

N-TOPO NILPOTENCY IN FUZZY NEIGHBORHOOD RINGS

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We introduce the notion of N-topo nilpotent fuzzy set in a fuzzy neighborhood ring and develop some fundamental results. Here we show that a fuzzy neighborhood ring is locally inversely bounded if and only if for all $0 < \alpha < 1$, the α -level topological rings are locally inversely bounded. This leads us to prove a characterization theorem which says that if a fuzzy neighborhood ring on a division ring is Wuyts-Lowen WNT_2 and locally inversely bounded, then the fuzzy neighborhood ring is a fuzzy neighborhood division ring. We also present another characterization theorem which says that a fuzzy neighborhood ring on a division ring is a fuzzy neighborhood division ring if the fuzzy neighborhood ring contains an N-topo nilpotent fuzzy neighborhood of zero.

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1. Introduction. This paper is a continuation into the investigation of the compatibility of the Lowen fuzzy neighborhood topologies with algebraic structures. In the present text, we introduce and study the notion of N-topo nilpotent fuzzy set (fuzzy neighborhood topologically nilpotent fuzzy set) in a fuzzy neighborhood ring. We prove that this notion is a good extension of its classical counterpart. We also prove that the notion of bounded fuzzy set introduced in [5] is a good extension. In [2], we introduced the concept of locally inversely bounded fuzzy neighborhood ring; here we show that a fuzzy neighborhood ring is locally inversely bounded if and only if its level topological rings are locally inversely bounded. This leads us to establish two characterization theorems, which in the first place, show that if a fuzzy neighborhood ring on a division ring is Wuyts-Lowen WNT_2 and locally inversely bounded, then it is a fuzzy neighborhood division ring; and secondly, a fuzzy neighborhood ring on a division ring is a fuzzy neighborhood division ring if the fuzzy neighborhood ring contains an N-topo nilpotent fuzzy set of fuzzy neighborhood of zero. In this regard, some pleasant properties related to N-topo nilpotency are achieved. Finally, we study the notion of N-topo quasi-regularity in connection with N-topo nilpotency in fuzzy neighborhood rings.

2. Preliminaries. As usual, we follow the standard notation for $I = [0, 1]$ the unit interval, $I_0 =]0, 1]$, $I_1 = [0, 1[$, and $I_{0,1} =]0, 1[$. If X is a set and $A \subseteq X$, then the characteristic function of A is given by 1_A , and if $A = \{x\}$ the singleton, we write $1_{\{x\}} = x$. By a fuzzy set we mean a function $\mu : X \rightarrow I$, and the set of all fuzzy sets is denoted by I^X . Throughout the text no distinction will be made between a constant fuzzy set $\underline{\alpha}$ and its value α .

If $*$ is a binary operation on X , then $\hat{*}$ is a binary operation on I^X . Thus for all $x \in X$ and for all $\mu, \nu \in I^X$, $\mu \hat{*} \nu$ is defined by

$$(\mu \hat{*} \nu)(x) = \sup_{s*t=x} \mu(s) \wedge \nu(t). \tag{2.1}$$

If $(\mathbf{R}, +, \cdot)$ is a ring and $x \in \mathbf{R}$, then the fuzzy set $x \oplus \nu$ is defined as

$$(x \oplus \nu)(y) = (1_{\{x\}} \oplus \nu)(y) = \nu(y - x), \tag{2.2}$$

and $(x \odot \nu)(y) = \sup_{xs=y} \nu(s)$ for all $y \in \mathbf{R}$.

If $n \in \mathbb{Z}^+$ and $\mu \in I^{\mathbf{R}}$, then we define

$$\mu^{(1)} = \mu, \quad \mu^{(n)} = \mu \odot \mu^{(n-1)}. \tag{2.3}$$

If G is an additive group, then a fuzzy set $\mu \in I^G$ is called symmetric if and only if $\mu = -\mu$, where $-\mu(x) = \mu(-x)$ for all $x \in G$; and if G is a multiplicative group, then for any fuzzy set $\nu \in I^G$, $\nu^{-1}: G \rightarrow I$, $x \mapsto \nu^{-1}(x) = \nu(x^{-1})$.

When \mathbf{R} is a division ring, \mathbf{R}^* denotes the set of all nonzero elements of \mathbf{R} and $(1 \oplus \nu)^{-1}$ is defined as

$$(1 \oplus \nu)^{-1}(z) = \nu(z^{-1} - 1) \tag{2.4}$$

for any $z \in \mathbf{R}^*$.

DEFINITION 2.1. Let X be a set, $\nu \in I^X$, and $\alpha \in I$. Then the strong (resp., weak) α -cut of ν is a crisp subset of X which is defined as

$$\nu^\alpha = \{x \in X \mid \nu(x) > \alpha\} \tag{2.5}$$

(resp., $\nu_\alpha = \{x \in X \mid \nu(x) \geq \alpha\}$).

LEMMA 2.2. If (X, \cdot) is a semigroup, $\mu, \nu \in I^X$, and $\alpha \in I$, then

$$(\nu \odot \mu)^\alpha = \nu^\alpha \cdot \mu^\alpha. \tag{2.6}$$

PROOF. We have $x \in (\nu \odot \mu)^\alpha$ if and only if $\sup_{st=x} \nu(s) \wedge \mu(t) > \alpha$, if and only if there exists s, t such that $st = x$, $\nu(s) > \alpha$, and $\mu(t) > \alpha$, if and only if there exists s, t , such that $st = x$, $s \in \nu^\alpha$, and $t \in \mu^\alpha$ if and only if $x \in \nu^\alpha \cdot \mu^\alpha$. \square

DEFINITION 2.3 [13]. If \mathcal{B} is a prefilterbasis in I^X , then the saturation operation of \mathcal{B} is written as $\tilde{\mathcal{B}}$ and defined by

$$\tilde{\mathcal{B}} = \{\nu : X \rightarrow I; \forall \epsilon > 0 \exists v_\epsilon \in \mathcal{B} \ni v_\epsilon - \epsilon \leq \nu\}. \tag{2.7}$$

If $\Sigma = (\Sigma(x))_{x \in X}$ is a fuzzy neighborhood system on a set X , then $t(\Sigma)$ is the fuzzy neighborhood topology on X , and the fuzzy topological space $(X, t(\Sigma))$ is called a fuzzy neighborhood space [11, 12, 13, 17].

DEFINITION 2.4 [13]. If $(X, t(\Sigma))$ and $(X', t(\Sigma'))$ are fuzzy neighborhood spaces and $f : X \rightarrow X'$, then f is said to be continuous at $x \in X$ if and only if for all $v' \in \Sigma'(f(x))$ and for all $\epsilon \in I_0$, there exists $v \in \Sigma(x)$ such that $v - \epsilon \leq f^{-1}(v')$.

DEFINITION 2.5 [5]. Let $(\mathbf{R}, +, \cdot)$ be a ring and Σ a fuzzy neighborhood system on \mathbf{R} . Then the quadruple $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is called a fuzzy neighborhood ring if and only if the following are satisfied:

- (FR1) the mapping $h : (\mathbf{R} \times \mathbf{R}, t(\Sigma_{\mathbf{R}}) \times t(\Sigma_{\mathbf{R}})) \rightarrow (\mathbf{R}, t(\Sigma_{\mathbf{R}})), (x, y) \mapsto x + y$, is continuous;
- (FR2) the mapping $k : (\mathbf{R}, t(\Sigma_{\mathbf{R}})) \rightarrow (\mathbf{R}, t(\Sigma_{\mathbf{R}})), x \mapsto -x$, is continuous;
- (FR3) the mapping $m : (\mathbf{R} \times \mathbf{R}, t(\Sigma_{\mathbf{R}}) \times t(\Sigma_{\mathbf{R}})) \rightarrow (\mathbf{R}, t(\Sigma_{\mathbf{R}})), (x, y) \mapsto xy$, is continuous.

PROPOSITION 2.6 [5]. Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring and let $x \in \mathbf{R}$. Then

- (a) the left homothety $L_x : (\mathbf{R}, t(\Sigma_{\mathbf{R}})) \rightarrow (\mathbf{R}, t(\Sigma_{\mathbf{R}})), z \mapsto xz$ (resp., right homothety $R_x : (\mathbf{R}, t(\Sigma_{\mathbf{R}})) \rightarrow (\mathbf{R}, t(\Sigma_{\mathbf{R}})), z \mapsto zx$) is continuous. If x is a unit element of \mathbf{R} , then each homothety is a homeomorphism;
- (b) the translation $T_x : (\mathbf{R}, t(\Sigma_{\mathbf{R}})) \rightarrow (\mathbf{R}, t(\Sigma_{\mathbf{R}})), z \mapsto x + z$, and the mapping $k : (\mathbf{R}, t(\Sigma_{\mathbf{R}})) \rightarrow (\mathbf{R}, t(\Sigma_{\mathbf{R}})), x \mapsto -x$, are homeomorphisms;
- (c) $v \in \Sigma_{\mathbf{R}}(0) \Leftrightarrow x \oplus v \in \Sigma_{\mathbf{R}}(x)$, that is, $T_x(v) \in \Sigma_{\mathbf{R}}(x)$;
- (d) $v \in \Sigma_{\mathbf{R}}(x) \Leftrightarrow -x \oplus v \in \Sigma_{\mathbf{R}}(0)$, that is, $T_{-x}(v) \in \Sigma_{\mathbf{R}}(0)$.

COROLLARY 2.7. Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring with unity and a invertible element. Then

$$v \in \Sigma_{\mathbf{R}}(0) \iff v \circ a \in \Sigma_{\mathbf{R}}(0) \iff a \circ v \in \Sigma_{\mathbf{R}}(0). \tag{2.8}$$

DEFINITION 2.8 [3]. Let $(\mathbf{R}, +, \cdot)$ be a division ring, and Σ a fuzzy neighborhood system on \mathbf{R} . Then the quadruple $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is called a fuzzy neighborhood division ring if and only if the following are true:

- (FD1) $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is a fuzzy neighborhood ring on a division ring;
- (FD2) the mapping $r : (\mathbf{R}^*, t(\Sigma|_{\mathbf{R}^*})) \rightarrow (\mathbf{R}^*, t(\Sigma|_{\mathbf{R}^*})), x \mapsto x^{-1}$, is continuous, where $\Sigma|_{\mathbf{R}^*}$ is the fuzzy neighborhood system on \mathbf{R}^* induced by \mathbf{R} .

THEOREM 2.9 [3]. Let $(\mathbf{R}, +, \cdot)$ be a division ring and Σ a fuzzy neighborhood system on \mathbf{R} . Then the quadruple $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is a fuzzy neighborhood division ring if and only if the following are satisfied:

- (1) for all $x \in \mathbf{R}$, $\Sigma(x) = \{T_x(v) : v \in \Sigma_{\mathbf{R}}(0)\}$;
- (2) for all $\mu \in \Sigma_{\mathbf{R}}(0)$, for all $\epsilon \in I_0$, there exists $v \in \Sigma_{\mathbf{R}}(0) \ni x \circ v \leq \mu + \epsilon$, that is, $z \mapsto zx$ is continuous at 0, and for all $\mu \in \Sigma_{\mathbf{R}}(0)$, for all $\epsilon \in I_0$, there exists $v \in \Sigma_{\mathbf{R}}(0) \ni v \circ x \leq \mu + \epsilon$, that is, $z \mapsto xz$ is continuous at 0;
- (3) for all $\mu \in \Sigma_{\mathbf{R}}(0)$, for all $\epsilon \in I_0$, there exists $v \in \Sigma_{\mathbf{R}}(0) \ni v \oplus v \leq \mu + \epsilon$, that is, $(x, y) \mapsto x + y$ is continuous at $(0, 0)$;
- (4) for all $\mu \in \Sigma_{\mathbf{R}}(0)$, for all $\epsilon \in I_0$, there exists $v \in \Sigma_{\mathbf{R}}(0) \ni v \circ v \leq \mu + \epsilon$, that is, $(x, y) \mapsto xy$ is continuous at $(0, 0)$;
- (5) for all $\mu \in \Sigma_{\mathbf{R}}(0)$, for all $\epsilon \in I_0$, there exists $v \in \Sigma_{\mathbf{R}}(0) \ni (1 \oplus v)^{-1} \leq (1 \oplus \mu) + \epsilon$, that is, $x \mapsto x^{-1}$ ($x \neq 0$) is continuous at 1.

DEFINITION 2.10 [15]. A fuzzy neighborhood space $(X, t(\Sigma))$ is called

- (a) WNT_1 if and only if for all $x \neq y \in X$ there exist $v_x \in \Sigma(x)$ and $\alpha \in I_1$ such that $v_x(y) \leq \alpha$;
- (b) WNT_2 if and only if for all $x \neq y \in X$ there exist $v_x \in \Sigma(x)$, $v_y \in \Sigma(y)$, and $\alpha \in I_1$ such that $v_x \wedge v_y \leq \alpha$.

THEOREM 2.11 [13]. Let $(X, t(\Sigma))$ be a fuzzy neighborhood space and let $\alpha \in I_1$. Then the α -level topological space $(X, \iota_\alpha(t(\Sigma)))$ has a neighborhood system $(\iota_\alpha(\Sigma(x)))_{x \in X}$ where, for all $x \in X$,

$$\iota_\alpha(\Sigma(x)) = \{v^\beta : v \in \Sigma(x), \beta < 1 - \alpha\}. \tag{2.9}$$

PROPOSITION 2.12 [13]. If $(X, t(\Sigma))$ is a fuzzy neighborhood space, then the family $(\iota_\alpha(t(\Sigma)))_{\alpha \in I_1}$ is a chain in the sense that $\alpha \leq \beta$ implies $\iota_\beta(t(\Sigma)) \subseteq \iota_\alpha(t(\Sigma))$.

THEOREM 2.13 [16]. If (X, Ω) is a fuzzy topological space, then (X, Ω) is a fuzzy neighborhood space if and only if for all $\theta \in \Omega$ and for all $\alpha \in I_1$, $\alpha \wedge 1_{\theta\alpha} \in \Omega$.

THEOREM 2.14 [2, 5, 6, 16]. The quadruple $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is a fuzzy neighborhood ring if and only if the quadruples $(\mathbf{R}, +, \cdot, \iota_\alpha(t(\Sigma_{\mathbf{R}})))$ are α -level topological rings for all $0 < \alpha < 1$.

3. N-topo nilpotent fuzzy set in a fuzzy neighborhood ring

DEFINITION 3.1. Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring and let $\mu \in I^{\mathbf{R}}$. Then μ is called N-topo nilpotent if and only if for all $v \in \Sigma_{\mathbf{R}}(0)$, and for all $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that for all $n \geq n_0$,

$$v \geq \mu^{(n)} - \epsilon. \tag{3.1}$$

We say that an element $t \in \mathbf{R}$ is N-topo nilpotent if and only if for all $\epsilon > 0$, for all $v \in \Sigma_{\mathbf{R}}(0)$, there exists $n_0 \in \mathbb{Z}^+ \ni \forall n \geq n_0, v(t^n) \geq 1 - \epsilon$.

REMARK 3.2. Following the notion of sequential convergence in a fuzzy neighborhood space [10], if we consider a sequence $(t^n)_{n \in \mathbb{Z}^+}$ in a fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$, then that t^n converges to 0 (written as $t^n \rightsquigarrow 0$) is equivalent to

$$\forall v \in \Sigma_{\mathbf{R}}(0), \forall \epsilon > 0, \exists n_0 \in \mathbb{Z}^+ \ni \forall n \geq n_0, v(t^n) \geq 1 - \epsilon. \tag{3.2}$$

This demonstrates the fact that the notion of N-topo nilpotency for an element $t \in \mathbf{R}$ in a fuzzy neighborhood ring can be represented in a similar manner as in the case of convergence of sequence. Moreover, the notion of N-topo nilpotency of an element in a fuzzy neighborhood ring is a good extension of its classical counterpart, which can be seen easily from [Theorem 3.3](#) below.

THEOREM 3.3. Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring and let $\mu \in I^{\mathbf{R}}$. Then μ is N-topo nilpotent if and only if for all $0 < \alpha < 1$, μ^α are topologically nilpotent in α -level topological rings $(\mathbf{R}, +, \cdot, \iota_\alpha(\Sigma_{\mathbf{R}}))$.

PROOF. Let μ be an N-topo nilpotent fuzzy set in a fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ and let $\alpha \in I_{0,1}$. We take $\sigma^\beta \in \iota_\alpha(\Sigma_{\mathbf{R}}(0))$ with $\sigma \in \Sigma_{\mathbf{R}}(0)$ and $\beta < \alpha$. Choose $\epsilon \in I_{0,1}$ such that $0 < \epsilon < \alpha - \beta$.

Since μ is N-topo nilpotent, there exists $n_0 \in \mathbb{Z}^+$ such that $\mu^{(n)} - \epsilon \leq \sigma$ for all $n \geq n_0$. Applying induction on n together with [Lemma 2.2](#), we arrive at

$$[\mu^\alpha]^{(n)} = [\mu^{(n)}]^\alpha \subseteq [\mu^{(n)}]^{\beta+\epsilon} = [\mu^{(n)} - \epsilon]^\beta \subseteq \sigma^\beta, \tag{3.3}$$

which yields that $[\mu^\alpha]^{(n)} \subseteq \sigma^\beta$, proving that μ^α is topologically nilpotent.

To prove the converse, we let $\nu \in \Sigma_{\mathbf{R}}(0)$ and $\epsilon \in I_0$. Then for each $\alpha \in I_{0,1}$, $\nu^\alpha \in \iota_\alpha(\Sigma_{\mathbf{R}}(0))$. Since μ^α is topologically nilpotent, there is $n_0 \in \mathbb{Z}^+$ such that $[\mu^\alpha]^{(n)} \subseteq \nu^\alpha$ implies $[\mu^{(n)}]^\alpha \subseteq \nu^\alpha$.

Consequently, we have

$$[\mu^{(n)} - \epsilon]^\alpha = [\mu^{(n)}]^{\alpha+\epsilon} \subseteq \nu^\alpha. \tag{3.4}$$

This implies

$$\mu^{(n)} - \epsilon \leq \nu, \tag{3.5}$$

which means that μ is N-topo nilpotent. □

DEFINITION 3.4 [5]. Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring. Then

- (a) a fuzzy set μ is called left bounded if and only if for all $\nu \in \Sigma_{\mathbf{R}}(0)$ and for all $\epsilon > 0$, there exists a $\theta \in \Sigma_{\mathbf{R}}(0)$ such that

$$\theta \odot \mu - \epsilon \leq \nu \tag{3.6}$$

(resp., right bounded if and only if for all $\nu \in \Sigma_{\mathbf{R}}(0)$ and for all $\epsilon > 0$, there exists a $\theta \in \Sigma_{\mathbf{R}}(0)$ such that $\mu \odot \theta - \epsilon \leq \nu$);

- (b) a fuzzy set μ is called bounded if and only if it is both left and right bounded;
- (c) a fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is called bounded if and only if $1_{\mathbf{R}} = \mathbf{R}$ is bounded.

In the sequel mostly, we will consider only the case for left boundedness, other parts will follow similarly.

THEOREM 3.5. *If $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is a fuzzy neighborhood division ring, then a fuzzy set μ is left bounded in $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ if and only if for all $0 < \alpha < 1$, μ^α are left bounded in α -level topological rings $(\mathbf{R}, +, \cdot, \iota_\alpha(t(\Sigma_{\mathbf{R}})))$.*

PROOF. Let μ be left bounded in $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ and let $\alpha \in I_{0,1}$. Let $\nu^\beta \in \iota_\alpha(\Sigma_{\mathbf{R}}(0))$ with $\nu \in \Sigma_{\mathbf{R}}(0)$ and $\beta < \alpha$.

Choose $0 < \epsilon < \alpha - \beta$. Then there exists $\theta \in \Sigma_{\mathbf{R}}(0)$ such that

$$\theta \odot \mu - \epsilon \leq \nu. \tag{3.7}$$

If $x \in \theta^{\epsilon+\beta} \cdot \mu^\alpha$, then there exists s, t such that $st = x$, $\theta(s) > \epsilon + \beta$, and $\mu(t) > \alpha$. Thus we have

$$\begin{aligned} \sup_{st=x} \theta(s) \wedge \mu(t) &> (\epsilon + \beta) \wedge \alpha \implies (\theta \odot \mu)(x) > \epsilon + \beta \\ &\implies x \in (\theta \odot \mu)^{\epsilon+\beta} = (\theta \odot \mu - \epsilon)^\beta \subseteq \nu^\beta \\ &\implies \theta^{\epsilon+\beta} \cdot \mu^\alpha \subseteq \nu^\beta. \end{aligned} \tag{3.8}$$

Now $\theta \in \Sigma_{\mathbf{R}}(0)$, and $(\alpha \wedge 1_{\theta^\alpha})^\beta = \theta^\alpha \in \iota_\alpha(\Sigma_{\mathbf{R}}(0))$. But $\theta^\alpha \subseteq \theta^{\epsilon+\beta}$, and as $\iota_\alpha(\Sigma_{\mathbf{R}}(0))$ is a filter, $\theta^{\epsilon+\beta} \in \iota_\alpha(\Sigma_{\mathbf{R}}(0))$. Hence it is proved that μ^α is left bounded.

Conversely, let $\nu \in \Sigma_{\mathbf{R}}(0)$ and let $\epsilon \in I_{0,1}$. Then $\nu^\beta \in \iota_\epsilon(\Sigma_{\mathbf{R}}(0))$ for some $\beta < 1 - \epsilon$. Since μ^ϵ is left bounded, there exists $G \in \iota_\epsilon(\Sigma_{\mathbf{R}}(0))$ such that $G \cdot \mu^\epsilon \subseteq \nu^\beta$.

Choose $\theta \in \Sigma_{\mathbf{R}}(0)$ and $0 < \beta_1 \leq \beta < 1 - \epsilon$ such that $G = \theta^{\beta_1}$ and

$$\theta^{\beta_1} \cdot \mu^\epsilon \subseteq \nu^\beta. \tag{3.9}$$

Now we have

$$\begin{aligned} [\theta \odot \mu - \epsilon]^{\beta_1} &= [\theta \odot \mu]^{\beta_1+\epsilon} \subseteq \theta^{\beta_1} \cdot \mu^\epsilon \subseteq \nu^\beta \subseteq \nu^{\beta_1} \\ &\implies \theta \odot \mu - \epsilon \leq \nu, \end{aligned} \tag{3.10}$$

which proves that μ is left bounded in $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$. □

THEOREM 3.6. *Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring and let $\mu \in I^{\mathbf{R}}$ be left bounded fuzzy set. Then the following are equivalent:*

- (1) μ is N-topo nilpotent;
- (2) $\mu^{(k)}$ is N-topo nilpotent for any $k \in \mathbb{Z}^+$;
- (3) there exists $k_0 \in \mathbb{Z}^+$ such that $\mu^{(k_0)}$ is N-topo nilpotent.

PROOF. It is obvious that (1) \implies (2) \implies (3).

(3) \implies (1). Let $\mu^{(k_0)}$ be N-topo nilpotent for some $k_0 \in \mathbb{Z}^+$, and let $\nu \in \Sigma_{\mathbf{R}}(0)$ and $\epsilon > 0$.

By [5, Proposition 4.4(c)], $\mu^{(2)}, \mu^{(3)}, \dots, \mu^{(k_0-1)}$ are left bounded.

Choose $\nu_1 \in \Sigma_{\mathbf{R}}(0)$ such that

$$\nu \geq \nu_1 \odot \mu^{(i)} - \frac{\epsilon}{2} \tag{3.11}$$

for $i = 1, 2, \dots, k_0 - 1$.

Since $\mu^{(k_0)}$ is N-topo nilpotent, there exists $n_0 \in \mathbb{Z}^+$ such that

$$\nu_1 \geq (\mu^{(k_0)})^{(n)} - \frac{\epsilon}{2} \quad \forall n \geq n_0. \tag{3.12}$$

Let $m \geq k_0 \cdot n_0$ ($= N_0$, say). Then $m = k_0 \cdot q + r$, $q \geq n_0$, and $k_0 > r \geq 0$. Therefore, in view of [8, Theorem 2.3.3], we have

$$\begin{aligned} \mu^{(m)} &= \mu^{(k_0 \cdot q+r)} = (\mu^{(k_0)})^{(q)} \odot \mu^{(r)} \\ &\leq \left(\nu_1 + \frac{\epsilon}{2} \right) \odot \mu^{(r)} \leq \nu_1 \odot \mu^{(r)} + \frac{\epsilon}{2} \leq \nu + \epsilon, \end{aligned} \tag{3.13}$$

implying that $\nu \geq \mu^{(m)} - \epsilon$.

This completes the proof that μ is N-topo nilpotent. □

LEMMA 3.7. *Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring and let $\nu_1, \nu_2 \in I^{\mathbf{R}}$. Then*

$$\overline{\nu_1 \odot \nu_2} \geq \overline{\nu_1} \odot \overline{\nu_2}. \tag{3.14}$$

PROOF. This follows from [1, Proposition 2.14]. □

PROPOSITION 3.8. *If $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is a fuzzy neighborhood ring, then*

$$\overline{\nu^{(k)}} \geq (\overline{\nu})^{(k)} \quad \forall k = 1, 2, \dots \tag{3.15}$$

PROOF. Applying induction on k , we prove the inequality (3.15) as follows: if $k = 1$, the inequality is trivially true, and if $k = 2$, it follows immediately from Lemma 3.7.

We assume that $\overline{\nu^{(k-1)}} \geq (\overline{\nu})^{(k-1)}$.

Then we have

$$\overline{\nu^{(k)}} = \overline{\nu \odot \nu^{(k-1)}} \geq \overline{\nu} \odot \overline{\nu^{(k-1)}} \geq \overline{\nu} \odot (\overline{\nu})^{(k-1)} = \overline{\nu}^{(k)}, \tag{3.16}$$

for all $k = 1, 2, \dots$ □

PROPOSITION 3.9. *Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring and let $\mu \in I^{\mathbf{R}}$ be N-topo nilpotent. Then $\overline{\mu}^{t(\Sigma_{\mathbf{R}})}$ is N-topo nilpotent.*

PROOF. Let μ be N-topo nilpotent, $\nu \in \Sigma_{\mathbf{R}}(0)$, and $\epsilon > 0$. Choose $\mu_1 \in \Sigma_{\mathbf{R}}(0)$ such that $\mu_1 \odot \mu_1 - \epsilon/2 \leq \nu$.

Then it follows from [1, Theorem 2.20] in conjunction with [13, Proposition 2.3] that

$$\overline{\mu_1}^{t(\Sigma_{\mathbf{R}})} = \inf_{\xi \in \Sigma_{\mathbf{R}}(0)} \mu_1 \odot \xi \leq (\mu_1 \odot \mu_1) \leq \nu + \frac{\epsilon}{2} \implies \overline{\mu_1}^{t(\Sigma_{\mathbf{R}})} \leq \nu + \frac{\epsilon}{2}. \tag{3.17}$$

By N-topo nilpotency of μ we can find $n_0 \in \mathbb{Z}^+$ such that for all $n \geq n_0$, $\mu^{(n)} \leq \mu_1 + \epsilon/2$, hence using Lemma 3.7 and Proposition 3.8, we arrive at the following:

$$\begin{aligned} [\overline{\mu}^{t(\Sigma)}]^{(n)} &\leq [\mu^{(n)}]^{t(\Sigma)} \leq \left[\mu_1 + \frac{\epsilon}{2} \right]^{t(\Sigma)} \leq \overline{\mu_1}^{t(\Sigma)} + \frac{\epsilon}{2} \leq \nu + \epsilon \\ &\implies [\overline{\mu}^{t(\Sigma)}]^{(n)} - \epsilon \leq \nu, \end{aligned} \tag{3.18}$$

proving that $\overline{\mu}^{t(\Sigma_{\mathbf{R}})}$ is N-topo nilpotent. □

PROPOSITION 3.10. *Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ and $(\mathbf{R}', +, \cdot, t(\Sigma_{\mathbf{R}'}))$ be fuzzy neighborhood rings and let $f : \mathbf{R} \rightarrow \mathbf{R}'$ be a continuous ring-homomorphism. If μ is an N -topo nilpotent fuzzy set of \mathbf{R} , then $f(\mu)$ is N -topo nilpotent fuzzy set of \mathbf{R}' .*

PROOF. Let $v' \in \Sigma_{\mathbf{R}'}(0)$ and $\epsilon > 0$. Then $f^{-1}(v') \in \Sigma_{\mathbf{R}}(0)$. Consequently, there exists $n_0 \in \mathbb{Z}^+$ such that for all $n \geq n_0$, $f^{-1}(v') \geq \mu^{(n)} - \epsilon$.

As $[f(\mu)]^{(n)} = f[\mu^{(n)}] \leq f[f^{-1}(v')] + \epsilon$, $[f(\mu)]^{(n)} - \epsilon \leq v'$.

The preceding equality is actually obtained by applying induction in conjunction with [1, Lemma 2.15] in the following way.

For $n = 1$, it is trivially true.

Now if $n = 2$, then we have

$$[f(\mu)]^{(2)} = f[\mu] \odot f[\mu] = f[\mu \odot \mu] = f[\mu^{(2)}]. \tag{3.19}$$

Assume $[f(\mu)]^{(k-1)} = f[\mu^{(k-1)}]$. Then

$$\begin{aligned} [f(\mu)]^{(k)} &= [f(\mu)] \odot [f(\mu)]^{(k-1)} = [f(\mu)] \odot f[\mu^{(k-1)}] \\ &= f[\mu \odot \mu^{(k-1)}] = f[\mu^{(k)}]. \end{aligned} \tag{3.20} \quad \square$$

DEFINITION 3.11 [2, 4]. If $(\mathbf{R}, +, \cdot)$ is a division ring and $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ a fuzzy neighborhood ring, then a fuzzy set $\mu \in I^{\mathbf{R}}$ is called inversely left bounded (resp., inversely right bounded) if and only if $\mu(0) = 1$ and $(1 - \mu)^{-1}$ is left bounded (resp., right bounded).

DEFINITION 3.12 [2, 4]. A fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ with unity is called locally left bounded (resp., locally right bounded) if and only if for any $v \in \Sigma_{\mathbf{R}}(0)$, v is left bounded (resp., right bounded).

DEFINITION 3.13 [4]. A fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is called locally inversely left bounded (resp., locally inversely right bounded) if and only if for any $v \in \Sigma_{\mathbf{R}}(0)$, v is inversely left bounded (resp., inversely right bounded).

It is called locally inversely bounded if and only if it is both locally inversely left bounded and locally inversely right bounded.

THEOREM 3.14. *A fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is locally inversely left bounded (resp., right bounded, bounded) if and only if for all $0 < \alpha < 1$, α -level topological rings $(\mathbf{R}, +, \cdot, t_{\alpha}(t(\Sigma_{\mathbf{R}})))$ are locally inversely left bounded (right bounded, bounded).*

PROOF. Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be locally inversely left bounded fuzzy neighborhood ring and $0 < \alpha < 1$. Then for any $v \in \Sigma_{\mathbf{R}}(0)$, v is inversely left bounded; meaning $v(0) = 1$ and $(1 - v)^{-1}$ is left bounded.

Choose $\beta \in I_{0,1}$ such that $\beta > \alpha$. Then $v^{1-\beta} \in t_{\alpha}(\Sigma_{\mathbf{R}}(0))$ and

$$\begin{aligned} (\mathbf{R} - v^{1-\beta}) &= (1 - v)_{\beta} \subseteq \cup_{\alpha < \beta} (1 - v)_{\beta} = (1 - v)^{\alpha} \\ &\Rightarrow (\mathbf{R} - v^{1-\beta}) \subseteq (1 - v)^{\alpha} \\ &\Rightarrow [\mathbf{R} - v^{1-\beta}]^{-1} \subseteq [(1 - v)^{-1}]^{\alpha}. \end{aligned} \tag{3.21}$$

As $(1 - \nu)^{-1}$ is left bounded, $[(1 - \nu)^{-1}]^\alpha$ is left bounded by [Theorem 3.5](#). Then $[\mathbf{R} - \nu^{1-\beta}]^{-1}$ is left bounded being a subset of a left bounded set. Therefore, $0 \in \nu^{1-\beta}$, and $\nu^{1-\beta}$ is inversely left bounded.

Next suppose that for all $0 < \alpha < 1$, the α -level topological rings $(\mathbf{R}, +, \cdot, \iota_\alpha(t(\Sigma_{\mathbf{R}})))$ are locally inversely left bounded. Then for any $U \in \iota_\alpha(\Sigma_{\mathbf{R}}(0))$, $0 \in U$, U is inversely left bounded, that is, $(\mathbf{R} - U)^{-1}$ is left bounded. Consequently, there is a $\sigma \in \Sigma_{\mathbf{R}}(0)$ and $\beta < 1 - \alpha$ such that $U = \sigma^\beta$.

But then one obtains

$$\mathbf{R} - \sigma^\beta = (1 - \sigma)_{(1-\beta)} \supseteq (1 - \sigma)^{(1-\beta)}. \tag{3.22}$$

That is, $[(1 - \sigma)^{(1-\beta)}]^{-1} \subseteq (\mathbf{R} - \sigma^\beta)^{-1}$ implies $[(1 - \sigma)^{-1}]^{(1-\beta)} \subseteq (\mathbf{R} - \sigma^\beta)^{-1}$.

Since $(\mathbf{R} - \sigma^\beta)^{-1}$ is left bounded in $(\mathbf{R}, +, \cdot, \iota_\alpha(t(\Sigma_{\mathbf{R}})))$, $[(1 - \sigma)^{-1}]^{(1-\beta)}$ is also left bounded there. Using [Theorem 3.5](#) again, we conclude that $(1 - \sigma)^{-1}$ is left bounded in $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$, and as obviously $\sigma(0) = 1$, it is proved that $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is locally inversely left bounded. \square

Now we are in a position to present a characterization theorem which says that a WNT_2 fuzzy neighborhood ring on a division ring having a property of locally inversely left boundedness (resp., locally inversely right boundedness) guarantees that the fuzzy neighborhood ring is in fact a fuzzy neighborhood division ring.

THEOREM 3.15. *Let $(\mathbf{R}, +, \cdot)$ be a division ring and $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ a WNT_2 locally inversely left bounded fuzzy neighborhood ring. Then $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is a fuzzy neighborhood division ring.*

PROOF. Since $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is WNT_2 , in view of [Proposition 2.6](#) and [Definition 2.10](#), we can find $\theta \in \Sigma(0)$ and $\alpha \in I_1$ such that

$$\begin{aligned} (1 \oplus \theta) \wedge \theta \leq \alpha &\implies (1 \oplus \theta)^\alpha \wedge \theta^\alpha = \mathbf{0} \\ &\implies (1 \oplus \theta)^\alpha \subseteq (\mathbf{R} - \theta^\alpha) \\ &\implies [(1 \oplus \theta)^\alpha]^{-1} \subseteq (\mathbf{R} - \theta^\alpha)^{-1}. \end{aligned} \tag{3.23}$$

Now in view of [Theorem 3.5](#), $(\mathbf{R} - \theta^\alpha)^{-1}$ is left bounded in α -level topological rings $(\mathbf{R}, +, \cdot, \iota_\alpha(t(\Sigma_{\mathbf{R}})))$, and hence $[(1 \oplus \theta)^{-1}]^\alpha$ is left bounded there too. Then again by [Theorem 3.5](#), $(1 \oplus \theta)^{-1}$ is left bounded in fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$.

It remains to be shown that the fuzzy neighborhood ring is a fuzzy neighborhood division ring. In order to prove this, we take $\nu \in \Sigma_{\mathbf{R}}(0)$ and $\epsilon > 0$. Choose $\sigma \in \Sigma_{\mathbf{R}}(0)$ symmetric such that

$$\sigma \odot (1 \oplus \theta)^{-1} \leq \nu + \epsilon, \tag{3.24}$$

and $\sigma \leq \theta$.

Then it follows immediately that $\sigma \odot (1 \oplus \sigma)^{-1} \leq \sigma \odot (1 \oplus \theta)^{-1}$.

For nonzero $z \in \mathbf{R}$, we have

$$\begin{aligned}
 (1 \oplus \sigma)^{-1}(z) &= \sup_{(1+x)^{-1}=z} \sigma(x) = \sup_{1-x/(1+x)=z} \sigma(x) = \sup_{1-x(1+x)^{-1}=z} \sigma(x) \\
 &\leq \sup_{-x(1+x)^{-1}=z-1} \sigma(-x) \wedge \sigma(x) \\
 &= \sup_{-x(1+x)^{-1}=z-1} \sigma(-x) \wedge (1 \oplus \sigma)^{-1}(1+x)^{-1} = [\sigma \odot (1 \oplus \sigma)^{-1}](z-1) \\
 &\leq [\sigma \odot (1 \oplus \theta)^{-1}](z-1) \leq \nu(z-1) + \epsilon \\
 &= (1 \oplus \nu)(z) + \epsilon.
 \end{aligned}
 \tag{3.25}$$

This implies that $(1 \oplus \sigma)^{-1} - \epsilon \leq (1 \oplus \nu)$, proving that the map $x \mapsto x^{-1}$ ($x \neq 0$) is continuous at $x = 1$. Hence the result follows from [Theorem 2.9](#). \square

LEMMA 3.16. *Let $(\mathbf{R}, +, \cdot)$ be a division ring and $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ a fuzzy neighborhood ring with unity. If $t(\Sigma_{\mathbf{R}})$ is nontrivial fuzzy neighborhood topology on \mathbf{R} , then $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is WNT_2 .*

PROOF. Since the fuzzy neighborhood system $\Sigma_{\mathbf{R}}$ is nontrivial, we can find a $\nu \in \Sigma_{\mathbf{R}}$ such that $\nu \neq 1_{\mathbf{R}}$. This means that there exists $\gamma \in \mathbf{R}$ such that $\nu(\gamma) < 1$. If $\nu \in \Sigma_{\mathbf{R}}(z)$ for any $z \in \mathbf{R}^*$, then $(-z \oplus \nu) \in \Sigma_{\mathbf{R}}(0)$ and $(-z \oplus \nu)(\gamma - z) = \nu(\gamma) < 1$. Now $(\gamma - z)^{-1} \odot (-z \oplus \nu)(1) = \nu(\gamma) < 1$. Let $\theta := (\gamma - z)^{-1} \odot (-z \oplus \nu)$, then in view of [Corollary 2.7](#), $\theta \in \Sigma_{\mathbf{R}}(0)$ and $\theta(1) < 1$. Let $s \in \mathbf{R}^*$, then $s \odot \theta \in \Sigma_{\mathbf{R}}(0)$ and $s \odot \theta(s) = \theta(1) < 1$, implying that $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is WNT_1 , and hence it is WNT_2 , this is so because the fuzzy neighborhood ring is fuzzy uniformizable (see, e.g., [\[9\]](#)). \square

Next we present another characterization theorem which says that a fuzzy neighborhood ring on a division ring becomes a fuzzy neighborhood division ring if the fuzzy neighborhood ring contains an N -topo nilpotent fuzzy neighborhood of zero.

THEOREM 3.17. *Let $(\mathbf{R}, +, \cdot)$ be a division ring and let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring. If the fuzzy neighborhood ring possesses an N -topo nilpotent fuzzy neighborhood of zero, then $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is a fuzzy neighborhood division ring.*

PROOF. The statement is obvious if the fuzzy neighborhood system is either discrete or trivial. We assume the fuzzy neighborhood system is proper, that is, nontrivial. Then due to [Lemma 3.16](#), it is WNT_2 .

Now we let $\theta \in \Sigma_{\mathbf{R}}(0)$ and let θ be an N -topo nilpotent fuzzy set in a fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$.

Consider an arbitrary basis $\mathcal{B}_{\mathbf{R}}(0)$ of symmetric fuzzy neighborhoods of zero for the system $\Sigma_{\mathbf{R}}(0)$.

Let $\nu \in \Sigma_{\mathbf{R}}(0)$ and $\epsilon \in I_{0,1}$. Choose $\nu_1 \in \Sigma_{\mathbf{R}}(0)$ such that

$$\nu \wedge \theta \geq \nu_1 \oplus \nu_1 \oplus \nu_1 - \epsilon
 \tag{3.26}$$

and $\nu_1(-1) < 1 - \epsilon$.

Since θ is N-topo nilpotent, there exists $n_0 \in \mathbb{Z}^+$ such that

$$v_1 \geq \theta^{(n)} - \epsilon \quad \forall n \geq n_0. \tag{3.27}$$

Now, since $\theta \in \Sigma_{\mathbf{R}}(0)$, $a^{n_0-1} \odot \theta \in \Sigma_{\mathbf{R}}(0)$ for some $a \neq 0 \in \mathbf{R}$. But then it follows easily that $\theta^{(n_0)} \geq a^{n_0-1} \odot \theta$, which yields that $\theta^{(n_0)} \in \Sigma_{\mathbf{R}}(0)$.

Consequently, there exists a $\sigma_\epsilon \in \mathcal{B}_{\mathbf{R}}(0)$ such that $\theta^{(n_0)} \geq \sigma_\epsilon - \epsilon$. Since $v_1 + \epsilon \geq \theta^{(n_0)} \geq \sigma_\epsilon - \epsilon$ implies $v_1 \geq \sigma_\epsilon - 2\epsilon \geq \sigma_\epsilon - 3\epsilon$. If we let $\sigma := \sup_{\epsilon \in I_0} (\sigma_\epsilon - 3\epsilon)$, then $\sigma \in \Sigma_{\mathbf{R}}(0)$, and as $\sigma < v_1$, we have $\sigma(-1) < 1$.

Now let $x \in \mathbf{R}$ be such that $\sigma(x) \geq 1 - \epsilon$.

Applying induction on k , we verify that

$$(v_1 \oplus v_1) \left(\sum_{i=1}^k x^i \right) \geq 1 - \epsilon \quad \text{for } k = 1, 2, \dots \tag{3.28}$$

If $k = 1$, then $(v_1 \oplus v_1)(x) \geq v_1(x) \geq \sigma(x) \geq 1 - \epsilon$, that is, $(v_1 \oplus v_1)(x) \geq 1 - \epsilon$. Suppose $(v_1 \oplus v_1)(\sum_{i=1}^{k-1} x^i) \geq 1 - \epsilon$, then with $z = \sum_{i=1}^k x^i$, we have

$$\begin{aligned} (v_1 \oplus v_1)(z) &= \sup_{a+b=z} v_1(a) \wedge v_1(b) \\ &\geq \sup_{a+b=z} v_1(a) \wedge \theta^{(n_0+1)}(b) - \epsilon \\ &\geq \sup_{a+b=z} v_1(a) \wedge [\theta^{(n_0)} \odot \theta](b) - \epsilon \\ &\geq \sup_{a+b=z} v_1(a) \wedge [\sigma \odot \theta](b) - 2\epsilon \\ &\geq \sup_{a+b=z} v_1(a) \wedge \theta(b) - 2\epsilon \\ &\geq \sup_{a+b=z} v_1(a) \wedge (v_1 \oplus v_1)(b) - 2\epsilon \\ &= v_1(x) \wedge (v_1 \oplus v_1) \left(\sum_{i=1}^{k-1} x^i \right) - 2\epsilon \\ &\geq \sigma(x) \wedge (v_1 \oplus v_1) \left(\sum_{i=1}^{k-1} x^i \right) - 2\epsilon \geq 1 - 3\epsilon, \end{aligned} \tag{3.29}$$

which implies that $(v_1 \oplus v_1)(z) \geq 1 - 3\epsilon$.

Since ϵ is arbitrary, we are done.

Now choose $v_2 \in \Sigma_{\mathbf{R}}(0)$ such that

$$v_1 \geq (1 - x)^{-1} \odot v_2 - \epsilon. \tag{3.30}$$

Since $\theta + \epsilon \geq v_1 \geq \theta^{(n_0)} - \epsilon \geq \sigma_\epsilon - 2\epsilon$ implies $\theta \geq \sigma_\epsilon - 3\epsilon$, then again, with $\sigma := \sup_{\epsilon \in I_0} (\sigma_\epsilon - 3\epsilon)$, $\sigma \in \Sigma_{\mathbf{R}}(0)$ and $\theta \geq \sigma$. As θ is N-topo nilpotent, so is σ . Now if we

let $\sigma := x$, there exists $m_0 \in \mathbb{Z}^+$ such that for all $n \geq m_0$,

$$v_2(x^n) \geq 1 - \epsilon. \tag{3.31}$$

If $z \in \mathbf{R}^*$, then

$$\begin{aligned} (1 \oplus \sigma)(z) &= (1 \ominus \sigma)^{-1}(z) \\ &= \sup_{(1-x)^{-1}=z, x \neq 1} \sigma(x) \\ &\leq \sup_{(1-x)^{-1}=z} v_1(x) + \epsilon \\ &\leq \sup_{(1-x)^{-1}=z} [v_1(x) \wedge 1] + \epsilon \\ &\leq \sup_{(1-x)^{-1}=z} v_1(x) \wedge v_2(x^{m_0+1}) + 2\epsilon \\ &\leq \sup_{(1-x)^{-1}=z} v_1(x) \wedge [(1-x)^{-1} \odot v_2((1-x)^{-1}x^{m_0+1})] + 2\epsilon \\ &= \sup_{(1-x)^{-1}=z} v_1(x) \wedge (1-x)^{-1} \odot v_2 \left[(1-x)^{-1} - \left(1 + \sum_{i=1}^{m_0} x^i \right) \right] + 2\epsilon \\ &\leq \sup_{(1-x)^{-1}=z} v_1(x) \wedge v_1 \left[(1-x)^{-1} - \left(1 + \sum_{i=1}^{m_0} x^i \right) \right] + 3\epsilon \\ &\leq \sup_{(1-x)^{-1}=z} v_1 \left[(1-x)^{-1} - \left(1 + \sum_{i=1}^{m_0} x^i \right) \right] + 3\epsilon \\ &\leq \sup_{(1-x)^{-1}=z} 1(1) \wedge (v_1 \oplus v_1) \left(\sum_{i=1}^{m_0} x^i \right) \wedge v_1 \left[(1-x)^{-1} - \left(1 + \sum_{i=1}^{m_0} x^i \right) \right] + 6\epsilon \\ &\leq \sup_{1+a+b=z} 1(1) \wedge (v_1 \oplus v_1)(a) \wedge v_1(b) + 6\epsilon \\ &= \sup_{a+b=z-1} (v_1 \oplus v_1)(a) \wedge v_1(b) + 7\epsilon \\ &\leq (v_1 \oplus v_1 \oplus v_1)(z-1) + 7\epsilon \\ &= 1 \oplus (v_1 \oplus v_1 \oplus v_1)(z) + 7\epsilon \\ &\leq (1 \oplus v)(z) + 8\epsilon, \end{aligned} \tag{3.32}$$

that is, $(1 \oplus v) \geq (1 \oplus \sigma)^{-1} - 8\epsilon$.

Since ϵ is arbitrary, the result follows from [Theorem 2.9](#). This completes the proof of the theorem. \square

PROPOSITION 3.18. *If a fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is locally left bounded, then $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ has a fuzzy neighborhood basis $\mathcal{B}_{\mathbf{R}}(0)$ containing a fuzzy*

neighborhood μ of 0 such that μ is a fuzzy subsemigroup of the multiplicative semigroup of the ring \mathbf{R} .

PROOF. Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a locally left bounded fuzzy neighborhood ring, $\nu \in \Sigma_{\mathbf{R}}(0)$ left bounded, and $\epsilon > 0$. Choose $\nu_1 \in \Sigma_{\mathbf{R}}(0)$ such that $\nu_1 \leq \nu$ and $\nu_1 \odot \nu \leq \nu + \epsilon$.

By induction on n , $\nu_1^{(n)} \leq \nu$, $n = 1, 2, \dots$. Indeed, due to the choice of ν and ν_1 , we get $\nu_1^{(1)} = \nu_1 \leq \nu$ which implies $\nu_1^{(1)} \leq \nu$. If $\nu_1^{(n-1)} \leq \nu$, then $\nu_1^{(n)} = \nu_1 \odot \nu_1^{(n-1)} \leq \nu_1 \odot \nu \leq \nu + \epsilon$. Since ϵ is arbitrary, $\nu_1^{(n)} \leq \nu$.

Let μ be a fuzzy subsemigroup of the multiplicative semigroup of the ring \mathbf{R} generated by ν_1 . Due to [8, Theorem 5.1.1], we get

$$\mu = \sup_{n=1}^{\infty} \nu_1^{(n)}. \tag{3.33}$$

Then it follows immediately that $\nu_1 \leq \mu \leq \nu$.

As ν is left bounded, so is μ , and $\mu \in \Sigma_{\mathbf{R}}(0)$ since $\Sigma_{\mathbf{R}}(0)$ is a prefilter. □

COROLLARY 3.19. *A locally left bounded (resp., locally right bounded) fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ has a basis of fuzzy neighborhoods of zero consisting of a fuzzy subsemigroup of the multiplicative semigroup of the ring \mathbf{R} .*

PROOF. This follows from Proposition 3.18. □

THEOREM 3.20. *Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a locally left bounded fuzzy neighborhood ring with unity. If there exists an invertible and N-topo nilpotent element $t \in \mathbf{R}$, then $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ contains an N-topo nilpotent neighborhood of zero.*

PROOF. From Corollary 3.19, it follows that there exists a $\nu \in \Sigma_{\mathbf{R}}(0)$ such that ν is left bounded and $\nu \odot \nu \leq \nu$.

Now as $\nu \in \Sigma_{\mathbf{R}}(0)$, in view of Corollary 2.7, $t \odot \nu \in \Sigma_{\mathbf{R}}(0)$, then due to left boundedness of ν , for any $\epsilon > 0$, there exists $\sigma \in \Sigma_{\mathbf{R}}(0)$ such that

$$\sigma \odot \nu - \frac{\epsilon}{2} \leq \nu \odot t. \tag{3.34}$$

Since t is N-topo nilpotent, there exists $k \in \mathbb{Z}^+$ such that, for all $n \geq k$,

$$\sigma(t^k) \geq 1 - \frac{\epsilon}{2}, \tag{3.35}$$

and since t^k is invertible, $t^k \odot \nu \in \Sigma_{\mathbf{R}}(0)$.

Applying induction on n , one obtains the following:

$$(t^k \odot \nu)^{(n)} - \epsilon \leq \nu \odot t \tag{3.36}$$

for any $n = 1, 2, \dots$

In fact, if $n = 1$, then for any $z \in \mathbf{R}$,

$$\begin{aligned}
 (t^k \circ v)(z) - \epsilon &\leq 1 \wedge (t^k \circ v)(z) - \epsilon \\
 &\leq \left[\sigma(t^k) + \frac{\epsilon}{2} \right] \wedge (t^k \circ v)(z) - \epsilon \\
 &\leq \sigma(t^k) \wedge (t^k \circ v)(z) - \frac{\epsilon}{2} \\
 &\leq \sup_{ab=t^k(t^k)^{-1}z} \sigma(a) \wedge v(b) - \frac{\epsilon}{2} \\
 &= (\sigma \circ v)(z) - \frac{\epsilon}{2} \\
 &\leq (v \circ t)(z).
 \end{aligned}
 \tag{3.37}$$

That is, $(t^k \circ v) - \epsilon \leq v \circ t$.

Suppose that

$$(t^k \circ v)^{(n-1)} - \epsilon \leq v \circ t^{n-1}.
 \tag{3.38}$$

Then

$$\begin{aligned}
 (t^k \circ v)^{(n)} &= (t^k \circ v) \circ (v \circ t^k)^{(n-1)} \\
 &\leq (t^k \circ v) \circ v \circ t^{n-1} \\
 &= t^k \circ (v \circ v) \circ t^{n-1} \\
 &\leq (t^k \circ v) \circ t^{n-1} \\
 &\leq [(v \circ t) + \epsilon] \circ t^{n-1} \\
 &\leq (v \circ t) \circ t^{n-1} + \epsilon \\
 &= v \circ t^n + \epsilon.
 \end{aligned}
 \tag{3.39}$$

This implies that $(t^k \circ v)^{(n)} - \epsilon \leq v \circ t^n$.

We verify that $t^k \circ v$ is N-topo nilpotent. Let $\theta \in \Sigma_{\mathbf{R}}(0)$ and $\theta_1 \in \Sigma_{\mathbf{R}}(0)$ be such that

$$v \circ \theta_1 - \epsilon \leq \theta.
 \tag{3.40}$$

Let $n_0 \in \mathbb{Z}^+$ be such that $\theta_1(t^n) \geq 1 - \epsilon$ for all $n \geq n_0$.

Then for any $x \in \mathbf{R}$,

$$\begin{aligned}
 (t^k \circ v)^{(n)} &\leq [v \circ t^n](x) + 2\epsilon \\
 &\leq [v \circ t^n](x) \wedge \theta_1(t^n) + 2\epsilon \\
 &\leq \sup_{ab=x(t^n)^{-1}t^n} v(a) \wedge \theta_1(b) + 2\epsilon \\
 &= v \circ \theta_1(x) + 2\epsilon \\
 &\leq \theta(x) + 3\epsilon.
 \end{aligned}
 \tag{3.41}$$

This implies that $(t^k \circ v)^{(n)} - 3\epsilon \leq \theta$, since ϵ is arbitrary, it is proved that $(t^k \circ v)$ is N-topo nilpotent. □

PROPOSITION 3.21. *Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a fuzzy neighborhood ring with unity and let $\nu \in \Sigma_{\mathbf{R}}(0)$ be left bounded (resp., right bounded). If $t \in \mathbf{R}$ is invertible and N-topo nilpotent, then*

$$\mathcal{B}_{\mathbf{R}}(0) = \{t^n \circ \nu \mid \nu \in \Sigma_{\mathbf{R}}(0), n = 1, 2, \dots\} \tag{3.42}$$

(resp., $\mathcal{B}_{\mathbf{R}}(0) = \{\nu \circ t^n \mid \nu \in \Sigma_{\mathbf{R}}(0), n = 1, 2, \dots\}$) is a basis for $\Sigma_{\mathbf{R}}(0)$.

PROOF. Let $\nu \in \Sigma_{\mathbf{R}}(0)$ be left bounded and $\epsilon > 0$. Since t^n is invertible, in view of [Corollary 2.7](#), $t^n \circ \nu \in \Sigma_{\mathbf{R}}(0)$ for any $n = 1, 2, \dots$

If $\theta \in \Sigma_{\mathbf{R}}(0)$, then there exist $\sigma \in \Sigma_{\mathbf{R}}(0)$ and $n_0 \in \mathbb{Z}^+$ such that

$$\sigma \circ \nu - \frac{\epsilon}{2} \leq \theta \tag{3.43}$$

and $\sigma(t^n) \geq 1 - \epsilon/2$ for any $n \geq n_0$.

Now for any $x \in \mathbf{R}$,

$$\begin{aligned} [t^n \circ \nu](x) &= \nu[(t^n)^{-1}x] \leq 1 \wedge \nu[(t^n)^{-1}x] \\ &\leq \sigma(t^n) \wedge \nu[(t^n)^{-1}x] + \frac{\epsilon}{2} \\ &\leq \sup_{ab=t^n(t^n)^{-1}x=x} \sigma(a) \wedge \nu(b) + \frac{\epsilon}{2} \\ &= \sigma \circ \nu(x) + \frac{\epsilon}{2} \\ &\leq \theta(x) + \epsilon \\ &\Rightarrow t^n \circ \nu - \epsilon \\ &\leq \theta. \end{aligned} \tag{3.44}$$

□

DEFINITION 3.22 [7]. An element $x \in \mathbf{R}$ in a fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is called N-topo left quasi-regular (resp., right quasi-regular) if and only if for all $\nu \in \Sigma_{\mathbf{R}}(0)$ and for all $\epsilon > 0$, there exists $y \in \mathbf{R}$ such that

$$\nu(x \circ y) \geq 1 - \epsilon \tag{3.45}$$

(resp., $\nu(y \circ x) \geq 1 - \epsilon$).

An element $x \in \mathbf{R}$ is called N-topo quasi-regular if and only if it is both left and right quasi-regular in $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$. Moreover, the ring \mathbf{R} is called N-topo quasi-regular if and only if all its elements are N-topo quasi-regular.

PROPOSITION 3.23. *Any N-topo nilpotent element in a fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is N-topo quasi-regular.*

PROOF. Let $x \in \mathbf{R}$ be a nilpotent element in a fuzzy neighborhood ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$, $\nu \in \Sigma_{\mathbf{R}}(0)$, and $\epsilon > 0$.

Then we can find an $n_0 \in \mathbb{Z}^+$ such that for all $n \geq n_0$, $\nu(x^n) \geq 1 - \epsilon$.

If we let $y = -\sum_{i=1}^{n-1} x^i$, then we have

$$x \circ y = x + \left(-\sum_{i=1}^{n-1} x^i\right) - x \left(-\sum_{i=1}^{n-1} x^i\right) = x^n, \tag{3.46}$$

which implies that $v(x \circ y) = v(x^n) \geq 1 - \epsilon$.

Similarly, one can show that $v(y \circ x) \geq 1 - \epsilon$, proving that x is N-topo quasi-regular. □

THEOREM 3.24. *Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be a bounded N-topo fuzzy neighborhood ring and Y a dense subring. Then the ring $(Y, +, \cdot, t(\Sigma_Y))$ is N-topo left quasi-regular (resp., right quasi-regular) if and only if the ring $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is N-topo left quasi-regular (resp., right quasi-regular).*

PROOF. Let $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ be bounded N-topo left quasi-regular neighborhood ring, $y \in Y$, $v_0 \in \Sigma_Y(0)$, and $\epsilon > 0$.

Then there exists $\theta_0 \in \Sigma_{\mathbf{R}}(0)$ such that

$$v_0 = \theta_0 \wedge 1_Y. \tag{3.47}$$

Consequently, we can find a symmetric $v_1 \in \Sigma_{\mathbf{R}}(0)$ such that

$$v_1 \oplus v_1 \oplus y \circ v_1 - \frac{\epsilon}{3} \leq \theta_0. \tag{3.48}$$

Since $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is N-topo left quasi-regular, there exists $z \in \mathbf{R}$ such that

$$v_1(y \circ z) \geq 1 - \frac{\epsilon}{3}. \tag{3.49}$$

As $(\mathbf{R}, +, \cdot, t(\Sigma_{\mathbf{R}}))$ is fuzzy uniformizable and Y is dense, it follows from [14, Lemma 2.2] that there is a $t \in Y$ such that $(z \oplus v_1)(t) \geq 1 - \epsilon/3$.

Now we have

$$\begin{aligned} 1 - \frac{\epsilon}{3} &\leq [y \oplus (z \oplus v_1) \oplus yz \oplus (y \circ v_1)](y \circ t) \\ &= [v_1 \oplus (y \circ v_1)][y \circ t - (y + z - yz)] \\ &\leq 1 \wedge [v_1 \oplus (y \circ v_1)](y \circ t - y \circ z) \\ &\leq \left[v_1(y \circ z) + \frac{\epsilon}{3}\right] \wedge [v_1 \oplus (y \circ v_1)](y \circ t - y \circ z) \\ &\leq v_1(y \circ z) \wedge [v_1 \oplus (y \circ v_1)](y \circ t - y \circ z) + \frac{\epsilon}{3} \\ &\leq \sup_{a+b=y \circ z + y \circ t - y \circ z = y \circ t} v_1(a) \wedge [v_1 \oplus (y \circ v_1)](b) + \frac{\epsilon}{3} \\ &= [v_1 \oplus v_1 \oplus (y \circ v_1)](y \circ t) + \frac{\epsilon}{3} \leq \theta_0(y \circ t) + \frac{2\epsilon}{3} \\ &\leq \theta_0(y \circ t) \wedge 1_Y(y \circ t) + \frac{2\epsilon}{3} \\ &= v_0(y \circ t) + \frac{2\epsilon}{3}. \end{aligned} \tag{3.50}$$

This implies that $v_0(y \circ t) \geq 1 - \epsilon$, which proves that $(Y, +, \cdot, t(\Sigma_Y))$ is N-topo left quasi-regular fuzzy neighborhood ring.

To prove the converse part we proceed as follows.

Let $x \in \mathbf{R}$, $\epsilon > 0$, and $v \in \Sigma(0)$. We need to produce $z \in \mathbf{R}$ such that

$$v(x \circ z) \geq 1 - \epsilon. \tag{3.51}$$

Choose $v_1 \in \Sigma_{\mathbf{R}}(0)$ symmetric such that

$$v_1 \oplus v_1 \oplus v_1 \circ \mathbf{R} - \frac{\epsilon}{3} \leq v. \tag{3.52}$$

Since Y is dense in \mathbf{R} , applying [14, Lemma 2.2], we get for any $x \in \mathbf{R}$, $\overline{1_Y}(x) = 1$ if and only if there exists $y \in Y \ni (y \oplus v_1)(x) \geq 1 - \epsilon/3$, this implies $(x \oplus v_1)(y) \geq 1 - \epsilon/3$.

N-topo left quasi-regularity of $(Y, +, \cdot, t(\Sigma_Y))$ ensures the existence of a $z \in Y$ such that

$$v_1(y \circ z) \geq 1 - \frac{\epsilon}{3}. \tag{3.53}$$

Now we have

$$\begin{aligned} 1 - \frac{\epsilon}{3} &\leq [(y \oplus v_1) \oplus z \oplus (y \oplus v_1) \circ z](x \circ z) \\ &= [v_1 \oplus v_1 \circ z](x \circ z - y \circ z) \\ &\leq 1 \wedge [v_1 \oplus v_1 \circ z](x \circ z - y \circ z) \\ &\leq \left[v_1(y \circ z) + \frac{\epsilon}{3} \right] \wedge [v_1 \oplus v \circ z](x \circ z - y \circ z) \\ &\leq [v_1(y \circ z) \wedge (v_1 \oplus v_1 \circ z)](x \circ z - y \circ z) + \frac{\epsilon}{3} \\ &\leq \sup_{a+b=y \circ z+x \circ z-y \circ z} v_1(a) \wedge [v_1 \oplus v_1 \circ z](b) + \frac{\epsilon}{3} \\ &= [v_1 \oplus v_1 \oplus v_1 \circ z](x \circ z) + \frac{\epsilon}{3} \\ &\leq v(x \circ z) + \frac{2\epsilon}{3}, \end{aligned} \tag{3.54}$$

which implies that $v(x \circ z) \geq 1 - \epsilon$.

This ends the proof. □

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