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SOME ANALYTICAL PROPERTIES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF NONINTEGER ORDER

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The analytical properties of solutions of the nonlinear differential equations $x^{(\alpha)}(t) = f(t,x)$, $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$ of noninteger order have been investigated. We obtained two results concerning the frame curves of solutions. Moreover, we proved a result on differential inequality with fractional derivatives.

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1. Introduction. The problem of existence and uniqueness of solutions of the nonhomogeneous differential equations with fractional derivatives

$$x^{(\alpha)}(t) = f(t,x), \quad \alpha \in \mathbb{R},\ 0 < \alpha \leq 1, \tag{1.1}$$

with the initial condition

$$x^{(\alpha-1)}(t_0) = x_0, \tag{1.2}$$

where $\mathbb{R}$ is the set of real numbers, $t \in I = [0,\infty)$, and f is a real-valued function on $D = I \times \mathbb{R}^n$ into $\mathbb{R}^n$ where $\mathbb{R}^n$ denotes the real n-dimensional Euclidean space, and $x_0 \in \mathbb{R}^n$, has been investigated by some authors (see [1, 2, 6, 9]).

In recent years, interest has increased concerning the numerical treatment of fractional differential equations (see [4, 5, 11, 12]). On the other hand, differential inequalities and comparison theorems with the unique solution are very important for the numerical solution of differential equations (see [8] for fractional differential equations, and [10] for ordinary differential equations).

In this note, we will obtain a differential inequality result of (1.1) and (1.2), our result is more general than that in [8]. Also, we obtain two results concerning frame curves, the lower and upper frame curves of the solutions of (1.1) and (1.2); these two results are extensions to those in [10] for ordinary differential equations.

We will use the definitions and terminology used in Barrett [3] and Al-Bassam [2].

It is worth mentioning that it was shown by Hadid and Alshamani [7] that the solutions of (1.1) and (1.2) satisfy the integral equation

$$x(t) = \frac{x_0(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x(s))\,ds, \tag{1.3}$$

where $0 < t_0 < t \leq t_0 + a$, provided that the integral exists in the Lebesgue sense, where Γ is the Gamma function.

2. The main theorems.

In this section, we will prove the main theorems.

Theorem 2.1. *Let $f(t,x)$ be a continuous function on the region*

$$\mathbb{R}(a,b) : 0 < t_0 < t \leq t_0 + a, \qquad \left| x - x_0 (t - t_0)^{\alpha-1} \right| \leq b. \tag{2.1}$$

Suppose $x_1(t)$ is a solution of the differential inequality

$$x^{(\alpha)}(t) \leq f(t, x_1(t)) \quad \text{on } (t_0, t_0 + a], \tag{2.2}$$

then there exists a solution $x_2(t)$ of the differential inequality

$$x_2^{(\alpha)}(t) \geq f(t, x_2(t)) \quad \text{on } (t_0, t_0 + a], \qquad x_1^{(\alpha-1)}(t_0) \leq x_2^{(\alpha-1)}(t_0) \tag{2.3}$$

such that on this interval, $x_1(t) \leq x_2(t)$.

Proof. Let $\psi(t, x_2) = f(t, \max(x_2, x_1(t)))$. Obviously, ψ is a continuous function on $\mathbb{R}$.

First we will prove the inequality

$$x_1(t) \leq w(t) \quad \text{for } t \in [t_0, t_0 + a], \tag{2.4}$$

where $w(t)$ satisfies the differential inequality

$$w^{(\alpha)}(t) \geq \psi(t, w(t)), \qquad w^{(\alpha-1)}(t_0) = x_2^{(\alpha-1)}(t_0). \tag{2.5}$$

Suppose that this is not true, that is, that for some value τ, $w(\tau) < x_1(\tau)$. Let τ_0 be the lower bound of numbers s for which we have $w(t) < x_1(t)$ for $s \leq t \leq \tau$. Then $w(\tau_0) = x_1(\tau_0)$ and $w(t) < x_1(t)$ for $\tau_0 < t < \tau$.

Therefore, we get

$$\psi(t, w(t)) = f(t, x_1(t)) \quad \text{on } t \in [\tau_0, \tau]. \tag{2.6}$$

Using inequality (2.2) and (1.3), it follows that

$$\begin{aligned} x_1(\tau) &\leq \frac{x_1(\tau_0)(\tau - \tau_0)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_{\tau_0}^{\tau} (\tau - s)^{\alpha-1} f(s, x_1(s)) ds \\ &= \frac{w(\tau_0)(\tau - \tau_0)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_{\tau_0}^{\tau} (\tau - s)^{\alpha-1} \psi(s, w(s)) ds, \end{aligned} \tag{2.7}$$

and from (2.5) and (2.7), we get $x_1(\tau) \leq w(\tau)$, which is in contradiction with our supposition. This proves inequality (2.4).

But now because (2.4) implies that a solution of inequality (2.5) is also a solution of inequality (2.3), we see that the result follows from (2.4). □

Remark 2.2. The above theorem means that the solution $x_1(t)$ is dominated by the solution $x_2(t)$. Moreover, if $x_2(t)$ is a bounded solution, then so is $x_1(t)$.

Theorem 2.3. *Let $\phi(t,y)$, $f(t,y)$, and $F(t,y)$ be continuous functions on the region*

$$\mathbb{R}_1(a,b): 0 < t_0 < t \le t_0 + a, \qquad |y - y_0(t-t_0)^{\alpha-1}| \le b \tag{2.8}$$

and satisfy

$$\phi(t,y) \le f(t,y) \le F(t,y). \tag{2.9}$$

Further let $x = x(t)$, $y = y(t)$, and $X = X(t)$ be solutions of the differential equations

$$x^{(\alpha)}(t) = \phi(t,x), \qquad y^{(\alpha)}(t) = f(t,y), \qquad X^{(\alpha)}(t) = F(t,X), \tag{2.10}$$

which pass through the point $(t_0, y_0(t-t_0)^{\alpha-1})$, defined on $[t_0, t_0+a]$, and which lie between $y_0(t-t_0)^{\alpha-1} - b$ and $y_0(t-t_0)^{\alpha-1} + b$.

If the function $f(t,y)$ satisfies the Lipschitz condition in the second parameter on $\mathbb{R}_1(a,b)$:

$$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2| \tag{2.11}$$

for some positive constant L, then

$$x(t) \le y(t) \le X(t). \tag{2.12}$$

Proof. It is clear from Theorem 2.1 and equations (2.9) and (2.10) that the following inequalities:

$$X^{(\alpha)}(t) - f(t,X) \ge 0, \quad x^{(\alpha)}(t) - f(t,x) \le 0, \quad x(t) \le y(t) \le X(t), \tag{2.13}$$

are satisfied if

$$X = y_0(t-t_0)^{\alpha-1} + Y, \qquad x = y_0(t-t_0)^{\alpha-1} - Y, \tag{2.14}$$

where

$$Y = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |f(s,y_0)|\, ds \le \frac{M}{\alpha\Gamma(\alpha)} (t-t_0)^{\alpha}, \tag{2.15}$$

and M is a positive real constant such that $|f(s,y)| \le M$.

Hence, the theorem is proved. □

Remark 2.4. The functions $X(t)$ and $x(t)$ are called "frame curves."

Theorem 2.5. *Let the functions $f(t,y)$, $F(t,y)$, $y(t)$, and $X(t)$ be defined as in Theorem 2.3. Set $h(t) = X^{(\alpha)}(t) - f(t,X(t))$, then the function*

$$X_1(t) = X(t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-L(t-s)} h(s)\, ds, \tag{2.16}$$

where L the Lipschitz constant for the function $f(t,y)$ is an upper frame curve on the interval $[t_0, t_0+a]$, and on that interval there exist the inequalities

$$y(t) \le X_1(t) \le X(t). \tag{2.17}$$

Proof. The inequality $X_1(t) \leq X(t)$ is obvious. On the other hand, as in [10], we have

$$\begin{aligned} &X^{(\alpha)}(t) - f(t, X_1(t)) \\ &\quad = X^{(\alpha)}(t) - h(t) + \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-L(t-s)} h(s)\,ds - f(t, X_1(t)) \\ &\quad = f(t, X_1(t)) - f(t, X(t)) + L[X(t) - X_1(t)] \geq 0. \end{aligned} \tag{2.18}$$

Thus $y(t) \leq X_1(t)$. □

Remark 2.6. By using the same above procedure, we can show that if $x(t)$ a lower frame and if we set

$$h_1(t) = x^{(\alpha)}(t) - f(t, x(t)), \tag{2.19}$$

then

$$x_1(t) = x(t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-L(t-s)} h_1(s)\,ds \tag{2.20}$$

is also a lower frame curve and we have

$$y(t) \leq x_1(t) \leq y(t). \tag{2.21}$$

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