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EXTENSION OF α-LABELINGS OF QUADRATIC GRAPHS

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First, a new proof for the existence of an α-labeling of the quadratic graph $Q(3,4k)$ is presented. Then the existence of α-labelings of special classes of quadratic graphs with some isomorphic components is shown.

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1. Introduction. In this paper, all graphs are finite without loops or multiple edges, and all parameters are positive integers. The symbols $|A|$, P_n, and C_n denote the cardinality of set A, a snake, and a cycle with n edges, respectively. A sequence of numbers in parentheses or square brackets indicates the labels of vertices of a graph or subgraph under consideration according to whether it is a snake or cycle, respectively.

Definition 1.1. A *graceful labeling* (or *β-labeling*) of a graph $G = (V,E)$, with m vertices and n edges, is a one-to-one mapping Ψ of the vertex set $V(G)$ into the set $\{0,1,2,\dots,n\}$ with this property: if we define, for any edge $e = \{u,v\} \in E(G)$, the value $\Phi(e) = |\Psi(u) - \Psi(v)|$, then Φ is a one-to-one mapping of the set $E(G)$ onto the set $\{1,2,\dots,n\}$.

A graph is called *graceful* if it has a graceful labeling. Not all graphs are graceful. For example, C_5 and K_5 are not graceful.

Definition 1.2. An *α-labeling* of a graph $G = (V,E)$ is a graceful labeling of G which satisfies the following additional condition: there exists a number $\gamma(0 \le \gamma|E(G)|)$ such that, for any edge $e \in E(G)$ with end vertices $u,v \in V(G)$, $\min[\Psi(u),\Psi(v)] \le \gamma < \max[\Psi(u),\Psi(v)]$. The values of an α-labeling Ψ which are less than or equal to γ are referred to as *"small values"* and the remaining values of Ψ as the *"large values"* of the given α-labeling.

The concepts of a graceful labeling and of an α-labeling were introduced by Rosa [8]. Rosa proved that any graceful Eulerian graph G satisfies the condition $|E(G)| \equiv 0$ or $3 \pmod 4$. This implies that any Eulerian graph G with an α-labeling satisfies the condition $|E(G)| \equiv 0 \pmod 4$ (G is bipartite). It is also known that these conditions are also sufficient if G is a cycle [8]. Abrham and Kotzig [3] proved that Rosa's condition is also sufficient for 2-regular graphs with two components. The author [4] proved the similar result for 2-regular graphs with three components with the exception of one special case.

A detailed history of the graph labeling problems and related results appears in Gallian [6]. One of the results of Abrham and Kotzig should be mentioned here.

TABLE 1.1. Some of the results about α-labelings of quadratic graphs.

Graph	Graceful labeling	α-labeling
$Q(1,s)$	It is graceful if and only if $s \equiv 0$ or $3 \pmod 4$ [8]	It has an α-labeling if and only if $s \equiv 0 \pmod 4$ [8]
$Q(2,s)$	It is graceful if and only if s is even and $s > 2$ [7]	It has an α-labeling if and only if s is even and $s > 2$ [7]
$Q(3,4k)$	It is graceful for each $k \geq 1$ [7]	It has an α-labeling for each $k > 1$ [7]
$Q(r,3)$	It is graceful if and only if $r = 1$ [7]	It has no α-labeling [7]
$Q(r,4)$	It is graceful for each $r \geq 1$ [2]	It has an α-labeling for each $r \geq 1$, $r \neq 3$ [2]
$Q(r,5)$	It is not graceful for any $r \geq 1$ [7]	It has no α-labeling [7]
$Q(5,4k)$	It is graceful for all $k \geq 1$ [5]	It has an α-labeling for all $k \geq 1$ [5]

DEFINITION 1.3. If G is a 2-regular graph on n vertices and n edges, which has a graceful labeling Ψ, then there exists exactly one number x $(0 < x < n)$ such that $\Psi(v) \neq x$ for all $v \in V(G)$; this number x is referred to as the missing value of the graceful labeling [1].

In the work done here, the problem of existence of an α-labeling of a special class of 2-regular graphs, called quadratic graph, is investigated.

DEFINITION 1.4. A quadratic graph $Q(r,s)$ is a 2-regular graph with r components, each of which is a cycle of length s.

Some of the results about α-labelings of quadratic graphs published in the literature are summarized in Table 1.1.

2. The existence of an α-labeling of $Q(3,4k)$

THEOREM 2.1. *A $Q(3,4k)$-graph has an α-labeling for each $k > 1$.*

PROOF. In this case we have a graph consisting of three cycles of length $4k$. We know that an α-labeling for this graph was constructed by Kotzig in [7]. We now present a different construction of an α-labeling of $3C_{4k}$ for $k > 1$; its advantage is that it makes it possible to obtain certain results in Section 3.

The vertices of the first C_{4k} are successively labeled as follows: $[0, 12k, 1, 12k-1, 2, 12k-2, \ldots, k-1, 11k+1, k+1, 11k, \ldots, 2k-1, 10k+2, 2k, 10k+1]$. The resulting edge values of the first C_{4k} are then $12k, 12k-1, 12k-2, \ldots, 10k+2, 10k, \ldots, 8k+2, 8k+1$, and $10k+1$.

The vertices of the second C_{4k} are consecutively labeled by the numbers $[4k, 8k, 4k+1, 8k-1, 4k+2, 8k-2, \ldots, 5k-1, 7k+1, 5k+1, 7k, \ldots, 6k-1, 6k+2, 6k, 6k+1]$. The

TABLE 2.1. An α-labeling of $Q(3,4k)$ for $2 \le k \le 5$.

k	$Q(3,4k)$	An α-labeling of $Q(3,4k)$
5	$Q(3,20)$	[0,51,10,52,9,53,8,54,7,55,6,56,4,57,3,58,2,59,1,60],[20,31,30,32,29,33,28,34,27,35,26,36,24,37,23,38,22,39,21,40,20],[5,44,17,43,14,42,18,41 16,50,19,49,11,48,12,47,25,46,13,45]
4	$Q(3,16)$	[0,41,8,42,7,43,6,44,5,45,3,46,2,47,1,48],[16,25,24,26,23,27,22,28,21,29,19,30,18,31,17,32],[4,35,15,40,11,38,20,37,9,39,13,34,10,33,14,36]
3	$Q(3,12)$	[0,31,6,32,5,33,4,34,2,35,1,36],[12,19,18,20,17,21,16,22,14,23,13,24],[3,26,10,25,8,30,11,29,15,28,7,27]
2	$Q(3,8)$	[0,21,4,22,3,23,1,24],[8,13,12,14,11,15,9,16],[2,17,5,19,10,20,7,18]

resulting edge values of the second C_{4k} are then $4k, 4k-1, 4k-2, \dots, 2k+2, 2k, 2k-1, \dots, 3, 2, 1$, and $2k+1$. The missing value of the first C_{4k} is equal to k and the missing value of the second C_{4k} is equal to $5k$. The missing value of the main graph is equal to $3k$ and $y = 6k$.

Now we construct the third cycle C_{4k}. First suppose that $k \ge 6$. Next we construct three snakes. The vertices of the first snake are successively labeled as $10k-1, 2k+1, 10k-2, 2k+2, \dots, 3k-4, 9k+3, 3k-3$, and $9k+2$. The resulting edge values of this snake are then $8k-2, 8k-3, 8k-4, \dots, 6k+7, 6k+6$, and $6k+5$. The vertices of the second snake are consecutively labeled by the numbers $9k-5, 3k-1, 9k-2, 3k+1$, and $9k-1$; this yields the edge values $6k-4, 6k-1, 6k-3$, and $6k-2$. Finally the vertices of the third snake are labeled as $9k-1, k, 9k, 3k-2, 9k+1, 5k$, and $9k+2$; this yields the edge values $8k-1, 8k, 6k+2, 6k+3, 4k+1$, and $4k+2$. Now we generate the edge labels $6k+4, 6k+1$, and $6k$ by connecting the following pairs of vertices to each other respectively: $4k-4$ and $10k$; $4k-1$ and $10k$; $4k-1$ and $10k-1$. In order to generate the rest of the edge labels, we need to use a special type of transforming of vertex labels, described in the appendix as "transformation of labels procedure." Therefore, in the next step, we apply the transformation of labels procedure to the remaining vertex labels, that is, $(3k+2, 3k+3, \dots, 4k-4, 4k-3, 4k-2)$ and $(8k+1, 8k+2, \dots, 9k-5, 9k-4, 9k-3)$ by considering the two vertices $4k-4$ and $9k-5$ as end vertices. This transformation generates the rest of the edge labels and the construction of the last C_{4k} is completed. The construction of an α-labeling of $Q(3,4k)$ with $x = 3k$ and $y = 6k$ for $2 \le k \le 5$ is illustrated in Table 2.1. □

3. Existence of α-labelings of general classes of quadratic graphs.

The following concept presented in [5] is very useful for further considerations in this section.

DEFINITION 3.1. The graph C_{4k} has a *standard labeling* if the values of the vertices of C_{4k} can be generated from an α-labeling of C_{4k} by adding constant factor(s) to the small or large values (or both) of an α-labeling of C_{4k}.

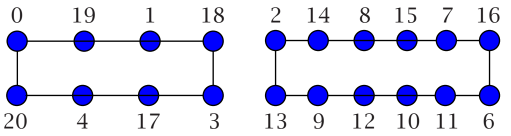

FIGURE 3.1. An α-labeling of the graph $C_8 \cup C_{12}$.

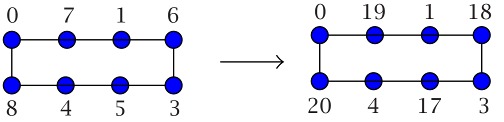

FIGURE 3.2. Transformations of an α-labeling of the graph C_8 to a standard labeling.

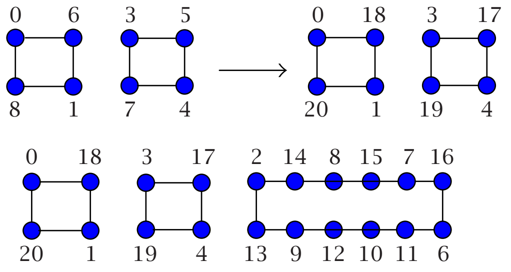

FIGURE 3.3. An α-labeling of $C_{12} \cup Q(2,4)$.

EXAMPLE 3.2. In Figure 3.1, an α-labeling of the graph $C_8 \cup C_{12}$ is presented. This graph consists of the disjoint union of two cycles and has 20 vertices. According to the results presented in [1], we know that in this graph the missing value is 5 and $y = 10$.

In the above α-labeling, C_8 has a standard labeling because it can be generated from an α-labeling of C_8 only by increasing the large values of this construction by 12, see Figure 3.2.

If a graph C_{4k} has a standard labeling, it can be replaced by any α-labeling of the disjoint union of cycles in the form of $\bigcup_{i=1}^{n} C_{4k_i}$ by considering the constant factor(s) if there is an α-labeling for $\bigcup_{i=1}^{n} C_{4k_i}$ and $k = k_1 + k_2 + \cdots + k_n$.

EXAMPLE 3.3. Since we know that $Q(2,4)$ has an α-labeling, the standard labeling of C_8 in Figure 3.1 can be replaced by an α-labeling of $Q(2,4)$ to form an α-labeling of $C_{12} \cup Q(2,4)$ if we increase the large values of an α-labeling $Q(2,4)$ by 12, see Figure 3.3.

In the construction of an α-labeling of $Q(3,4k)$, the first and second C_{4k} have standard labelings because the first cycle can be generated by adding $8k$ to the large values of an α-labeling of C_{4k} with $x = k$, $y = 2k$, and the second cycle can be generated by adding $4k$ to the small and large values of an α-labeling of C_{4k} with $x = k$, $y = 2k$.

In the following theorems, we use this property to extend the class of quadratic graphs with isomorphic components that have α-labelings.

Theorem 3.4. *The following graphs have α-labelings if $k = 3k_1$, $k_i = 3k_{i+1}$, $k_i > 1$, $i = 1,2,3,\dots,n-1$:*

(i) $\bigcup_{i=1}^{n} Q(2,4k_i) \cup Q(2,4k) \cup C_{4k_n}$,

(ii) $\bigcup_{i=1}^{n} Q(4,4k_i) \cup Q(2,4k_n) \cup C_{4k}$.

Proof. It is shown that in the construction of an α-labeling of $Q(3,4k)$, two isomorphic components C_{4k} have standard labelings. Now we apply the following transformations in order to obtain the proof of each part of the theorem.

In the construction of an α-labeling of $Q(3,4k)$, substitute one of the components of C_{4k} with standard labeling by $Q(3,4k_1)$, $k = 3k_1$. Then, since at least one component of $Q(3,4k_1)$ still has a standard labeling, we are able to replace it again by $Q(3,4k_2)$, $k_1 = 3k_2$. In the next stages, we continue to replace one component of each $Q(3,4k_i)$ by $Q(3,4k_{i+1})$, $k_i = 3k_{i+1}$, for $i = 2,3,\dots,n-1$, to obtain an α-labeling of the first graph of the theorem.

The proof of the second part is similar to the proof of the first part. This time we use the replacements for both components with standard labelings in an α-labeling of $Q(3,4k_i)$. □

Example 3.5. The following classes of graphs have α-labelings according to Theorem 2.1, for $k = 6$ and $k_1 = 2$:

$$Q(3,8) \cup Q(2,24), \qquad Q(6,8) \cup C_{24}. \tag{3.1}$$

Theorem 3.6. *The following graphs have α-labelings if $k = \sum_{i=1}^{n} k_i$ and $k_i \geq \sum_{t=i+1}^{n} k_t$ for $i = 1,2,3,\dots,n-1$:*

(i) $\bigcup_{i=1}^{n} C_{4k_i} \cup Q(2,4k)$,

(ii) $\bigcup_{i=1}^{n} Q(2,4k_i) \cup C_{4k}$.

Proof. In the construction of an α-labeling of $Q(3,4k)$, at least two cycles C_{4k} have standard α-labelings. In order to obtain the different parts of Theorem 3.4, apply the following replacements.

(i) Consider one of the standard labelings of C_{4k}. First we replace it by $C_{4k_1} \cup C_{4q_1}$, where $q_1 \leq k_1$ and $k = k_1 + q_1$. Then, since C_{4q_1} still has a standard labeling [3], it can be replaced again by $C_{4k_2} \cup C_{4q_2}$, where $q_2 \leq k_2$ and $q_1 = k_2 + q_2$. In the next stages, we continue to replace each C_{4q_i} by $C_{4k_{i+1}} \cup C_{4q_{i+1}}$, $q_{i+1} \leq k_{i+1}$, where $q_i = k_{i+1} + q_{i+1}$ for $i = 2,3,\dots,n-2$, and $k_n = q_{n-1}$.

(ii) We apply the replacement procedure of the first part for both C_{4k} which have standard labelings in an α-labeling of $Q(3,4k)$. □

Example 3.7. The following classes of graphs have α-labelings, for $r,t \geq 1$:

$$C_{4r} \cup C_{4t} \cup Q(2,4(r+t)), \qquad C_{4(r+t)} \cup Q(2,4r) \cup Q(2,4t). \tag{3.2}$$

Theorem 3.8. *The following graphs have α-labelings if $k = 3k_1$, $k_i = 3k_{i+1}$, $k_1 > 1$, $i = 1,2,3,\dots,n-1$:*

(i) $\bigcup_{i=1}^{n} Q(2,4k_i) \cup Q(4,4k) \cup C_{4k_n}$,

(ii) $\bigcup_{i=1}^{n} Q(4,4k_i) \cup Q(2,4k_n) \cup Q(3,4k)$,

(iii) $\bigcup_{i=1}^{n} Q(6,4k_i) \cup Q(3,4k_n) \cup Q(2,2k)$.

Proof. It has been shown that in the construction of an α-labeling of $Q(5,4k)$, at least three isomorphic components C_{4k} have standard labelings [5].

First consider one of the components of C_{4k} with standard labeling in the construction of an α-labeling of $Q(5,4k)$. Then substitute it by $Q(3,4k_1)$, where $k = 3k_1$, $k_1 > 1$. Since at least one component of $Q(3,4k_1)$ still has a standard labeling, it can be replaced again by $Q(3,4k_2)$, $k_1 = 3k_2$. In the next stages, we continue to replace one component of each $Q(3,4k_i)$ by $Q(3,4k_{i+1})$, where $k_i = 3k_{i+1}$, for $i = 2,3,\dots,n-1$. Finally we obtain an α-labeling of the graph in the first part of the theorem.

The proof of the second (and third) part of the theorem can be easily obtained by applying the above replacements to the second (and third) isomorphic component of C_{4k} with standard labelings in an α-labeling of $Q(5,4k)$. □

Example 3.9. The following classes of graphs have α-labelings according to Theorem 3.8, for $k = 18$, $k_1 = 6$, and $k_2 = 2$:

$$\begin{gathered} Q(3,8) \cup Q(2,24) \cup Q(4,72), \\ Q(6,8) \cup Q(4,24) \cup Q(3,72), \\ Q(9,8) \cup Q(6,24) \cup Q(2,36). \end{gathered} \tag{3.3}$$

Theorem 3.10. *The following graphs have α-labelings if $k = 5k_1$, $k_i = 5k_{i+1}$, $i = 1,2,3,\dots,n-1$:*

(i) $\bigcup_{i=1}^{n} Q(4,4k_i) \cup Q(2,4k) \cup C_{4k_n}$,

(ii) $\bigcup_{i=1}^{n} Q(8,4k_i) \cup Q(2,4k_n) \cup C_{4k}$.

Proof. In the first part of the theorem, consider one of the components of C_{4k} with standard labeling in an α-labeling of $Q(3,4k)$. Then substitute it by $Q(5,4k_1)$, $k = 5k_1$. We know that in the construction of an α-labeling of $Q(5,4k)$, at least three isomorphic components C_{4k} have standard labelings [5]. Then, since at least one component of $Q(5,4k_1)$ still has a standard labeling, we are able to replace it again by $Q(5,4k_2)$, $k_1 = 5k_2$. In the next stages, we continue to replace one component of each $Q(3,4k_i)$ by $Q(3,4k_{i+1})$, where $k_i = 5k_{i+1}$, for $i = 2,3,\dots,n-1$. Finally we obtain an α-labeling of the graph in the first part of the theorem.

The proof of the second part is similar to the proof of the first part. This time we use the replacements for two isomorphic components of C_{4k} with standard labelings in an α-labeling of $Q(5,4k)$. □

Example 3.11. The following classes of graphs have α-labelings for $r \geq 1$:

$$Q(5,4r) \cup Q(2,20r), \qquad Q(10,4r) \cup C_{20r}. \tag{3.4}$$

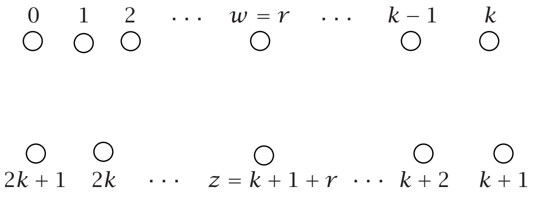

FIGURE A.1. Arrangement of vertex labels of snake P_{2k+1} according to Lemma A.1.

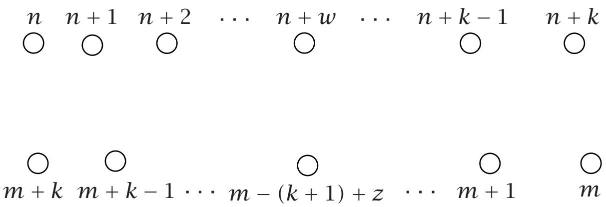

FIGURE A.2. Arrangement of vertex labels in the transformation.

THEOREM 3.12. *The following classes of graphs have α-labelings for $r, t \geq 1$:*
(i) $C_{4r+2} \cup C_{4t+2} \cup Q(2, 4(r+t+1))$,
(ii) $C_{4(r+t)} \cup Q(2, 4r+2) \cup Q(2, 4t+2)$.

PROOF. In the first part, we need to replace one of the standard labelings of C_{4k} in the construction of $Q(3, 4k)$ by $C_{4r+2} \cup C_{4t+2}$, $r, t \geq 1$ and $r+t+1 = k$, because we know that the graph $C_{4r+2} \cup C_{4t+2}$ has an α-labeling for $r, t \geq 1$ [3]. In the second part of the theorem, we replace both the standard α-labelings of C_{4k} and the construction of $Q(3, 4k)$ by $C_{4r+2} \cup C_{4t+2}$, $r, t \geq 1$, and $r+t+1 = k$, respectively. □

Appendix

Transformation of labels procedure. The transformation presented below is used in Theorem 2.1.

LEMMA A.1 (Abrham and Kotzig [3]). *Let r be a nonnegative integer and let s be an odd integer, $s = 2k+1 \geq 2r+1$. Then P_s has an α-labeling Ψ with endpoints labelled w and z that satisfy the conditions $z - w = k+1$ and $w = r$. (Without loss of generality, assume that $w < z$.)*

Given any $0 \leq w \leq k$, $k+1 \leq z \leq 2k+1$, and $z - w = k+1$, we can always construct an α-labeling for a bipartite snake P_{2k+1} with edge labels 1 through $2k+1$ and endpoints w and z, with $y = k$, $w = r$, and $z = k+r+1$, see Figure A.1.

Now suppose we add n to the upper half and add $m - (k+1)$ to the lower half for any positive integers m and n, where $m - 1 > n + k$, see Figure A.2.

Then the edge labels increase by precisely $m - (k+1) - n$. The transformation produces the edge labels from $[m-k-n]$ through $[m+k-n]$ according to Lemma A.1.

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