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q-RIEMANN ZETA FUNCTION

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We consider the modified q-analogue of Riemann zeta function which is defined by $\zeta_q(s) = \sum_{n=1}^{\infty}(q^{n(s-1)}/[n]^s)$, $0 < q < 1$, $s \in \mathbb{C}$. In this paper, we give q-Bernoulli numbers which can be viewed as interpolation of the above q-analogue of Riemann zeta function at negative integers in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers. Also, we will treat some identities of q-Bernoulli numbers using non-Archimedean q-integration.

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1. Introduction. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$.

The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we normally assume $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. We use the notation

$$[x] = [x:q] = \frac{1-q^x}{1-q} = 1+q+q^2+\cdots+q^{x-1}. \tag{1.1}$$

Note that $\lim_{q\to 1}[x] = x$ for $x \in \mathbb{Z}_p$ in the p-adic case.

Let $UD(\mathbb{Z}_p)$ be denoted by the set of uniformly differentiable functions on $\mathbb{Z}_p$.

For $f \in UD(\mathbb{Z}_p)$, we start with the expression

$$\frac{1}{[p^N]}\sum_{0\le j<p^N} q^j f(j) = \sum_{0\le j<p^N} f(j)\mu_q(j+p^N\mathbb{Z}_p) \tag{1.2}$$

representing the analogue of Riemann's sums for f (cf. [4]).

The integral of f on $\mathbb{Z}_p$ will be defined as the limit $(N\to\infty)$ of these sums, which exists. The p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by (see [4])

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]}\sum_{0\le j<p^N} f(j)q^j. \tag{1.3}$$

For d that is a fixed positive integer with $(p,d)=1$, let

$$\begin{aligned} X = X_d &= \varprojlim_N \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \quad X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a (\bmod dp^N)\}, \end{aligned} \tag{1.4}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$.

Let $\mathbb{N}$ be the set of positive integers. For $m,k \in \mathbb{N}$, the q-Bernoulli polynomials, $\beta_m^{(-m,k)}(x,q)$, of higher order for the variable x in $\mathbb{C}_p$ are defined using p-adic q-integral by (cf. [4])

$$\begin{aligned} &\beta_m^{(-m,k)}(x,q) \\ &\quad = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k \text{ times}} [x+x_1+x_2+\cdots+x_k]^m \\ &\qquad \cdot q^{-x_1(m+1)-x_2(m+2)-\cdots-x_k(m+k)} d\mu_q(x_1)d\mu_q(x_2)\cdots d\mu_q(x_k). \end{aligned} \tag{1.5}$$

Now, we define the q-Bernoulli numbers of higher order as follows (cf. [2, 4, 7]):

$$\beta_m^{(-m,k)} (= \beta_m^{(-m,k)}(q)) = \beta_m^{(-m,k)}(0,q). \tag{1.6}$$

By (1.5), it is known that (cf. [4])

$$\begin{aligned} \beta_m^{(-m,k)} &= \lim_{N\to\infty} \frac{1}{[p^N]^k} \sum_{x_1=0}^{p^N-1}\cdots\sum_{x_k=0}^{p^N-1} [x_1+\cdots+x_k]^m q^{-x_1 m - x_2(m+1)+\cdots-x_k(m+k-1)} \\ &= \frac{1}{(1-q)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^i \frac{(i-m)(i-m-1)\cdots(i-m-k+1)}{[i-m][i-m-1]\cdots[i-m-k+1]}, \end{aligned} \tag{1.7}$$

where $\binom{m}{i}$ are the binomial coefficients.

Note that $\lim_{q\to 1}\beta_m^{(-m,k)} = B_m^{(k)}$, where $B_m^{(k)}$ are ordinary Bernoulli numbers of order k (cf. [2, 3, 5, 7, 9]). By (1.5) and (1.7), it is easy to see that

$$\begin{aligned} \beta_m^{(-m,1)}(x,q) &= \sum_{i=0}^{m}\binom{m}{i} q^{xi}\beta_i^{(-m,1)}[x]^{m-i} \\ &= \frac{1}{(1-q)^m}\sum_{j=0}^{m} q^{jx}\binom{m}{j}(-1)^j \frac{j-m}{[j-m]}. \end{aligned} \tag{1.8}$$

We modify the q-analogue of Riemann zeta function which is defined in [1] as follows: for $q \in \mathbb{C}$ with $0<q<1$, $s \in \mathbb{C}$, define

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{(s-1)n}}{[n]^s}. \tag{1.9}$$

The numerator ensures the analytic continuation for $\Re(s) > 1$. In (1.9), we can consider the following problem.

"Are there q-Bernoulli numbers which can be viewed as interpolation of $\zeta_q(s)$ at negative integers in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers?"

In this paper, we give the value $\zeta_q(-m)$ for $m \in \mathbb{N}$, which is the answer of the above problem, and construct a new complex q-analogue of Hurwitz's zeta function and q-L-series. Also, we will treat some interesting identities of q-Bernoulli numbers.

2. Some identities of q-Bernoulli numbers $\beta_m^{(-m,1)}$. In this section, we assume $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. By (1.5), we have

$$\begin{aligned} \beta_n^{(-n,1)}(x,q) &= \int_X q^{-(n+1)t}[x+t]^n d\mu_q(t) \\ &= [d]^{n-1}\sum_{i=0}^{d-1} q^{-ni}\int_{\mathbb{Z}_p} q^{-(n+1)dx}\left[\frac{x+i}{d} : q^d\right]^n d\mu_{q^d}(x). \end{aligned} \tag{2.1}$$

Thus, we have

$$\beta_n^{(-n,1)}(x,q) = [d]^{n-1}\sum_{i=0}^{d-1} q^{-ni}\beta_n^{(-n,1)}\left(\frac{x+i}{d}, q^d\right), \tag{2.2}$$

where d, n are positive integers.

If we take $x = 0$, then we have

$$[n]\beta_m^{(-m,1)} - n[n]^m\beta_m^{(-m,1)}(q^n) = \sum_{k=0}^{m-1}\binom{m}{k}[n]^k\beta_k^{(-m,1)}(q^n)\sum_{j=1}^{n-1} q^{-(m-j)k}[j]^{m-k}. \tag{2.3}$$

It is easy to see that $\lim_{q\to 1}\beta_m^{(-m,1)} = B_m$, where B_m are ordinary Bernoulli numbers (cf. [7]).

Remark 2.1. By (2.3), note that

$$n(1-n^m)B_m = \sum_{k=0}^{m-1}\binom{m}{k}n^k B_k \sum_{j=1}^{n-1} j^{m-k}. \tag{2.4}$$

Let $F_q(t)$ be the generating function of $\beta_n^{(-n,1)}$ as follows:

$$F_q(t) = \sum_{k=0}^{\infty}\beta_k^{(-k,1)}\frac{t^k}{k!}. \tag{2.5}$$

By (1.7) and (2.5), we easily see that

$$F_q(t) = -\sum_{m=0}^{\infty}\left(m\sum_{n=0}^{\infty} q^{-mn}[n]^{m-1}\right)\frac{t^m}{m!}. \tag{2.6}$$

Through differentiating both sides with respect to t in (2.5) and (2.6), and comparing coefficients, we obtain the following proposition.

Proposition 2.2. *For $m > 0$, there exists*

$$-\frac{\beta_m^{(-m,1)}}{m} = \sum_{n=1}^{\infty} q^{-nm}[n]^{m-1}. \tag{2.7}$$

Moreover, $\beta_0^{(0,1)} = (q-1)/\log q$.

Remark 2.3. Note that Proposition 2.2 is a q-analogue of $\zeta(1-2m)$ for any positive integer m.

Let χ be a primitive Dirichlet character with conductor $f \in \mathbb{N}$.
For $m \in \mathbb{N}$, we define

$$\beta_{m,\chi}^{(-m,1)} = \int_X q^{-(m+1)x}\chi(x)[x]^m d\mu_q(x), \quad \text{for } m \geq 0. \tag{2.8}$$

Note that

$$\beta_{m,\chi}^{(-m,1)} = [d]^{m-1}\sum_{i=0}^{d-1}\chi(i)q^{-mi}\beta_m^{(-m,1)}\left(\frac{i}{d},q^d\right). \tag{2.9}$$

3. q-analogs of zeta functions. In this section, we assume $q \in \mathbb{C}$ with $|q| < 1$. In [1], the q-analogue of Riemann zeta function was defined by (cf. [1])

$$\zeta_q^*(s) = \sum_{n=1}^{\infty}\frac{q^{ns}}{[n]^s}, \quad \Re(s) > 0. \tag{3.1}$$

Now, we modify the above q-analogue of Riemann zeta function as follows: for $q \in \mathbb{C}$ with $0 < |q| < 1$, $s \in \mathbb{C}$, define

$$\zeta_q(s) = \sum_{n=1}^{\infty}\frac{q^{(s-1)n}}{[n]^s}. \tag{3.2}$$

By (2.5), (2.6), and (2.7), we obtain the following proposition.

Proposition 3.1. *For $m \in \mathbb{N}$, there exists*
(i) $\zeta_q(1-m) = -\beta_m^{(-m,1)}/m$, *for $m \geq 1$;*
(ii) $\zeta_q(s)$ *having simple pole at $s = 1$ with residue $(q-1)/\log q$.*

By (1.7) and (1.8), we see that

$$\beta_n^{(-n,1)}(x,q) = -n\sum_{k=0}^{\infty}\left([k]q^x + [x]\right)^{n-1}q^{-n(k+x)}, \quad \text{where } 0 \leq x < 1. \tag{3.3}$$

Hence, we can define q-analogue of Hurwitz ζ-function as follows: for $s \in \mathbb{C}$, define

$$\zeta_q(s,x) = \sum_{n=0}^{\infty} \frac{q^{(s-1)(n+x)}}{([n]q^x + [x])^s}. \tag{3.4}$$

Note that $\zeta_q(s,x)$ has an analytic continuation in $\mathbb{C}$ with only one simple pole at $s = 1$.
By (3.3) and (3.4), we have the following theorem.

Theorem 3.2. *For any positive integer k, there exists*

$$\zeta_q(1-k,x) = -\frac{\beta_k^{(-k,1)}(x,q)}{k}. \tag{3.5}$$

Let χ be Dirichlet character with conductor $d \in \mathbb{N}$. By (2.9), the generalized q-Bernoulli numbers with χ can be defined by

$$\beta_{m,\chi}^{(-m,1)} = [d]^{m-1} \sum_{i=0}^{d-1} \chi(i) q^{-mi} \beta_m^{(-m,1)}\left(\frac{i}{d}, q^d\right). \tag{3.6}$$

For $s \in \mathbb{C}$, we define

$$L_q(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n) q^{(s-1)n}}{[n]^s}. \tag{3.7}$$

It is easy to see that

$$L_q(\chi,s) = [d]^{-s} \sum_{a=1}^{d} \chi(a) q^{(s-1)a} \zeta_{q^d}\left(s, \frac{a}{d}\right). \tag{3.8}$$

By (3.6), (3.7), and (3.8), we obtain the following theorem.

Theorem 3.3. *Let k be a positive integer. Then there exists*

$$L_q(1-k,\chi) = -\frac{\beta_{k,\chi}^{(-k,1)}}{k}. \tag{3.9}$$

Let a and F be integers with $0 < a < F$. For $s \in \mathbb{C}$, we consider the functions $H_q(s,a,F)$ as follows:

$$H_q(s,a,F) = \sum_{m \equiv a(F), m>0} \frac{q^{m(s-1)}}{[m]^s} = [F]^{-s} \zeta_{q^F}\left(s, \frac{a}{F}\right). \tag{3.10}$$

Then we have

$$H_q(1-n,a,F) = -\frac{[F]^{n-1}}{n} \beta_n^{(-n,1)}\left(\frac{a}{F}, q^F\right), \tag{3.11}$$

where n is any positive integer.

Therefore, we obtain the following theorem.

Theorem 3.4. *Let a and F be integers with $0 < a < F$. For $s \in \mathbb{C}$, there exists*
(i) $H_q(1-n,a,F) = -([F]^{n-1}/n)\beta_n^{(-n,1)}(a/F,q^F)$;
(ii) $H_q(s,a,F)$ *having a simple pole at $s = 1$ with residue* $(1/[F]F)((q^F-1)/\log q)$.

In a recent paper, the q-analogue of Riemann zeta function was studied by Cherednik (see [1]). In [1], we can consider the q-Bernoulli numbers which can be viewed as an interpolation of the q-analogue of Riemann zeta function at negative integers. In this paper, we have shown that the q-analogue of zeta function interpolates q-Bernoulli numbers at negative integers in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers (cf. [2, 5, 7]).

Remark 3.5. Let $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. Then the p-adic q-gamma function was defined as (see [8])

$$\Gamma_{p,q}(n) = (-1)^n \prod_{1 \le j < n,\,(j,p)=1} [j]. \tag{3.12}$$

For all $x \in \mathbb{Z}_p$, we have

$$\Gamma_{p,q}(x+1) = \epsilon_{p,q}(x)\Gamma_{p,q}(x), \tag{3.13}$$

where $\epsilon_{p,q}(x) = -[x]$ for $|x|_p = 1$, and $\epsilon_{p,q}(x) = -1$ for $|x|_p < 1$, (see [8]). By (3.13), we easily see that (cf. [6])

$$\log \Gamma_{p,q}(x+1) = \log \epsilon_{p,q}(x) + \log \Gamma_{p,q}(x). \tag{3.14}$$

By the differentiation of both sides in (3.14), we have (cf. [6])

$$\frac{\Gamma'_{p,q}(x+1)}{\Gamma_{p,q}(x+1)} = \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} + \frac{\epsilon'_{p,q}(x)}{\epsilon_{p,q}(x)}. \tag{3.15}$$

By (3.15), we easily see that (cf. [6])

$$\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} = \left(\sum_{j=1}^{x-1} \frac{q^j}{[j]}\right)\frac{\log q}{q-1} + \frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}. \tag{3.16}$$

Define

$$L_{p,q}(x) = \sum_{j=0}^{x-1} \frac{\epsilon'_{p,q}(j)}{\epsilon_{p,q}(j)}. \tag{3.17}$$

It is easy to check that $L_{p,q}(1) = 0$. By (3.15), we also see that

$$\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} = L_{p,q}(x) + \frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}, \quad \text{for } x \in \mathbb{Z}_p, \tag{3.18}$$

where $L_{p,q}(x)$ denotes the indefinite sum of $\epsilon'_{p,q}(x)/\epsilon_{p,q}(x)$. By using (3.18) after substituting $x=1$, we obtain $L_{p,q}(1)=0$. The classical Euler constant was known as $\gamma = -\Gamma'(1)/\Gamma(1)$. In [8], Koblitz defined the p-adic q-Euler constant $\gamma_{p,q} = -\Gamma'_{p,q}(1)/\Gamma_{p,q}(1)$ (cf. [6, 8]). By using (3.16) and the congruence of Andrews (cf. [3]), we obtain the following congruence:

$$\frac{q-1}{\log q}\left(\frac{\Gamma'_{p,q}(p)}{\Gamma_{p,q}(p)}-\gamma_{p,q}\right)=\sum_{j=1}^{p-1}\frac{q^j}{[j]}\equiv\frac{p-1}{2}(q-1)(\mathrm{mod}[p]). \tag{3.19}$$

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