$ is a maximal element of $\mathrm{Tors}(\mathscr{A}) - \mathscr{A}$. By Proposition 3.2, $\mathscr{A}$ is a maximal localizing category of $\mathscr{A}$ if and only if $\mathscr{A}/\mathscr{T}$ is an atomical Grothendieck category.

Recall that an object C in $\mathscr{A}$ is called a *cogenerator* if for each nonzero morphism $f : X \to Y$ in $\mathscr{A}$, there exists a morphism $g : Y \to C$ such that $gf \neq 0$. This is equivalent to the existence of a monomorphism $A \to C^I$ for some index set I, for every object $A \in \mathscr{A}$. It is clear that an injective object E of $\mathscr{A}$ is a cogenerator if and only if for each nonzero object $A \in \mathscr{A}$, there exists a nonzero morphism $f : A \to E$.

Proposition 4.2. *If $\mathscr{A}$ is a Grothendieck category, then $\mathscr{A}$ is an atomical category if and only if every nonzero injective object of $\mathscr{A}$ is a cogenerator.*

Moreover, if the category has enough projectives, then $\mathscr{A}$ is an atomical category if and only if every nonzero projective object of $\mathscr{A}$ is a generator.

Proof. Assume that $\mathscr{A}$ is atomical, then any nonzero injective object cogenerates a nonzero torsion-free class. Hence, this torsion-free class must be the whole category and this injective is a cogenerator. Since any localizing subcategory of $\mathcal{A}$ is cogenerated by an injective object, the converse is clear. □

It is clear that for an atomical Grothendieck category $\mathcal{A}$, we have that the Goldie torsion theory has to be either $\{0\}$ or $\mathcal{A}$. In the first case, we say that $\mathcal{A}$ is a nonsingular Grothendieck category and we characterize this type of simple Grothendieck categories. Recall that a Grothendieck category $\mathcal{A}$ is called spectral if any short exact sequence splits and a spectral Grothendieck category is called discrete if every object is semisimple.

Proposition 4.3. *Let $\mathcal{A}$ be a Grothendieck category. The category $\mathcal{A}$ is nonsingular atomical if and only if $\mathcal{A}$ is a spectral category which is equivalent to $R\text{-Mod}/\mathcal{G}$, where R is a regular prime self-injective ring and $\mathcal{G}$ is the Goldie localizing subcategory. Moreover, $\mathcal{A}$ contains a simple object if and only if R is isomorphic to the ring of all linear transformations of a left vector space over a division ring.*

Proof. Suppose $\mathcal{A}$ is nonsingular and atomical. Since $\mathcal{A}$ is nonsingular, then $\mathcal{G} = 0$. Hence, $X \subseteq' E(X)$ with $E(X)/X$ singular, a matter which implies that $X = E(X)$ and any object is injective. Thus, $\mathcal{A}$ is a spectral Grothendieck category.

Let U be a generator of $\mathcal{A}$ and $R = \mathrm{Hom}_{\mathcal{A}}(U,U)$, by the Gabriel-Oberst theorem [5, Chapter XII, Theorem 1.3] $\mathcal{A}$ is equivalent to $R\text{-Mod}/\mathcal{G}$, where R is a regular self-injective ring and $\mathcal{G}$ is the Goldie's localizing subcategory. Since $R\text{-Mod}/\mathcal{G}$ is atomical, then $\mathcal{G}$ is maximal. Hence, by [1, Theorem 2.2], $0 = t_{\mathcal{G}}(R)$ is a prime ideal.

Conversely, assume that R is a prime self-injective regular ring. Since R is prime, then it is nonsingular. Thus, $t_{\mathcal{G}}(R) = 0$ is a prime ideal, where $\mathcal{G}$ is a maximal localizing subcategory by [1, Theorem 2.2]. Therefore, $R\text{-Mod}/\mathcal{G}$ is an atomical Grothendieck category.

Assume that $\mathcal{A}$ contains a simple object, then $\mathcal{A}$ coincides with the localizing subcategory generated by this simple object. Hence, as an object in $R\text{-Mod}/\mathcal{G}$, R contains a simple object. Therefore, there exists a $\mathcal{G}$-cocritical left ideal C of R. If C is not simple as a left R-module, then we can find a finitely generated left ideal $I \neq 0$ contained in C. Since R is regular, there exists a left ideal J such that $I \oplus J = R$. Thus, $C = I \oplus (J \cap C)$, which is a contradiction since I is essential in C. Therefore, C is a simple left ideal and $\mathrm{Soc}(R) \neq 0$. By [3, Theorem 9.12], R is the ring of all linear transformations of any left vector space over a division ring.

Conversely, if R is the ring of all linear transformations of any left vector space over a division ring, then $\mathrm{Soc}(R)$ is not zero. Any simple left ideal will produce a simple object in the quotient category. □

We will now consider the case where the Goldie torsion theory coincides with the whole category. When the Grothendieck category contains simple objects, we have the following characterization. Recall that a Grothendieck category $\mathscr{A}$ is called semi-Artinian if every nonzero object of $\mathscr{A}$ contains a simple object.

Proposition 4.4. *Let $\mathscr{A}$ be a singular Grothendieck category. If $\mathscr{A}$ is atomical, and it has simple objects, then $\mathscr{A}$ is a semi-Artinian Grothendieck category with a unique isomorphic class of simple objects.*

Proof. Since $\mathscr{A}$ is atomical, the localizing subcategory generated by a simple object coincides with category $\mathscr{A}$. Hence, the result follows. □

Proposition 4.5. *Let $\mathscr{A}$ be a locally finitely generated Grothendieck category. Then $\mathscr{A}$ is atomical if and only if any object of $\mathscr{A}$ is S-primary, and $\mathscr{A}$ is semisimple or singular.*

We now specialize our discussion to the module category R-Mod. In this case, we have the following result.

Proposition 4.6. R-Mod *is an atomical category if and only if the ring R is local right perfect.*

Proof. If R is local right perfect, then R-Mod is clearly atomical. Conversely, if R-Mod is atomical and R is nonsingular, then the Goldie torsion theory is trivial. Hence, any module is injective and R is semisimple. Since there is only an isomorphic class of simple modules, R is simple Artinian. We only need to consider the case when R is singular. But then $R\text{-Mod} = (R\text{-Mod})_S$ for some simple left R-module S and there is only an isomorphic class of left simple R-modules. Thus, R is semi-Artinian and $J = \operatorname{ann}(S)$. We will see that R/J is a simple Artinian ring. In fact, consider $\operatorname{Soc}(R/J) = A/J \neq 0$. If $A \neq R$, then $A \subseteq M$ for some maximal left ideal M. Therefore, $A(R/M) = 0$ and $A \subseteq \operatorname{ann}(S) = J$, a contradiction. Hence, $A = R$ and R/J is simple Artinian. Since R is semi-Artinian, J is T-nilpotent. Since R/J is simple Artinian and J is T-nilpotent, then R is a local right perfect ring. □

Now, we consider the case of closed subcategories of R-Mod.

Corollary 4.7. *Let M be a left R-module. Then $\sigma[M]$ is an atomical category if and only if either M is semisimple or M is S-primary with S a simple singular left R-module.*

Finally, we present an example of a singular atomical Grothendieck category without simple objects.

Example 4.8. We consider the same ring as in Example 3.4. Then the quotient category $\mathscr{T}/(R\text{-Mod})_{(R/M)}$ is an atomical singular Grothendieck category without simple objects.

Proof. We have proved that $R\text{-Mod}/(R\text{-Mod})_{(R/M)}$ has no simple objects, then $\mathscr{T}/(R\text{-Mod})_{(R/M)}$ has no simple objects. We also know from Example 3.4 that this category is atomical. We will denote by $T : \mathscr{T} \to \mathscr{T}/(R\text{-Mod})_{(R/M)}$ the canonical functor. Let $0 \neq I \subset M$ be an ideal of R such that $I \neq M$. It is clear that $R/I \in \mathcal{T}$, and $R/I \notin (R\text{-Mod})_{(R/M)}$. Let J/I be the torsion part of $R/I \in (R\text{-Mod})_{(R/M)}$. Since $M^2 = M$, then $J \subset M$ and $J \neq M$. By the exact sequence

$$0 \longrightarrow J/I \longrightarrow R/I \longrightarrow R/J \longrightarrow 0, \tag{4.1}$$

it follows that $T(R/I) \simeq T(R/J)$. Since R is a valuation ring, we have that R/J is a uniform (coirreducible) R-module, so $T(R/J)$ is still uniform in the quotient category. Denote $X = T(R/J) \simeq T(R/I)$. Then X is uniform and contains no simple objects (because the category does not have nonzero simple objects). Then we can consider Y as a nonzero subobject of X such that $Y \neq X$. It is clear that X/Y belongs to the Goldie torsion theory (of the quotient category) and $X/Y \neq 0$. As the quotient category is an atomical category, it must be the same as the Goldie torsion theory. □

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