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PARTIAL SUMS OF FUNCTIONS OF BOUNDED TURNING

JAY M. JAHANGIRI and K. FARAHMAND

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We determine conditions under which the partial sums of the Libera integral operator of functions of bounded turning are also of bounded turning.

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1. Introduction. Let $\mathcal{A}$ denote the family of functions f which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and are normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathcal{U}. \tag{1.1}$$

For $0 \le \alpha < 1$, let $\mathcal{B}(\alpha)$ denote the class of functions f of the form (1.1) so that $\Re(f') > \alpha$ in $\mathcal{U}$. The functions in $\mathcal{B}(\alpha)$ are called functions of bounded turning (cf. [4]). By the Nashiro-Warschowski theorem (see, e.g., [3]), the functions in $\mathcal{B}(\alpha)$ are univalent and also close-to-convex in $\mathcal{U}$.

For f of the form (1.1), the Libera integral operator F is given by

$$F(z) = \frac{2}{z}\int_0^z f(\zeta)\,d\zeta = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k. \tag{1.2}$$

The nth partial sums $F_n(z)$ of the Libera integral operator $F(z)$ are given by

$$F_n(z) = z + \sum_{k=2}^{n} \frac{2}{k+1} a_k z^k. \tag{1.3}$$

In [6] it was shown that if $f \in \mathcal{A}$ is starlike of order α, $\alpha = 0.294, \ldots$, so is the Libera integral operator F. We also know that (see, e.g., [1]) there are functions which are univalent or spiral-like in $\mathcal{U}$ so that their Libera integral operators are not univalent or spiral-like in $\mathcal{U}$. Li and Owa [5] proved that if $f \in \mathcal{A}$ is univalent in $\mathcal{U}$, then $F_n(z)$ is starlike in $|z| < 3/8$. The number $3/8$ is sharp. In this note we make use of a result of Gasper [2] to provide a simple proof for the following theorem.

Main theorem. *If* $1/4 \le \alpha < 1$ *and* $f \in \mathcal{B}(\alpha)$, *then* $F_n \in \mathcal{B}((4\alpha - 1)/3)$.

2. Preliminary lemmas. To prove our Main theorem, we will need the following three lemmas. The first lemma is due to Gasper (see [2, Theorem 1]) and the third lemma is a well-known and celebrated result (cf. [3]) that can be derived from the Herglotz' representation for positive real part functions.

Lemma 2.1. *Let θ be a real number and let m and k be natural numbers. Then*

$$\frac{1}{3}+\sum_{k=1}^{m}\frac{\cos(k\theta)}{k+2}\geq 0. \tag{2.1}$$

Lemma 2.2. *For $z \in \mathcal{U}$,*

$$\Re\left(\sum_{k=1}^{m}\frac{z^k}{k+2}\right) > -\frac{1}{3}. \tag{2.2}$$

Proof. For $0 \le r < 1$ and for $0 \le |\theta| \le \pi$, write $z = re^{i\theta} = r(\cos(\theta)+i\sin(\theta))$. By DeMoivre's law and the minimum principle for harmonic functions, we have

$$\Re\left(\sum_{k=1}^{m}\frac{z^k}{k+2}\right) = \sum_{k=1}^{m}\frac{r^k\cos(k\theta)}{k+2} > \sum_{k=1}^{m}\frac{\cos(k\theta)}{k+2}. \tag{2.3}$$

Now by Abel's lemma (cf. Titchmarsh [7]) and condition (2.1) of Lemma 2.1 we conclude that the right-hand side of (2.3) is greater than or equal to $-1/3$. □

Lemma 2.3. *Let $P(z)$ be analytic in $\mathcal{U}$, $P(0) = 1$ and let $\Re(P(z)) > 1/2$ in $\mathcal{U}$. For functions Q analytic in $\mathcal{U}$, the convolution function $P * Q$ takes values in the convex hull of the image on $\mathcal{U}$ under Q.*

The operator "$*$" stands for the Hadamard product or convolution of two power series $f(z)=\sum_{k=1}^{\infty}a_kz^k$ and $g(z)=\sum_{k=1}^{\infty}b_kz^k$ denoted by $(f*g)(z)=\sum_{k=1}^{\infty}a_kb_kz^k$.

3. Proof of Main theorem. Let f be of the form (1.1) and belong to $\mathcal{B}(\alpha)$ for $1/4 \le \alpha < 1$. Since $\Re(f'(z)) > \alpha$, we have

$$\Re\left(1+\frac{1}{2(1-\alpha)}\sum_{k=2}^{\infty}ka_kz^{k-1}\right) > \frac{1}{2}. \tag{3.1}$$

Applying the convolution properties of power series to $F_n'(z)$, we may write

$$\begin{aligned} F_n'(z) &= 1+\sum_{k=2}^{n}\frac{2k}{k+1}a_kz^{k-1} \\ &= \left(1+\frac{1}{2(1-\alpha)}\sum_{k=2}^{\infty}ka_kz^{k-1}\right) * \left(1+(1-\alpha)\sum_{k=2}^{n}\frac{4}{k+1}z^{k-1}\right) \\ &= P(z) * Q(z). \end{aligned} \tag{3.2}$$

From Lemma 2.2 for $m = n-1$, we obtain

$$\Re\left(\sum_{k=2}^{n}\frac{z^{k-1}}{k+1}\right) > -\frac{1}{3}. \tag{3.3}$$

Applying a simple algebra to inequality (3.3) and $Q(z)$ in (3.2) yields

$$\Re(Q(z)) = \Re\left(1+(1-\alpha)\sum_{k=2}^{n}\frac{4}{k+1}z^{k-1}\right) > \frac{4\alpha-1}{3}. \tag{3.4}$$

On the other hand, the power series $P(z)$ in (3.2) in conjunction with the condition (3.1) yield $\Re(P(z)) > 1/2$. Therefore, by Lemma 2.3, $\Re(F_n'(z)) > (4\alpha-1)/3$. This concludes the Main theorem.

Remark 3.1. The Main theorem also holds for $\alpha < 1/4$. We also note that $\mathcal{B}(\alpha)$ for $\alpha < 0$ is no longer a bounded turning family.

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Jay M. Jahangiri: Department of Mathematics, Kent State University, Burton, Ohio 44021-9500, USA

E-mail address: jay@geauga.kent.edu

K. Farahmand: School of Computing and Mathematics, University of Ulster, Jordanstown Campus, Newtownabbey Co. Antrim BT37 0QB, UK

E-mail address: k.farahmand@ulster.ac.uk