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ON CAUCHY-TYPE FUNCTIONAL EQUATIONS

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Let G be a Hausdorff topological locally compact group. Let $M(G)$ denote the Banach algebra of all complex and bounded measures on G. For all integers $n \geq 1$ and all $\mu \in M(G)$, we consider the functional equations $\int_G f(xty)d\mu(t) = \sum_{i=1}^n g_i(x)h_i(y)$, $x, y \in G$, where the functions f, $\{g_i\}$, $\{h_i\}$: $G \to \mathbb{C}$ to be determined are bounded and continuous functions on G. We show how the solutions of these equations are closely related to the solutions of the μ-spherical matrix functions. When G is a compact group and μ is a Gelfand measure, we give the set of continuous solutions of these equations.

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1. Introduction. Let G be a locally compact Hausdorff group, that is, a locally compact group which satisfies the following separation axiom: every pair of distinct points in G have disjoint neighborhoods. Let μ be a complex bounded measure on G. We consider the functional equation

$$\int_G f(xty)d\mu(t) = \sum_{i=1}^n g_i(x)h_i(y), \quad x, y \in G. \tag{1.1}$$

In the particular case where $\mu = \delta_e$ (the Dirac complex measure concentrated at the identity element of G) (i.e., $\langle f, \delta_e \rangle = \int_G f(t)d\delta_e(t) = f(e)$, for all $f: G \to \mathbb{C}$), (1.1) reduces to Levi-Civita equation

$$f(xy) = \sum_{i=1}^n g_i(x)h_i(y), \quad x, y \in G, \tag{1.2}$$

a special case of which is Cauchy's functional equation $f(xy) = f(x)f(y)$, for all $x, y \in G$. This explains the choice of the title of this note.

Solutions of general equations like (1.2) were studied by many authors. The trigonometric addition and subtraction formulas and their relations have been studied by Wilson [19], Vietoris [17], and Vincze [18].

In their work, Chung et al. [7] found the solutions of the equation

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in G, \tag{1.3}$$

in which the group G need not be abelian.

O'Connor [11] studied a solution of the equation

$$f(xy^{-1}) = \sum_{i=1}^{n} a_i(x)\overline{a_i(y)}, \quad x,y \in G, \tag{1.4}$$

on a locally compact abelian group.

Poulsen and Stetkær [12] introduced and solved the following functional equations:

$$\begin{aligned} f(x\sigma(y)) &= f(x)f(y)+g(x)g(y), \quad x,y \in G, \\ f(x\sigma(y)) &= f(x)g(y)+g(x)f(y), \quad x,y \in G, \end{aligned} \tag{1.5}$$

where σ is a homomorphism of G such that $\sigma(\sigma(x)) = x$, for all $x \in G$.

For other references and more information about (1.2), we can see the monographs by Aczél [1], by Aczél and Dhombres [2], and by Székelyhidi [16].

Let K be a compact subgroup of the group $\mathrm{Aut}(G)$ of all mappings of G onto G that are simultaneously automorphisms and homeomorphisms. Let dk be the normalized Haar measure on K, that is, the normalized nonnegative measure on K which is invariant by translations of K (see [10]) and consider

$$\int_K f(xk(y))dk = \sum_{i=1}^{n} g_i(x)h_i(y), \quad x,y \in G. \tag{1.6}$$

Equation (1.6) were considered by Stetkær in several works (see, e.g., [13, 14, 15]) and it was solved in the particular case when G is compact and commutative (see [13]). Furthermore, on an abelian group G, the bounded solutions of equations like

$$\int_K f(xk(y))dk = f(x)f(y), \quad x,y \in G, \tag{1.7}$$

were discussed by Chojnacki [6] and Badora [5].

Consider the group $\tilde{G} = G \times_s K$, the semidirect product of G and K, where the topology is the product topology and the group operation is given by

$$(g_1,k_1)(g_2,k_2) = (g_1k_1(g_2),k_1k_2), \tag{1.8}$$

$K_s = \{e\} \times K$ is a closed compact subgroup of $\tilde{G}$. So the functional equation (1.6) on G is closely related to the functional equation

$$\int_{K_s} f(xky)dk = \sum_{i=1}^{n} g_i(x)h_i(y) \tag{1.9}$$

on $\tilde{G}$, which is a particular case of (1.1).

When $n = 2$, there are two interesting cases of (1.1):

$$\int_G f(xty)d\mu(t) = f(x)g(y)+f(y)g(x), \quad x,y \in G, \tag{1.10}$$

$$\int_G f(xty)d\mu(t) = f(x)f(y)+g(y)g(x), \quad x,y \in G, \tag{1.11}$$

which cover the functional equation for sinh and cosh that were studied by Vincze [18].

The aim of this note is to study (1.1). Our discussion is organized as follows. In Section 2, we make a general setup and recall some definitions. In Section 3, we establish some general properties about the solutions of (1.1). In Section 4, we suppose that G is compact and μ is a Gelfand measure (see [3]). In this case, when $\{g_1,\dots,g_n\}$ and $\{h_1,\dots,h_n\}$ are two sets of linearly independent functions and f is μ-invariant, we completely solve (1.1). The solutions are described in Theorem 4.2. As a particular case, we obtain the result given by Stetkær when G is compact and commutative (see [13]).

In Sections 4.3 and 4.4, we solve (1.10) and (1.11) without any assumption of μ-invariance nor of independence of the unknown functions f and g. We notice that the solutions of (1.1), (1.10), and (1.11) are expressed in terms of μ-spherical function of the compact group G characterized in [3]. The approach adopted here is based on a general process of diagonalization of matricial μ-spherical functions.

The results obtained in this note may be viewed as some generalizations of important works studied in the literature (see [1, 2, 5, 6, 7, 11, 12, 13, 14, 15, 19]). Our work unifies many of the results presented in these references.

2. Setup and notations. Throughout this note, G will be a Hausdorff topological locally compact group. Let $M(G)$ denote the Banach algebra of complex bounded measures; it is the dual of $C_0(G)$, the Banach space of continuous functions vanishing at infinity. For all $\mu,\nu\in M(G)$, we recall that the convolution product $\mu*\nu$ is the measure given by

$$\langle f,\mu*\nu\rangle=\int_G\int_G f(ks)\,d\mu(k)\,d\nu(s), \tag{2.1}$$

and the involution is defined in $M(G)$ by $\mu^*=\check{\bar{\mu}}$, where

$$\langle f,\bar{\mu}\rangle=\overline{\langle\bar{f},\mu\rangle},\quad \langle f,\check{\mu}\rangle=\langle\check{f},\mu\rangle,\quad \text{with } \check{f}(x)=f(x^{-1}), \tag{2.2}$$

for all $x\in G$.

Let $C(G)$ (resp., $C_b(G)$) designate the Banach space of continuous (resp., continuous and bounded) complex valued functions.

For $f\in C_b(G)$, we define

$$f_\mu(x)=\int_G\int_G f(kxt)\,d\mu(t)\,d\mu(k),\quad x\in G, \tag{2.3}$$

and we say that f is μ-invariant if $f_\mu(x)=f(x)$, for all $x\in G$.

For every $n\in\mathbb{N}$, $M_{(n,n)}(\mathbb{C})$ (resp., $GL_n(\mathbb{C})$) will be the algebra of all complex $n\times n$ matrices (resp., invertible matrices). If $A\in M_{(n,n)}(\mathbb{C})$, we let A^t denote the transpose of the matrix A.

Let $\mu\in M(G)$, μ is called a Gelfand measure (see [3, 4]) if $\mu^*=\mu=\mu*\mu$ and the Banach algebra $\mu*M(G)*\mu$ is commutative under the convolution.

Remark 2.1. If K is a compact subgroup of G and $\mu = dk$ is the normalized Haar measure of K, then dk is a Gelfand measure on G if and only if (G,K) is a Gelfand pair (see [8, 9]).

Definition 2.2. A nonzero function $\Phi \in C_b(G)$ is a μ-spherical function if it satisfies the functional equation $\int_G \Phi(xty)d\mu(t) = \Phi(x)\Phi(y)$, for all $x, y \in G$.

A function $\Phi \in M_{(n,n)}(\mathbb{C})$ $(n \geq 2)$, is a μ-spherical matrix function if $\Phi(e) = I_n$ and Φ satisfies the functional equation $\int_G \Phi(xty)d\mu(t) = \Phi(x)\Phi(y)$, for all $x, y \in G$.

3. Cauchy-type functional equations. In this section, we study the general properties of the functional equation (1.1) where $\mu \in M(G)$ and the unknown functions $f, g_i, h_i \in C_b(G)$, for all $i \in \{1,2,\dots,n\}$.

The following useful lemma produces a necessary condition for (1.1) to have a solution.

Lemma 3.1. *If $(f, \{g_i\}, \{h_i\})$ are solutions of (1.1), then*

$$\sum_{i=1}^{n}\left(\int_G g_i(xty)d\mu(t)\right)h_i(z) = \sum_{i=1}^{n}\left(\int_G h_i(ytz)d\mu(t)\right)g_i(x), \tag{3.1}$$

for all $x, y, z \in G$.

Proof. Using Fubini's theorem, (1.1) shows that

$$\begin{aligned}
\sum_{i=1}^{n}\left(\int_G g_i(xty)d\mu(t)\right)h_i(z) &= \int_G\left(\sum_{i=1}^{n} g_i(xty)h_i(z)\right)d\mu(t) \\
&= \int_G\left(\int_G f(xtysz)d\mu(s)\right)d\mu(t) \\
&= \int_G\left(\int_G f(xtysz)d\mu(t)\right)d\mu(s) \\
&= \int_G\left(\sum_{i=1}^{n} g_i(x)h_i(ysz)\right)d\mu(s) \\
&= \sum_{i=1}^{n}\left(\int_G h_i(ytz)d\mu(t)\right)g_i(x),
\end{aligned} \tag{3.2}$$

which proves equality (3.1). □

Assumptions. For the remainder of the note, we will make the following assumptions:

(H_1) $\mu \in M(G)$ and $\mu * \mu = \mu$;

(H_2) the two sets of functions $\{g_i\}$, $\{h_i\}$ are linearly independent.

We recall from [2] that assumption (H_2) implies that there exist $\{a_i\}_{i\in\{1,2,\dots,n\}} \in G \times G \times \cdots \times G$ and $\{b_i\}_{i\in\{1,2,\dots,n\}} \in G \times G \times \cdots \times G$ such that the matrices $\{g_j(a_i)\}$, $\{h_i(b_j)\}$ are invertible.

In the next theorem, we illustrate how the μ-spherical matrix functions can be useful in the study of the functional equation (1.1).

Theorem 3.2. *Let $\mu \in M(G)$ such that $\mu * \mu = \mu$. Let $(f, \{g_i\}, \{h_i\})$ be a solution of (1.1). Then*

(i) *for all $z \in G$, the following identity holds in $M_{(n,n)}(\mathbb{C})$:*

$$\left\{\int_G g_j(x_i t z)\, d\mu(t)\right\}\{h_i(y_j)\} = \{g_j(x_i)\}\left\{\int_G h_i(z t y_j)\, d\mu(t)\right\}; \tag{3.3}$$

(ii) *let $\Phi(z)$ for $z \in G$ be the $n \times n$ matrix function defined by*

$$\begin{aligned}\Phi(z) &= \{g_j(a_i)\}^{-1}\left\{\int_G g_j(a_i t z)\, d\mu(t)\right\}\\ &= \left\{\int_G h_i(z t b_j)\, d\mu(t)\right\}\{h_i(b_j)\}^{-1},\end{aligned} \tag{3.4}$$

then Φ is a μ-spherical matrix function. Furthermore, there exist the following identities:

$$\left\{\int_G g_j(x_i t z)\, d\mu(t)\right\} = \{g_j(x_i)\}\Phi(z), \tag{3.5}$$

$$\left\{\int_G h_i(z t x_j)\, d\mu(t)\right\} = \Phi(z)\{h_i(x_j)\}, \quad \forall z \in G; \tag{3.6}$$

(iii) *let*

$$(g_1', \ldots, g_n') = (g_1, \ldots, g_n)A^{-1}, \qquad (h_1', \ldots, h_n') = (h_1, \ldots, h_n)A^t, \tag{3.7}$$

where A is an invertible matrix of $M_{(n,n)}(\mathbb{C})$. Then $(g_1', \ldots, g_n')$ and $(h_1', \ldots, h_n')$ are linearly independent sets. There exists

$$\sum_{i=1}^{n} g_i(x)h_i(y) = \sum_{i=1}^{n} g_i'(x)h_i'(y), \tag{3.8}$$

for all $x, y \in G$, and the μ-spherical matrix function Φ' corresponding to $\{g_i'\}$ and $\{h_i'\}$ defined in (ii) above is given by

$$\Phi'(z) = A\Phi(z)A^{-1}; \tag{3.9}$$

(iv) *assume that $\Phi'(z)$ has diagonal form, that is,*

$$\Phi'(z) = \operatorname{Diag}\left(\Phi_i'(z)\right)_{i \in \{1, \ldots, n\}}, \tag{3.10}$$

then $\Phi_1', \ldots, \Phi_n'$ are μ-spherical functions with complex values and

$$\begin{aligned}\int_G g_i'(x t y)\, d\mu(t) &= g_i'(x)\Phi_i'(y),\\ \int_G h_i'(x t y)\, d\mu(t) &= \Phi_i'(x)h_i'(y),\end{aligned} \tag{3.11}$$

for all $x, y \in G$ and for all $i = 1, \ldots, n$.

Proof. (i) From Lemma 3.1, we can easily derive (3.3).

(ii) For all $x_1,\dots,x_n,\ z\in G$,

$$\{g_j(x_i)\}\Phi(z)=\{g_j(x_i)\}\left\{\int_G h_i(ztb_j)d\mu(t)\right\}\{h_i(b_j)\}^{-1}. \tag{3.12}$$

By using (3.3), we get

$$\begin{aligned}\{g_j(x_i)\}\Phi(z)&=\left\{\int_G g_j(x_itz)d\mu(t)\right\}\{h_i(b_j)\}\{h_i(b_j)\}^{-1}\\&=\left\{\int_G g_j(x_itz)d\mu(t)\right\},\end{aligned} \tag{3.13}$$

which proves (3.5).

From the assumed form of Φ, we compute

$$\begin{aligned}\int_G\Phi(xsy)d\mu(s)&=\{g_j(a_i)\}^{-1}\left\{\int_G\int_G g_j(a_itxsy)d\mu(t)d\mu(s)\right\}\\&=\{g_j(a_i)\}^{-1}\left\{\int_G g_j(a_itx)d\mu(t)\right\}\Phi(y)\\&=\{g_j(a_i)\}^{-1}\{g_j(a_i)\}\Phi(x)\Phi(y)=\Phi(x)\Phi(y).\end{aligned} \tag{3.14}$$

Using (1.1) and assumption (H_1), we obtain

$$\sum_{i=1}^{n}\int_G g_i(xs)d\mu(s)h_i(y)=\sum_{i=1}^{n}g_i(x)h_i(y),\quad\forall x,y\in G. \tag{3.15}$$

In view of assumption (H_2), we deduce that $\int_G g_i(xs)d\mu(s)=g_i(x)$, for all $x\in G$ and $\Phi(e)=I_n$. Therefore, according to Definition 2.2, Φ is a μ-spherical matrix function.

Now, by applying (3.3) and (3.4), we get

$$\begin{aligned}\Phi(z)\{h_i(x_j)\}&=\{g_j(a_i)\}^{-1}\left\{\int_G g_j(a_itz)d\mu(t)\right\}\{h_i(x_j)\}\\&=\{g_j(a_i)\}^{-1}\{g_j(a_i)\}\left\{\int_G h_j(ztx_j)d\mu(t)\right\}\\&=\left\{\int_G h_i(ztx_j)d\mu(t)\right\},\end{aligned} \tag{3.16}$$

which proves (3.6). Thus (i) and (ii) are proved.

To prove (iii), we write $(g_1',\dots,g_n')=(g_1,\dots,g_n)A^{-1}$ and $(h_1',\dots,h_n')=(h_1,\dots,h_n)A^t$, where A is an invertible $n\times n$ matrix. Then by a simple computation we get (3.8).

As mentioned above, the matrices $\{g_j(a_i)\}$, $\{h_i(b_j)\}$ are invertible, then $\{g_j'(a_i)\}$, $\{h_i'(b_j)\}$ are also invertible, and from (ii), the matrix μ-spherical function Φ' corresponding to $\{g_i'\}$ and $\{h_i'\}$ is defined by

$$\begin{aligned}\Phi(z)'&=\{g_j'(a_i)\}^{-1}\left\{\int_G g_j'(a_itz)d\mu(t)\right\}\\&=A\{g_j(a_i)\}^{-1}\left\{\int_G g_j(a_itz)d\mu(t)\right\}A^{-1}=A\Phi(z)A^{-1}.\end{aligned} \tag{3.17}$$

We notice that (3.11) come from (3.5) and (3.6). This ends the proof of Theorem. □

In the next section, we study (1.1) in the case of compact groups.

4. Cauchy-type functional equations on compact groups

4.1. The aim of this section is to treat (1.1) in the case where G is compact (Hausdorff) not necessarily commutative. We suppose only that G is endowed with a Gelfand measure μ (see [3]). For such a measure, we let Σ_μ denote the set of all μ-spherical functions on G; it is the Gelfand spectrum of the commutative Banach algebra $L_1^\mu(G) = \mu * L_1(G,dx) * \mu$. We recall (see [3]) that in compact groups, a μ-spherical function ω is also a positive definite function and in particular, $\breve{\omega}(x) = \overline{\omega(x)}$, for all $x \in G$.

4.2. Solutions of (1.1) for compact groups. We start by the following result concerning μ-spherical matricial function.

Theorem 4.1. *Let $\Phi : G \to M_{(n,n)}(\mathbb{C})$ be a continuous μ-spherical matrix function, that is,*

$$\begin{gathered}\int_G \Phi(xty)\,d\mu(t) = \Phi(x)\Phi(y), \quad x,y \in G,\\ \Phi(e) = I_n.\end{gathered} \tag{4.1}$$

Then there exists $\omega_1,\dots,\omega_n \in \Sigma_\mu$ and an invertible matrix $A \in M_{(n,n)}(\mathbb{C})$ such that

$$A\Phi A^{-1} = \operatorname{Diag}(\omega_i)_{i\in\{1,\dots,n\}}. \tag{4.2}$$

Proof. Let $\hat{G}$ denote the set of irreducible characters of G (see [10]) and consider the mapping from $\hat{G}$ to $M_{(n,n)}(\mathbb{C})$ defined by

$$\hat{\Phi}(\chi_\pi) = \int_G \Phi(x)\overline{\chi_\pi(x)}\,dx. \tag{4.3}$$

From (4.1), we have $\int_G \Phi(xt)\,d\mu(t) = \Phi(x)$. Consequently, we get

$$\begin{aligned}\int_G \Phi(x)\overline{\chi_\pi(x)}\,dx &= \int_G\int_G \Phi(xt)\overline{\chi_\pi(x)}\,d\mu(t)\,dx\\ &= \int_G \Phi(x)\Big(\int_G \overline{\chi_\pi(xt^{-1})}\,d\mu(t)\Big)dx\\ &= \int_G \Phi(x)\Big(\int_G \chi_\pi(tx^{-1})\,d\mu(t)\Big)\\ &= \int_G \Phi(x)\omega_\pi(x^{-1})\,dx\\ &= \int_G \Phi(x)\overline{\omega_\pi(x)}\,dx,\end{aligned} \tag{4.4}$$

where

$$\omega_\pi(x) = \int_G \chi_\pi(tx)\,d\mu(t). \tag{4.5}$$

We recall (see [3]) that the μ-spherical functions in the compact group are exactly those given by (4.5) with $\chi_\pi \in \hat{G}$.

For all $y \in G$, we have

$$\begin{aligned}\Phi(y)\hat{\Phi}(\chi_\pi) &= \int_G \Phi(y)\Phi(x)\overline{\omega_\pi}(x)dx \\ &= \int_G\int_G \Phi(ytx)\overline{\omega_\pi}(x)d\mu(t)dx.\end{aligned} \tag{4.6}$$

In view of $\int_G \Phi(kx)d\mu(k) = \Phi(x)$, it follows that

$$\begin{aligned}\Phi(y)\hat{\Phi}(\chi_\pi) &= \int_G\int_G\int_G \Phi(kytx)\overline{\omega_\pi(x)}d\mu(k)d\mu(t)dx \\ &= \int_G\int_G\int_G \Phi(x)\overline{\omega_\pi(t^{-1}y^{-1}k^{-1}x)}d\mu(k)d\mu(t)dx \\ &= \int_G\int_G\int_G \Phi(x)\omega_\pi(x^{-1}kyt)d\mu(k)d\mu(t)dx \\ &= \int_G \Phi(x)\overline{\omega_\pi(x)}\omega_\pi(y)dx \\ &= \omega_\pi(y)\int_G \Phi(x)\overline{\omega_\pi(x)}dx = \omega_\pi(y)\hat{\Phi}(\chi_\pi).\end{aligned} \tag{4.7}$$

By the Peter-Weyl theorem (see, e.g., [10]), we know that the irreducible characters of the compact groups G form a basis of the Banach space $L^2(G,dx)$. Therefore, the Span $\{\hat{\Phi}(\chi_\pi)\varsigma,\ \chi_\pi \in \hat{G},\ \varsigma \in \mathbb{C}^n\} = \mathbb{C}^n$, and it follows that there exists $\chi_{\pi_1},\chi_{\pi_2},\dots,\chi_{\pi_n} \in \hat{G}$ and $\varsigma_1,\varsigma_2,\dots,\varsigma_n \in \mathbb{C}^n$ such that $\{\hat{\Phi}(\chi_{\pi_i})\varsigma_i,\ i = 1,\dots,n\}$ is a basis of $\mathbb{C}^n$. By combining this result with (4.7), we obtain the desired conclusion. □

By using Theorems 4.1 and 3.2, we get the following result.

Theorem 4.2. *Let $(f,\{g_i\},\{h_i\})$ be a solution of (1.1) such that f is μ-invariant. Then there exists $A \in GL_n(\mathbb{C})$, and for each $i = 1,\dots,n$, there exist positive definite μ-spherical functions ω_i and $\alpha_i,\beta_i \in \mathbb{C}$ such that*

$$\begin{aligned}(g_1,\dots,g_n) &= (\alpha_1\omega_1,\dots,\alpha_n\omega_n)A, \\ (h_1,\dots,h_n) &= (\beta_1\omega_1,\dots,\beta_n\omega_n)(A^t)^{-1},\end{aligned} \tag{4.8}$$

$$f = \sum_{i=1}^{n} \alpha_i\beta_i\omega_i. \tag{4.9}$$

Conversely, formulas (4.8) and (4.9) define a solution of (1.1).

To solve the functional equation (1.1) in compact groups, we note the following general result that describes its solutions in case of $n = 1$.

Proposition 4.3. *Let $\mu \in M(G)$. Let $f,g,h \in C(G)\setminus\{0\}$ be a solution of the functional equation*

$$\int_G f(xty)d\mu(t) = g(x)h(y), \quad x,y \in G. \tag{4.10}$$

Then there exists a μ-spherical function ϕ on G such that

$$\begin{aligned}\int_G g(xty)d\mu(t) &= g(x)\phi(y), \quad x,y \in G,\\ \int_G h(xty)d\mu(t) &= \phi(x)h(y), \quad x,y \in G,\end{aligned} \tag{4.11}$$

for all $x, y \in G$.

Furthermore, if $\mu * \mu = \mu$ and f is μ-invariant, then there exist $\alpha, \beta \in \mathbb{C}^*$ such that

$$g = \alpha\phi, \qquad h = \beta\phi, \qquad f = \alpha\beta\phi. \tag{4.12}$$

Proof. Let $a, b \in G$ such that $g(a) \neq 0$ and $h(b) \neq 0$. In view of (4.10) and Lemma 3.1, we get

$$h(b)\int_G g(atx)d\mu(t) = g(a)\int_G h(xtb)d\mu(t), \tag{4.13}$$

and

$$\phi(x) = \frac{1}{h(b)}\int_G h(xtb)d\mu(t) = \frac{1}{g(a)}\int_G g(atx)d\mu(t) \tag{4.14}$$

is a μ-spherical function.

Now according to Theorem 3.2, we have the rest of the proof. □

Proof of Theorem 4.2. By Theorem 3.2, there exists a μ-spherical $n \times n$ matrix function $\Phi(z)$, $z \in G$ such that (3.5) and (3.6) hold. Since G is compact, then by using Theorem 4.1, there exist $\omega_1, \dots, \omega_n \in \Sigma_\mu$ and $A \in GL_n(\mathbb{C})$ such that

$$A\Phi A^{-1} = \begin{pmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_n \end{pmatrix}. \tag{4.15}$$

Now, if we put $(g_1', \dots, g_n') = (g_1, \dots, g_n)A^{-1}$ and $(h_1', \dots, h_n') = (h_1, \dots, h_n)A^t$, it follows immediately from Theorem 3.2(ii) that the functions h_i' and g_i' are solutions of the system of functional equations

$$\begin{aligned}\int_G g_i'(xty)d\mu(t) &= g_i'(x)\omega_i(y),\\ \int_G h_i'(xty)d\mu(t) &= \omega_i(x)h_i'(y),\end{aligned} \tag{4.16}$$

for all $x, y \in G$ and for all $i = 1, \dots, n$.

In view of Proposition 4.3, there exists $\alpha_i, \beta_i \in \mathbb{C}^*$ such that

$$g_i' = \alpha_i\omega_i, \qquad h_i' = \beta_i\omega_i, \tag{4.17}$$

which implies that

$$\begin{aligned}(g_1,\dots,g_n) &= (\alpha_1\omega_1,\dots,\alpha_n\omega_n)A,\\ (h_1,\dots,h_n) &= (\beta_1\omega_1,\dots,\beta_n\omega_n)(A^t)^{-1},\end{aligned} \tag{4.18}$$

and $f=\sum_{i=1}^n \alpha_i\beta_i\omega_i$. This ends the proof of the theorem. □

The subject of the following subsections is to treat some particular cases with $n=2$. More precisely, we are interested in solving (1.10) and (1.11). We will see that the assumptions of independence or μ-invariance required in Theorem 4.2 are not needed here.

4.3. Solutions of (1.10). By using Theorems 4.2 and 3.2, and without assuming the μ-invariance of f, we get the following result.

Theorem 4.4. *The complete list of functions $f,g\in C(G)\setminus\{0\}$ satisfying the functional equation (1.10) consists of the following two cases, where Φ_1,Φ_2, $(\Phi_1\neq\Phi_2)$, Φ are positive definite μ-spherical functions and $\alpha,\beta\in\mathbb{C}\setminus\{0\}$:*

(i) $f=(1/2\alpha)\Phi$, $g=\Phi/2$;

(ii) $f=\beta(\Phi_1-\Phi_2)/2$, $g=(\Phi_1+\Phi_2)/2$.

To solve this equation we need to prove the following lemmas.

Lemma 4.5. *If $f\in C(G)$ is a solution of the functional equation*

$$\int_G f(xty)\,d\mu(t)=f(x)\omega(y)+f(y)\omega(x),\quad x,y\in G, \tag{4.19}$$

in which ω is a μ-spherical function such that f and ω are linearly independent, then $f(x)=0$, for all $x\in G$.

Proof. Let f satisfy (4.19). By a small computation, we show that f is μ-invariant. Multiplying (4.19) by $\overline{\omega(x)}$ and integrating the result over G, we get

$$\int_G\int_G f(xty)\overline{\omega(x)}\,dx\,d\mu(t)=f(y)\int_G|\omega(x)|^2dx+\omega(y)\int_G f(x)\overline{\omega(x)}\,dx. \tag{4.20}$$

Since $\int_G|\omega(x)|^2dx\neq 0$ (see [3]), and

$$\begin{aligned}\int_G\int_G & f(xty)\overline{\omega(x)}\,dx\,d\mu(t)\\ &=\int_G\int_G\int_G f(xtyk)\overline{\omega(x)}\,dx\,d\mu(t)\,d\mu(k),\\ \int_G\int_G\int_G & f(x)\overline{\omega(xk^{-1}y^{-1}t^{-1})}\,dx\,d\mu(t)\,d\mu(k)\\ &=\int_G\int_G\int_G f(x)\omega(tykx^{-1})\,dx\,d\mu(t)\,d\mu(k)\\ &=\omega(y)\int_G f(x)\overline{\omega(x)}\,dx,\end{aligned} \tag{4.21}$$

we obtain that $f(x)=0$, for all $x\in G$. This completes the proof of Lemma 4.5. □

Lemma 4.6. *If $f,g \in C(G)$ with $f \neq 0$ constitute a solution of the functional equation (1.10), then there exists $\alpha \in \mathbb{C}\setminus\{0\}$ such that*

$$\int_G g(xty)\,d\mu(t) = g(x)g(y) + \alpha f(x)f(y), \quad x,y \in G. \tag{4.22}$$

Proof. Equation (1.10) shows that

$$f(x)\left[\int_G g(ytz)\,d\mu(t) - g(y)g(z)\right] = f(z)\left[\int_G g(xty)\,d\mu(t) - g(y)g(x)\right], \tag{4.23}$$

for all $x,y,z \in G$.

Let $a \in G$ such that $f(a) \neq 0$, $\Phi(x,y) = \int_G g(xty)\,d\mu(t) - g(x)g(y)$, then we get $f(x)\Phi(y,a) = f(a)\Phi(x,y)$ and $\Phi(x,y) = f(x)\Psi(y)$, where $\Psi(y) = \Phi(y,a)/f(a)$. Consequently, $f(x)f(y)\Psi(a) = f(y)f(a)\Psi(x)$, from which we see that there exists $\alpha \in \mathbb{C}$ such that $\alpha = \omega(a)/f(a)$ and $\int_G g(xty)\,d\mu(t) = g(y)g(x) + \alpha f(x)f(y)$, for all $x,y \in G$.

Now, by Lemma 4.5, $\alpha \neq 0$. This proves Lemma 4.6. □

Proof of Theorem 4.4. If f, g are linearly independent, by using Lemma 4.6, the matrix μ-spherical function defined in Theorem 3.2 is

$$\Phi(z) = \begin{pmatrix} g(z) & \alpha f(z) \\ f(z) & g(z) \end{pmatrix}, \quad z \in G. \tag{4.24}$$

Since $\alpha \neq 0$, then we can diagonalize $\Phi(z)$ as follows:

$$\Phi(z) = \begin{pmatrix} g(z)+\beta f(z) & 0 \\ 0 & g(z)-\beta f(z) \end{pmatrix}, \tag{4.25}$$

where $\beta^2 = \alpha$. This implies that $g(z)+\beta f(z) = \Phi_1$ and $g(z)-\beta f(z) = \Phi_2$ are μ-spherical functions. Consequently, we obtain case (ii). The rest of the proof is obvious. □

4.4. Solutions of (1.11)

Theorem 4.7. *The complete list of functions $f,g \in C(G)\setminus\{0\}$ satisfying the functional equation (1.11) consists of the following two cases, where Φ_1,Φ_2, $(\Phi_1 \neq \Phi_2)$, Φ are positive definite μ-spherical functions and $\alpha,\beta \in \mathbb{C}\setminus\{0,\pm i\}$:*

(i) $f = (1/(1+\alpha^2))\Phi$, $g = \alpha\Phi/(1+\alpha^2)$

(ii) $f = (\beta\Phi_1 + \beta^{-1}\Phi_2)/(\beta+\beta^{-1})$, $g = (\Phi_1 - \Phi_2)/(\beta+\beta^{-1})$.

To solve (1.11) we need the following lemma.

Lemma 4.8. *If $f,g \in C(G)$ constitute a solution of the functional equation (1.11), then there exists $\alpha \in \mathbb{C}$ such that*

$$\int_G g(xty)\,d\mu(t) = f(x)g(y) + g(x)f(y) + \alpha g(x)g(y), \quad x,y \in G. \tag{4.26}$$

Proof of Lemma 4.8. Let f, g satisfy (1.11). A small computation shows that

$$g(x)\left(\int_G g(ytz)\,d\mu(t) - f(y)g(z) - g(y)f(z)\right) = g(z)\left(\int_G g(xty)\,d\mu(t) - g(x)f(y) - f(x)g(y)\right). \tag{4.27}$$

The case $g = 0$ is trivial. Now we assume that $g \neq 0$. Let $a \in G$ such that $g(a) \neq 0$ and $\Phi(x,y) = \int_G g(xty)\,d\mu(t) - f(x)g(y) - g(x)f(y)$, then we get $g(x)\Phi(y,a) = g(a)\Phi(x,y)$ and $\Phi(x,y) = g(x)\Psi(y)$, where $\Psi(y) = \Phi(y,a)/g(a)$. Consequently, for all $x,y \in G$, we have $g(a)g(y)\Psi(x) = g(x)g(y)\Psi(a)$, which proves that $g(a)\Psi(x) = g(x)\Psi(a)$ and $\Psi(x) = \alpha g(x)$, where $\alpha = \Psi(a)/g(a)$, from which we get $\Phi(x,y) = \alpha g(x)g(y)$. This completes the proof of Lemma 4.8. □

Proof of Theorem 4.7. If f, g are linearly independent, then by using Lemma 4.8, the matrix μ-spherical function defined in Theorem 3.2 is

$$\begin{pmatrix} f(z) & g(z) \\ g(z) & f(z)+\alpha g(z) \end{pmatrix}, \tag{4.28}$$

and from Lemma 4.5, $\alpha \in \mathbb{C}\backslash\{0,\pm i\}$, then we may diagonalize $\Phi(z)$ as follows:

$$\Phi(z) = \begin{pmatrix} f(z)+\left(\dfrac{\alpha}{2}+\sqrt{\dfrac{\alpha^2}{4}+1}\right)g(z) & 0 \\ 0 & f(z)+\left(\dfrac{\alpha}{2}-\sqrt{\dfrac{\alpha^2}{4}+1}\right)g(z) \end{pmatrix}. \tag{4.29}$$

Now if we take $\beta = (\alpha/2+\sqrt{\alpha^2/4+1})$, by a small computation, we produce the solution formulas of (1.11). □

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