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QUATERNIONIC REPRESENTATION OF THE MOVING FRAME FOR SURFACES IN EUCLIDEAN THREE-SPACE AND LAX PAIR

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The moving frame and associated Gauss-Codazzi equations for surfaces in three-space are introduced. A quaternionic representation is used to identify the Gauss-Weingarten equation with a particular Lax representation. Several examples are given, such as the case of constant mean curvature.

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The study of surfaces in three- and higher-dimensional spaces has seen a resurgence of interest recently due to various applications of these surfaces to various areas of mathematical physics, especially to the area of integrable systems [1, 6, 8]. The particular class of surfaces known as minimal surfaces with constant mean curvature has many applications to various physical problems. It is the intention here to review and establish the Gauss-Codazzi equations for surfaces in Euclidean three-space. Next, a quaternionic representation is introduced for the moving frame of the conformally parametrized surface. It will be shown how the frame equations can be written using quaternions by means of an SU(2) matrix. The main new element here is a straightforward derivation of a Lax pair based on the use of quaternions, and an application of this result to the generalized Weierstrass representation [2]. Some specific examples of solutions for the resulting equations are given, and a particular application to the case of constant mean curvature surfaces under Konopelchenko's generalization of the Weierstrass representation is presented [7].

We begin by establishing some general notions with regard to orientable surfaces in three-dimensional Euclidean space. Under such a parametrization, which is called conformal, the surface S can be given by a vector-valued function

$$F = (F_1, F_2, F_3) : \mathcal{R} \longrightarrow \mathbb{R}^3. \tag{1}$$

The metric is conformal so that $g = e^{u_i} dz_i \, d\bar{z}_i$, where z_i is the local coordinate on the Riemann surface.

The vectors F_z, $F_{\bar{z}}$ as well as the normal N such that $(F_z, N) = (F_{\bar{z}}, N) = 0$ and $(N, N) = 1$ define a moving frame on the surface. The bracket represents the Euclidean inner product $(a, b) = a_1 b_1 + a_2 b_2 + a_3 b_3$. The moving frame satisfies the Gauss-Weingarten equations

$$\sigma_z = \mathcal{U}\sigma, \quad \sigma_{\bar{z}} = \mathcal{V}\sigma, \quad \sigma = (F_z, F_{\bar{z}}, N)^T, \tag{2}$$

and the matrices $\mathcal{U}$ and $\mathcal{V}$ are defined by

$$\mathcal{U}=\begin{pmatrix} u_z & 0 & Q \\ 0 & 0 & \frac{1}{2}He^u \\ -H & -2Qe^{-u} & 0 \end{pmatrix}, \qquad \mathcal{V}=\begin{pmatrix} 0 & 0 & \frac{1}{2}He^u \\ 0 & u_{\bar z} & Q \\ -2\bar{Q}e^{-u} & 0 & 0 \end{pmatrix}. \tag{3}$$

Moreover, we have the relations

$$Q=(F_{zz},N), \qquad \frac{1}{2}He^u=(F_{z\bar z},N). \tag{4}$$

The first and second fundamental forms are given by the matrices

$$M_I=e^u\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_{II}=\begin{pmatrix} Q+\bar{Q}+He^u & i(Q-\bar{Q}) \\ i(Q-\bar{Q}) & -(Q+\bar{Q})+He^u \end{pmatrix}. \tag{5}$$

The principal curvatures k_1 and k_2 are the eigenvalues of the matrix $M_{II}\cdot M_I^{-1}$. The characteristic polynomial of this matrix is given by

$$\lambda^2-2He^u\lambda-4|Q|^2+H^2e^{2u}=0. \tag{6}$$

This polynomial has the two roots $\lambda=He^u\pm 2|Q|$, and so the principle curvatures are given by $k_{1,2}=H\pm 2e^{-u}|Q|$. Then the mean curvature is given by the average of $k_{1,2}$ and the Gaussian curvature is given by their product

$$K=k_1k_2=H^2-4e^{-2u}|Q|^2. \tag{7}$$

The Gauss-Codazzi equations, which are the compatibility conditions for (2), are obtained by calculating

$$\mathcal{U}_{\bar z}-\mathcal{V}_z+[\mathcal{U},\mathcal{V}]=0. \tag{8}$$

Using $\mathcal{U},\mathcal{V}$ given in (3), the expression in (8) reduces to the following matrix:

$$\begin{pmatrix} u_{z\bar z}-2|Q|^2e^{-u}+\frac{1}{2}H^2e^u & 0 & Q_{\bar z}-\frac{1}{2}H_ze^u \\ 0 & -u_{z\bar z}-\frac{1}{2}H^2e^u+2|Q|^2e^{-u} & \frac{1}{2}H_{\bar z}e^u-\bar{Q}_z \\ -H_{\bar z}+2e^{-u}\bar{Q}_z & -2Q_{\bar z}e^{-u}+H_z & 0 \end{pmatrix}. \tag{9}$$

Requiring that all of the elements in the matrix given in (9) vanish as required by (8) gives rise to the following set of equations:

$$u_{z\bar z}+\frac{1}{2}H^2e^u-2|Q|^2e^{-u}=0, \qquad Q_{\bar z}=\frac{1}{2}H_ze^u, \qquad \bar{Q}_z=\frac{1}{2}H_{\bar z}e^u. \tag{10}$$

The first equation in (10) is referred to as the Gauss equation and the last pair as the Codazzi equations.

There exists a connection between quaternions and surfaces in $\mathbb{R}^3$, that is, there is a quaternionic description of surfaces in $\mathbb{R}^3$, which we introduce now. The matrix

$\Phi \in SU(2)$ transforms the quaternionic basis $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ into the moving frame F_x, F_y, N. Equations (2) for the moving frame are rewritten using the Lie algebra isomorphism between so(3) and so(2) in terms of 2×2 matrices. The quaternionic representation of surfaces permits the identification of the Gauss-Weingarten equations of certain surfaces with the Lax representations for Painlevé equations. This quaternionic description will be useful for analytic studies of curves and surfaces in three- and four-dimensional spaces.

The algebra of quaternions is denoted by $\mathbb{H}$ and the multiplicative quaternion group by $\mathbb{H}_* = \mathbb{H}\setminus\{0\}$ such that the standard basis is written as $\{\mathbf{1},\hat{\imath},\hat{\jmath},\hat{k}\}$, where the elements in this set satisfy $\hat{\imath}\hat{\jmath} = \hat{k}$, $\hat{\jmath}\hat{k} = \hat{\imath}$, and $\hat{k}\hat{\imath} = \hat{\jmath}$. The Pauli matrices can be identified with this basis under the following association: $\sigma_1 = i\hat{\imath}$, $\sigma_2 = i\hat{\jmath}$, $\sigma_3 = i\hat{k}$, and $\mathbf{I} = \mathbf{1}$. The multiplication in terms of Pauli matrices is simply matrix multiplication. Then the matrix $\Phi \in SU(2)$ transforms the basis $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ into the frame F_x, F_y, N as follows:

$$F_x = e^{u/2}\Phi^{-1}\hat{\imath}\Phi, \qquad F_y = e^{u/2}\Phi^{-1}\hat{\jmath}\Phi, \qquad N = \Phi^{-1}\hat{k}\Phi. \tag{11}$$

These equations imply that by means of the identification

$$\beta = \beta_0\mathbf{1} + \beta_1\hat{\imath} + \beta_2\hat{\jmath} + \beta_3\hat{k} \longleftrightarrow \begin{pmatrix} \beta_0-\beta_3 & -\beta_1+\beta_2 \\ \beta_1+\beta_2 & \beta_0+\beta_3 \end{pmatrix}, \tag{12}$$

the moving frame $(e^{-u/2}F_x, e^{-u/2}F_y, N)$ of the surface is described by the expression

$$\mathrm{Ad}(\Phi)(\hat{\imath},\hat{\jmath},\hat{k}) = (e^{-u/2}F_x, e^{-u/2}F_y, N). \tag{13}$$

The complex representation for the first derivatives of F will be required to be used with (2) and can be calculated as follows:

$$\begin{aligned} F_z &= \frac{1}{2}(F_x - iF_y) = -ie^{u/2}\Phi^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi, \\ F_{\bar{z}} &= \frac{1}{2}(F_x + iF_y) = -ie^{u/2}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi. \end{aligned} \tag{14}$$

The quaternion Φ satisfies linear differential equations. To obtain these equations, we introduce the matrices U, V given by

$$U = \Phi_z\Phi^{-1}, \qquad V = \Phi_{\bar{z}}\Phi^{-1}. \tag{15}$$

The quantities U, V must satisfy the compatibility condition

$$U_{\bar{z}} - V_z + [U,V] = 0. \tag{16}$$

Differentiating (14) and using the definition of V in (15), the following second-order derivatives of F are obtained:

$$F_{z\bar{z}} = -\frac{i}{2}u_{\bar{z}}e^{u/2}\Phi^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi - ie^{u/2}\Phi^{-1}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, V\right]\Phi, \tag{17}$$

and similarly for the mixed derivative,

$$F_{\bar{z}z} = -\frac{i}{2}u_z e^{u/2}\Phi^{-1}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\Phi - ie^{u/2}\Phi^{-1}\left[\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, U\right]\Phi. \tag{18}$$

By differentiating $\Phi\Phi^{-1} = \mathbf{I}$ with respect to $\bar{z}$, we obtain an expression for the derivative of Φ^{-1}, namely, $\Phi_{\bar{z}}^{-1} = -\Phi^{-1}\Phi_{\bar{z}}\Phi^{-1}$. This is used to obtain the results in (17) and (18). The second derivatives of F can also be obtained in terms of other quantities from the equations for σ_z in (2) using the matrices in (3). The following results will be required in due course:

$$F_{zz} = u_z F_z + QN, \qquad F_{\bar{z}z} = \frac{1}{2}e^u HN, \qquad N_z = -HF_z - 2e^{-u}QF_{\bar{z}}. \tag{19}$$

Moreover, let the matrices U and V defined in (15) above have matrix elements:

$$U = \begin{pmatrix}U_{11} & U_{12}\\U_{21} & U_{22}\end{pmatrix}, \qquad V = \begin{pmatrix}V_{11} & V_{12}\\V_{21} & V_{22}\end{pmatrix}, \tag{20}$$

where U and V are traceless matrices such that $U_{11} + U_{22} = 0$ and $V_{11} + V_{22} = 0$. The entries of the matrices in (20) can be determined by using the Gauss-Weingarten equations (2) and the compatibility conditions.

In fact, we can work out $F_{z\bar{z}}$ in terms of the V_{ij} which appear in V in (20) and then equate the result to the quantity $(e^u/2)HN$ as dictated by (19) to give the elements of the matrix V explicitly:

$$\begin{pmatrix}0 & 0\\u_{\bar{z}} & 0\end{pmatrix} + 2\left[\begin{pmatrix}0 & 0\\V_{11} & V_{12}\end{pmatrix} - \begin{pmatrix}V_{12} & 0\\V_{22} & 0\end{pmatrix}\right] = e^{u/2}H\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}. \tag{21}$$

Equating the corresponding elements of the resulting matrices on both sides of this equation, we obtain that

$$-u_{\bar{z}} = 2V_{11} - 2V_{22}, \qquad -V_{12} = \frac{1}{2}e^{u/2}H, \qquad V_{12} = -\frac{1}{2}e^{u/2}H. \tag{22}$$

When the matrix V is traceless, we must have $V_{11} = -u_{\bar{z}}/4 = -V_{22}$. Using the compatibility condition $F_{\bar{z}z} = F_{z\bar{z}}$, we also have

$$\begin{pmatrix}0 & u_z\\0 & 0\end{pmatrix} + 2\left[\begin{pmatrix}U_{21} & U_{22}\\0 & 0\end{pmatrix} - \begin{pmatrix}0 & U_{11}\\0 & U_{21}\end{pmatrix}\right] = e^u H\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}. \tag{23}$$

This result produces the following set of conditions which give U_{ij} to be $2U_{21} = e^u H$, $u_z + 2U_{22} - 2U_{11} = 0$, and $-2U_{21} = -e^u H$. These results imply that

$$U_{11} = \frac{1}{4}u_z, \qquad U_{21} = \frac{1}{2}e^u H. \tag{24}$$

To obtain an equation which contains F_{zz}, we begin by differentiating the expression for F_z in (14) with respect to z to obtain

$$F_{zz} = -\frac{i}{2}e^{u/2}u_z\Phi^{-1}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\Phi - ie^{u/2}\Phi_z^{-1}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\Phi - ie^{u/2}\Phi^{-1}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\Phi_z. \tag{25}$$

From the Gauss-Weingarten equations (19), the equation for F_{zz} yields

$$\begin{aligned}\frac{u_z}{2}\Phi^{-1}\begin{pmatrix}0&0\\1&0\end{pmatrix}\Phi-\Phi^{-1}U\begin{pmatrix}0&0\\1&0\end{pmatrix}\Phi+\Phi^{-1}\begin{pmatrix}0&0\\1&0\end{pmatrix}U\Phi\\ =u_z\Phi^{-1}\begin{pmatrix}0&0\\1&0\end{pmatrix}\Phi+e^{-u/2}Q\Phi^{-1}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\Phi.\end{aligned} \tag{26}$$

In terms of the required unknown matrix elements, (26) implies that

$$U_{12}=-e^{-u/2}Q,\qquad U_{11}-U_{22}+\frac{1}{2}u_z=u_z, \tag{27}$$

and therefore,

$$U_{12}=-e^{-u/2}Q,\qquad U_{22}=-U_{11}=-\frac{1}{4}u_z. \tag{28}$$

The results which have been obtained here in the form of (22), (24), and (28) can be summarized in the form of the following theorem.

Theorem 1. *Under the isomorphism $Y=-i\sum_{\alpha=1}^{3}X_\alpha\sigma_\alpha\to(X_1,X_2,X_3)$ in Euclidean three-space, the moving frame F_z, $F_{\bar z}$, N of the conformally parametrized surface is described by F_z, $F_{\bar z}$ given by (14), where $\Phi\in\mathrm{SU}(2)$ satisfies (15) and the matrices U and V are given in the form*

$$U=\begin{pmatrix}\frac{1}{4}u_z & -e^{-u/2}Q\\ \frac{1}{2}e^{u/2}H & -\frac{1}{4}u_z\end{pmatrix},\qquad V=\begin{pmatrix}-\frac{1}{4}u_{\bar z} & -\frac{1}{2}e^{u/2}H\\ e^{-u/2}\bar Q & \frac{1}{4}u_{\bar z}\end{pmatrix}. \tag{29}$$

The quantity Φ, which can be considered to be $\mathbb{H}$-valued, satisfies the pair of equations

$$\Phi_z=U\Phi,\qquad \Phi_{\bar z}=V\Phi, \tag{30}$$

which are an equivalent form of the equations in (15).

As an example, it is possible to exhibit solutions to the set of equations in (30). We consider the choice $H=1/2$, $Q=-\lambda/4$, and $\bar Q=-1/4\lambda$. Then $u=0$ is a global solution of the Gauss-Codazzi equations when $|\lambda|^2=1$, defined on the whole plane and referred to as the vacuum solution. The deformed equations corresponding to (30) for this state are

$$\Phi_{\lambda,z}=\begin{pmatrix}0&\frac{\lambda}{4}\\ \frac{1}{4}&0\end{pmatrix}\Phi_\lambda,\qquad \Phi_{\lambda,\bar z}=\begin{pmatrix}0&-\frac{1}{4}\\ -\frac{1}{4\lambda}&0\end{pmatrix}\Phi_\lambda. \tag{31}$$

Under the initial condition $\Phi_\lambda(0,0)=\mathbf{1}$, these equations can be solved explicitly.

Theorem 2. *The following Φ_λ satisfies (31) when $\lambda = \mu^2$:*

$$\Phi_\lambda = \begin{pmatrix} \cosh\left(\frac{1}{4}(\mu z - \mu^{-1}\bar{z})\right) & \mu \sinh\left(\frac{1}{4}(\mu z - \mu^{-1}\bar{z})\right) \\ \mu^{-1}\sinh\left(\frac{1}{4}(\mu z - \mu^{-1}\bar{z})\right) & \cosh\left(\frac{1}{4}(\mu z - \mu^{-1}\bar{z})\right) \end{pmatrix}. \tag{32}$$

Theorem 3. *The following Φ_λ satisfies (31) when $\lambda = -\mu^2$:*

$$\Phi_\lambda = \begin{pmatrix} \cos\left(\frac{1}{4}(\mu z + \mu^{-1}\bar{z})\right) & -\mu \sin\left(\frac{1}{4}(\mu z + \mu^{-1}\bar{z})\right) \\ \mu^{-1}\sin\left(\frac{1}{4}(\mu z + \mu^{-1}\bar{z})\right) & \cos\left(\frac{1}{4}(\mu z + \mu^{-1}\bar{z})\right) \end{pmatrix}. \tag{33}$$

As another important example, consider the case of constant mean curvature surfaces which can be generated by means of solutions of the generalized Weierstrass representation. It has been shown recently in [3, 5] that surfaces with constant mean curvature can be generated by calculating explicit solutions for the following system of nonlinear Dirac type equations:

$$\begin{aligned} \partial\psi_1 &= p\psi_2, & \bar{\partial}\psi_2 &= -p\psi_1, \\ \bar{\partial}\bar{\psi}_1 &= p\bar{\psi}_2, & \partial\bar{\psi}_2 &= -p\bar{\psi}_1, \end{aligned} \tag{34}$$

where $p = |\psi_1|^2 + |\psi_2|^2$ and the equations in (34) have been normalized so that $H = 1/2$. There exists a conserved current for system (34) which is given by

$$J = \psi_2\partial\bar{\psi}_1 - \bar{\psi}_1\partial\psi_2. \tag{35}$$

Using the generalized system (34), it is easy to show that

$$\bar{\partial}J = \partial\bar{J} = 0. \tag{36}$$

The Gaussian curvature K of the surface can be calculated from

$$K = -\frac{1}{p^2}\partial\bar{\partial}(\ln p). \tag{37}$$

The coordinate functions of the surface are found by substituting explicit solutions of (34) and evaluating the following integrals:

$$\begin{aligned} X_1 + iX_2 &= 2i\int_\gamma (\bar{\psi}_1^2 dz' - \bar{\psi}_2^2 d\bar{z}'), \\ X_1 - iX_2 &= 2i\int_\gamma (\psi_2^2 dz' - \psi_1^2 d\bar{z}), \\ X_3 &= -2\int_\gamma (\bar{\psi}_1\psi_2 dz' + \psi_1\bar{\psi}_2 d\bar{z}'). \end{aligned} \tag{38}$$

The integrals are then evaluated and on account of system (34), the right-hand sides of (38) do not depend on the choice of contour γ in $\mathbb{C}$. The functions $X_i(z,\bar{z})$ are then treated as the coordinates of a surface immersed into $\mathbb{R}^3$.

Moreover, it is straightforward to show that system (34) is equivalent to the system of equations

$$\partial\bar{\partial}\ln p = \frac{|J|^2}{p^2} - p^2, \qquad \bar{\partial}J = \partial\bar{J} = 0. \tag{39}$$

The equations in (39) are the corresponding Gauss-Codazzi equations, which essentially correspond to the compatibility conditions for (3) if we identify $u = \ln p^2$ and make some other reparametrizations [4].

With respect to the generalized Weierstrass system (34), the linear problem (3) can be rewritten in terms of ψ_1 and ψ_2 as follows. Differentiating $p = |\psi_1|^2 + |\psi_2|^2$, we have modulo (34),

$$\partial p = \psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2. \tag{40}$$

Using the definition of the current J in (35), we can solve for the derivatives $\partial\bar{\psi}_1$ and $\partial\psi_2$. Using these with (34), we obtain that the set of derivatives satisfies

$$\partial\begin{pmatrix}\bar{\psi}_1\\ \bar{\psi}_2\end{pmatrix} = \begin{pmatrix}\partial\ln p & \dfrac{J}{p}\\ -p & 0\end{pmatrix}\begin{pmatrix}\bar{\psi}_1\\ \bar{\psi}_2\end{pmatrix}, \qquad \partial\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}0 & p\\ -\dfrac{J}{p} & \partial\ln p\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{41}$$

The compatibility conditions for (41) should coincide with (39). Theorem 1 can be applied to this case by taking $H = 1/2$, $Q = J$ and identifying $u = \ln p^2$. Then the matrices U and V are given by

$$U = \begin{pmatrix}\dfrac{\partial p}{2p} & -\dfrac{J}{p}\\ \dfrac{p}{4} & -\dfrac{\partial p}{2p}\end{pmatrix}, \qquad V = \begin{pmatrix}-\dfrac{\bar{\partial}p}{2p} & -\dfrac{p}{4}\\ \dfrac{\bar{J}}{p} & \dfrac{\bar{\partial}p}{2p}\end{pmatrix}. \tag{42}$$

These are the matrices which would appear in (30) and which satisfy (16).

As a final point, we would like to show that the classic Enneper surface for which $H = 0$ can be produced as a solution to system (34). In this instance, since the right-hand side of system (34) is proportional to H, it reduces to the simple linear system $\partial\psi_1 = 0$ and $\bar{\partial}\psi_2 = 0$, since H cannot be scaled out in this case. These equations have the general solutions $\psi_1 = f(\bar{z})$ and $\psi_2 = g(z)$. Consider this specific case of system (34) where the particular solution $\psi_1 = a\bar{z}$ and $\psi_2 = b$, with $a, b \in \mathbb{R}$ is taken. Using these solutions in (38) and integrating, the coordinates of the following Enneper-type surface is obtained:

$$\begin{aligned} X_1 &= 2a^2u^2v - \frac{2}{3}a^2v^3 + \frac{v}{2a^2},\\ X_2 &= 2a^2uv^2 - \frac{2}{3}a^2u^3 + \frac{u}{2a^2},\\ X_3 &= u^2 - v^2. \end{aligned} \tag{43}$$

We have taken $b = 1/(2a)$ and substituted $z = u + iv$ after integration to obtain the equations in (43). By rescaling the coordinates $(u,v) \to (u/2a^2, v/2a^2)$ and then X_i, the standard classic form of Enneper's surface is obtained. It is usually written in the following form:

$$\vec{\mathbf{X}} = \left(v - \frac{v^3}{3} + u^2 v,\ u - \frac{u^3}{3} + u v^2,\ u^2 - v^2 \right). \tag{44}$$

It is known that a surface has constant mean curvature of zero in $\mathbb{R}^3$ if and only if its parametrized coordinate functions satisfy Laplace's equation. In this case, it is easy to verify that the results in both (43) and (44) do satisfy Laplace's equation, namely, $\vec{\mathbf{X}}_{uu} + \vec{\mathbf{X}}_{vv} = 0$ as required. By taking solutions of the form $\psi_1 = \bar{a}\bar{z} + \bar{b}$, $\psi_2 = cz + d$, with $a, b, c, d \in \mathbb{C}$ in (38), the coordinate expressions of the more generalized Enneper-type surfaces can be calculated.

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