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TRAVELLING WAVE SOLUTIONS TO SOME PDEs OF MATHEMATICAL PHYSICS

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Nonlinear operations such as multiplication of distributions are not allowed in the classical theory of distributions. As a result, some ambiguities arise when we want to solve nonlinear partial differential equations such as differential equations of elasticity and multifluid flows, or some new cosmological models such as signature changing space-times. Colombeau's new theory of generalized functions can be used to remove these ambiguities. In this paper, we first consider a simplified model of elasticity and multifluid flows in the framework of Colombeau's theory and show how one can handle such problems, investigate their jump conditions, and resolve their ambiguities. Then we consider as a new proposal the case of cosmological models with signature change and use Colombeau's theory to solve Einstein equation for the beginning of the Universe.

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1. Introduction. Classical theory of distributions, based on Schwartz-Sobolev theory of distributions, does not allow nonlinear operations of distributions [4]. In Colombeau's theory, a mathematically consistent way of multiplying distributions is proposed. Colombeau's motivation is the inconsistency in multiplication and differentiation of distributions. Take, as it is given in the classical theory of distributions,

$$\theta^n = \theta \quad \forall n = 2, 3, \ldots, \tag{1.1}$$

where θ is the Heaviside step function. Differentiation of (1.1) gives

$$n\theta^{n-1}\theta' = \theta'. \tag{1.2}$$

Taking $n = 2$, we obtain $2\theta\theta' = \theta'$. Multiplication by θ gives $2\theta^2\theta' = \theta\theta'$. Using (1.2), it follows that

$$\frac{2}{3}\theta' = \frac{1}{2}\theta' \tag{1.3}$$

which is unacceptable because $\theta' \neq 0$. The trouble arises at the origin being the unique singular point of θ and θ'. If one accepts to consider $\theta^n \neq \theta$ for $n = 2, 3, \ldots$, the inconsistency can be removed. The difference $\theta^n - \theta$, being infinitesimal, is the essence of Colombeau's theory of generalized functions. Colombeau considers $\theta(t)$ as a function with "microscopic structure" at $t = 0$, making θ not to be a sharp step function, but having a width ϵ. $\theta(t)$ can cross the normal axis at any value of τ where $0 < \tau < 1$. It is interesting to note that the behavior of $\theta^n(t)$ around $t = 0$ is not the same as $\theta(t)$, that is, $\theta^n(t) \neq \theta(t)$ around $t = 0$ [8]. In the following we give a short formulation of Colombeau's theory.

2. New generalized functions: Colombeau's algebra. Suppose that $\Phi \in D(\mathbb{R}^n)$ with $D(\mathbb{R}^n)$ the space of smooth (i.e., C^∞) C-valued test functions on $\mathbb{R}^n$. For $f : \mathbb{R}^n \to C$, not necessarily continuous, we define the smoothing process for f as one of the convolutions

$$\tilde{f}(x) := \int f(y)\Phi(y-x)d^n y, \tag{2.1}$$

or

$$\tilde{f}_\epsilon(x) := \int f(y)\frac{1}{\epsilon^n}\Phi\left(\frac{y-x}{\epsilon}\right)d^n y. \tag{2.2}$$

This smoothing procedure is valid for distributions too. Take the distribution R, then by smoothing of R we mean one of the two convolutions (2.1) or (2.2), with f replaced by R. Now we can perform the product Rf of the distribution R with the discontinuous function f through the action of the product on a test function Ψ as follows

$$(Rf, \Psi) = \lim_{\epsilon \to 0} \int \tilde{R}_\epsilon(x)\tilde{f}_\epsilon(x)\Psi(x)d^n x. \tag{2.3}$$

The multiplication so defined does not coincide with the ordinary multiplication even for continuous functions. To resolve this difficulty consider one-parameter families (f_ϵ) of C^∞ functions used to construct the algebra

$$\begin{aligned}\mathcal{E}_M(\mathbb{R}^n) = \Big\{(f_\epsilon) \mid f_\epsilon \in C^\infty(\mathbb{R}^n)\ \forall K \subset \mathbb{R}^n \text{ compact},\\ \forall \alpha \in \mathbb{N}^n\ \exists N \in \mathbb{N},\ \exists \eta > 0,\ \exists c > 0 : \sup_{x \in K}\left|D^\alpha f_\epsilon(x)\right| \le c\epsilon^{-N}\ \forall 0 < \epsilon < \eta\Big\},\end{aligned} \tag{2.4}$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}}, \tag{2.5}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n. \tag{2.6}$$

Accordingly, C^∞-functions are embedded into $\mathcal{E}_M(\mathbb{R}^n)$ as constant sequences. Now, we have to identify different embeddings of C^∞ functions.

Take a suitable ideal $\mathcal{N}(\mathbb{R}^n)$ defined as

$$\begin{aligned}\mathcal{N}(\mathbb{R}^n) = \Big\{(f_\epsilon) \mid (f_\epsilon) \in \mathcal{E}_M(\mathbb{R}^n)\ \forall K\mathbb{R}^n \text{ compact},\\ \forall \alpha \in \mathbb{N}^n,\ \forall N \in \mathbb{N}\ \exists \eta > 0,\ \exists c > 0 : \sup_{x \in K}\left|D^\alpha f_\epsilon(x)\right| \le c \in^N\ \forall 0 < \epsilon < \eta\Big\},\end{aligned} \tag{2.7}$$

containing negligible functions such as

$$f(x)-\int d^n y \frac{1}{\epsilon^n}\varphi\left(\frac{y-x}{\epsilon}\right)f(y). \tag{2.8}$$

Now, the Colombeau's algebra $\mathcal{G}(\mathbb{R}^n)$ is defined as

$$\mathcal{G}(\mathbb{R}^n)=\frac{\mathcal{E}_M(\mathbb{R}^n)}{\mathcal{N}(\mathbb{R}^n)}. \tag{2.9}$$

A Colombeau's generalized function is thus a moderate family $(f_\epsilon(x))$ of C^∞ functions modulo negligible families. Two Colombeau's objects (f_ϵ) and (g_ϵ) are said to be associate (written as $(g_\epsilon)\approx(f_\epsilon)$) if

$$\lim_{\epsilon\to 0}\int d^n x\left(f_\epsilon(x)-g_\epsilon(x)\right)\varphi(x)=0 \quad \forall\varphi\in D(\mathbb{R}^n). \tag{2.10}$$

For example, if $\varphi(x)=\varphi(-x)$, then $\delta\theta\approx(1/2)\delta$, where δ is Dirac delta function and θ is Heaviside step function. Moreover, we have in this algebra $\theta^n\approx\theta$ and not $\theta^n=\theta$. For an extensive introduction to Colombeau's theory, see [1, 2, 3, 6, 7, 8, 9].

3. Theory of elasticity and Colombeau's algebra. In this section, we consider a simplified model of elasticity to apply new generalized functions for investigating jump conditions of nonlinear PDEs arising in this theory. Jump conditions are important in elasticity since these conditions contain information about the behavior of the system on the boundary and wavefront. On the other hand, a part of the structural properties about the medium are contained in these jump conditions. In the system of elasticity, Hooke's law in terms of the stress σ can be expressed as $(d/dt)\sigma=k^2u_x$ where $d/dt=\partial/\partial t+u(\partial/\partial x)$ and lowercase indices show derivatives with respect to these indices. Now the equations of system of elasticity are

$$\begin{aligned}
\rho_t+(\rho u)_x &\approx 0 \quad \text{balance of mass},\\
(\rho u)_t+(\rho u^2)_x &\approx \sigma_x \quad \text{balance of momentum},\\
\sigma_t+u\sigma_x &\approx k^2u_x \quad \text{Hooke's law},
\end{aligned} \tag{3.1}$$

where $\rho=$ density, $u=$ velocity, and $k^2=$ constant. Equations (3.1) are stated with three associations since we know this statement is a faithful generalization of the concept of weak solutions of systems in conservative form.

3.1. Jump conditions. Now we want to consider jump conditions for this model of elasticity. For this end we seek travelling waves solutions for (3.1) of the form

$$\begin{aligned}
u(x,t)&=(\Delta u)H(x-ct)+u_l,\\
\sigma(x,t)&=(\Delta\sigma)K(x-ct)+\sigma_l,\\
\rho(x,t)&=(\Delta\rho)L(x-ct)+\rho_l,
\end{aligned} \tag{3.2}$$

with H, K, and L three Heaviside generalized functions and Δu, $\Delta\sigma$, and $\Delta\rho$ the differences between two values of the velocity, stress, and density in two different sides of jump surface (wavefront). Also u_l, σ_l, and ρ_l are the values of the corresponding quantities in the left side of the jump surface. Putting (3.2) into the first equation of (3.1), we get (assuming that $\Delta\rho \neq 0$)

$$c - u_l = \Delta u + \rho_l \frac{\Delta u}{\Delta \rho}, \tag{3.3}$$

where c is the speed of shock wave in the medium. The second equation of (3.1) gives

$$(c - u_l - \Delta u)(u_l \Delta\rho + \rho_l \Delta u + \Delta\rho\Delta u) = u_l \rho_l \Delta u - \Delta\sigma. \tag{3.4}$$

These two equations are exactly the classical Rankine-Hugoniot jump conditions [5], since these equations are in conservative form. Putting (3.2) into the last equation of (3.1), we get

$$c - u_l = A\Delta u - k^2 \frac{\Delta u}{\Delta \sigma}, \tag{3.5}$$

$$HK' \approx A\delta, \tag{3.6}$$

where A is a real number. In derivation of (3.5) we have used the fact that Dirac δ-function and Heaviside step functions are linearly independent so the terms containing delta function in two different sides of equation must be equal. Now (3.3), (3.4), and (3.5) can be rewritten as

$$\begin{gathered} c = u_l + A\Delta u - k^2 \frac{\Delta u}{\Delta \sigma}, \\ k^2 \frac{\Delta u}{\Delta \sigma} - \frac{1}{2}\left[\frac{1}{\rho_l} + \frac{1}{\rho_r}\right]\frac{\Delta u}{\Delta \sigma} = \left[A - \frac{1}{2}\right]\Delta u, \\ \rho_l \rho_r (\Delta u)^2 = -\Delta\sigma\Delta\rho. \end{gathered} \tag{3.7}$$

In these equations ρ_r and ρ_l are the values of ρ in the right and left-hand side of the wavefront and $[F]$ is the jump of any quantity F. As usual, we find that the jump conditions of (3.1) depend on an arbitrary parameter, the real number A. Since A is an arbitrary parameter, we encounter infinite number of possible jump conditions, each of which gives a different physical interpretation. But physical phenomena must have single interpretation, so we must remove this ambiguity and it is interesting that this ambiguity arises because of difficulties inherent in the classical theory of distributions.

3.2. Resolution of the ambiguities. Now according to Colombeau's theory we can state (3.1) in more precise form

$$\rho_t + (\rho u)_x = 0, \qquad (\rho u)_t + (\rho u^2)_x = \sigma_x, \qquad \sigma_t + u\sigma_x \approx k^2 u_x. \tag{3.8}$$

The first two equations of (3.8) are equivalent to

$$\rho_t + (\rho u)_x = 0, \qquad u_t + uu_x = \frac{1}{\rho}\sigma_x, \tag{3.9}$$

since $\rho \neq 0$. It is convenient to set $v = 1/\rho$, where v is called the specific volume. Then (3.9) takes the form

$$\begin{aligned} v_t + uv_x - vu_x &= 0, \\ u_t + uu_x - v\sigma_x &= 0. \end{aligned} \tag{3.10}$$

Now we can restate (3.2) in the following form:

$$\begin{aligned} u(x,t) &= (\Delta u)H(x-ct) + u_l, \\ \sigma(x,t) &= (\Delta\sigma)K(x-ct) + \sigma_l, \\ v(x,t) &= (\Delta v)M(x-ct) + v_l, \end{aligned} \tag{3.11}$$

with $H,K,M \approx \theta$, the Heaviside function. The first equation in (3.10) gives

$$(c - u_l - \Delta u H)M' + \Delta u H' M + \Delta u \frac{v_l}{\Delta v} H' = 0. \tag{3.12}$$

In this relation the prime denotes usual differentiation with respect to the argument of the functions. The jump condition for the first equation of (3.8) is

$$\frac{c - u_l}{\Delta u} = \frac{\rho_r}{\Delta\rho} \quad \text{with } \rho_r = \Delta\rho + \rho_l, \tag{3.13}$$

which can be written as

$$\frac{c - u_l}{\Delta u} = -\frac{v_l}{\Delta v}. \tag{3.14}$$

Then (3.12) gives

$$\left[\frac{v_l}{\Delta v} + H\right]M' = \left[\frac{v_l}{\Delta v} + M\right]H'. \tag{3.15}$$

Putting (3.11) into the second equation of (3.10), we can obtain

$$\frac{c - u_l}{\Delta u}H' - HH' + (\Delta v M + v_l)\frac{\Delta\sigma}{\Delta u^2}K' = 0. \tag{3.16}$$

The jump condition of the second equation in (3.8) gives

$$\Delta v = \frac{(\Delta u)^2}{\Delta\sigma}. \tag{3.17}$$

Now we can consider (3.14), (3.16), and (3.17) together to find

$$\left[\frac{v_l}{\Delta v} + H\right]H' = \left[\frac{v_l}{\Delta v} + M\right]K'. \tag{3.18}$$

Setting $\alpha = v_l/\Delta v > 0$, then (3.15) and (3.18) are the system

$$(\alpha+H)M' = (\alpha+M)H',\\ (\alpha+H)H' = (\alpha+M)K'. \tag{3.19}$$

Now these equation can be rewritten as

$$(\alpha+H)M' - H'H - \alpha H' = 0,\\ K' = \frac{\alpha+H}{\alpha+M}H'. \tag{3.20}$$

Since H and M are null on $(-\infty,0[$ and identical to 1 on $]0,+\infty)$, an application of the classical formula for the solution of ordinary differential equation

$$a(x)y' + b(x)y + c(x) = 0 \tag{3.21}$$

allows to compute M as a function of H from the first equation of (3.20). One finds $M = H$ since α is a constant. This method relies upon the extension to $\mathcal{G}$ for the classical study of ordinary differential equations of the above kind. One can check that in this case the classical formula makes sense and provides a unique solution in the sense of equality in $\mathcal{G}$. This approach can be considered as a particular case of a much deeper study of linear hyperbolic systems with coefficients in $\mathcal{G}$. There, one can prove the uniqueness of the solutions of the Cauchy problem; this argument of uniqueness gives at once the result that $M = H$. Then the second equation in (3.19) gives $K = H$. Now we have resolved the ambiguities

$$HK' = HH' = \frac{1}{2}\delta \tag{3.22}$$

and therefore $A = 1/2$.

As a conclusion to the above argument we can state the following theorem.

Theorem 3.1. *The system of two equations and three unknowns*

$$\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + (\rho u^2)_x - \sigma_x = 0, \tag{3.23}$$

is equivalent to the system ($v = 1/\rho$),

$$v_t + uv_x - vu_x = 0,\\ u_t + uu_x - v\sigma_x = 0. \tag{3.24}$$

Further, travelling waves of the form

$$w(x,t) = (\Delta w)H_w(x-ct) + w_l, \tag{3.25}$$

with $w = v,u,\sigma$ successively ($\Delta w, c, w_l \in \mathbb{R}$ and H_w a Heaviside generalized function), are solution of (3.18) if and only if $H_v = H_u = H_\sigma$ plus the classical jump condition of (3.18).

4. Theory of multifluid flows

4.1. Equations of multifluid flows. Consider a mixture of two fluids. Let p denote the pressure of the mixture, and ρ_i, u_i with $i = 1, 2$ denote the respective density and velocity of the fluid number i at each point (x, t), let α denote the volumic proportion of the fluid number 1 in the mixture where $0 \le \alpha \le 1$. Now equations of the system are

$$\begin{gathered}
(\alpha\rho_1)_t + (\alpha\rho_1 u_1)_x = 0 \quad \text{balance of mass for fluid 1,} \\
((1-\alpha)\rho_2)_t + ((1-\alpha)\rho_2 u_2)_x = 0 \quad \text{balance of mass for fluid 2,} \\
(\alpha\rho_1 u_1)_t + (\alpha\rho_1 u_1^2)_x + \alpha p_x = 0 \quad \text{balance of momentum for fluid 1,} \\
((1-\alpha)\rho_2 u_2)_t + ((1-\alpha)\rho_2 u_2^2)_x + (1-\alpha) p_x = 0 \quad \text{balance of momentum for fluid 2,} \\
\rho_1 \approx \rho_1(p), \qquad \rho_2 \approx \rho_2(p) \quad \text{equations of state of fluids 1 and 2,}
\end{gathered} \tag{4.1}$$

where the terms αp_x and $(1-\alpha)p_x$ show multiplications of distributions of the form $\theta\delta$ in the case of shock waves. Now we must state the four first equations with equality in $\mathcal{G}$. This means that we assume that their relevance in space-time volumes are smaller than the width of shock waves. Consider the following change of variables:

$$\begin{aligned}
r_1(x,t) &= \alpha(x,t)\rho_1(x,t), \\
r_2(x,t) &= (1-\alpha(x,t))\rho_2(x,t),
\end{aligned} \tag{4.2}$$

then the first four equations in (4.1) become

$$\begin{aligned}
(r_1)_t + (r_1 u_1)_x &= 0, \\
(r_2)_t + (r_2 u_2)_x &= 0, \\
(r_1 u_1)_t + (r_1 u_1^2)_x + \alpha p_x &= 0, \\
(r_2 u_2)_t + (r_2 u_2^2)_x + (1-\alpha) p_x &= 0.
\end{aligned} \tag{4.3}$$

The first and third equations of (4.3) give

$$r_1(u_1)_t + r_1(u_1^2)_x + \alpha p_x = 0, \tag{4.4}$$

and assuming that α is nowhere vanishing, we can define

$$\begin{aligned}
v_1 &= \frac{1}{r_1} = \frac{1}{\alpha\rho_1}, \\
v_2 &= \frac{1}{r_2} = \frac{1}{(1-\alpha)\rho_2}.
\end{aligned} \tag{4.5}$$

Therefore (4.3) can be written as

$$\begin{aligned}
(v_1)_t + u_1(v_1)_x - v_1(u_1)_x &= 0, \\
(v_2)_t + u_2(v_2)_x - v_2(u_2)_x &= 0, \\
(u_1)_t + u_1(u_1)_x + \alpha v_1 p_x &= 0, \\
(u_2)_t + u_2(u_2)_x + (1-\alpha) v_2 p_x &= 0.
\end{aligned} \tag{4.6}$$

4.2. Jump conditions. We want to consider travelling wave solutions for the set of nonlinear partial differential equations and investigate their jump conditions in the framework of Colombeau's theory. Let the solutions to these equations be of the forms:

$$\begin{aligned} u_i(x,t) &= \Delta u_i H_i(x-ct)+u_{il}, \quad i=1,2,\\ v_i(x,t) &= \Delta v_i M_i(x-ct)+v_{il}, \quad i=1,2,\\ p(x,t) &= \Delta p K(x-ct)+p_l,\\ \alpha(x,t) &= \Delta\alpha L(x-ct)+\alpha_l, \end{aligned} \tag{4.7}$$

where c, Δu_i, Δv_i, Δp, $\Delta\alpha$, u_{il}, p_{il}, and α_l are real numbers and $H_i, M_i, K, L \in \mathcal{G}(\mathbb{R}^n)$ are Heaviside generalized functions.

From the first equation of (4.7) we have

$$M_1' - \frac{H_1'}{a+H_1}M_1 - \frac{aH_1'}{a+H_1} = 0 \quad \text{if } a = \frac{v_{1l}}{\Delta v_1},\ \Delta v_1 \neq 0, \tag{4.8}$$

where one deduces $M_1 = H_1$. Similarly, the second equation of (4.7) yields $M_1 = H_1$. Therefore the two first equations of (4.6) can be written respectively as

$$\begin{aligned} M_1 &= H_1, \qquad c-u_{1l} = -v_{1l}\frac{\Delta u_1}{\Delta v_1},\\ M_2 &= H_2, \qquad c-u_{2l} = -v_{2l}\frac{\Delta u_2}{\Delta v_2}. \end{aligned} \tag{4.9}$$

Now the third and fourth equations of (4.1) according to (4.7) and (4.9) become

$$\begin{aligned} (\Delta\alpha L+\alpha_l)K' &= -\frac{(\Delta u_1)^2}{\Delta p\Delta v_1}H_1',\\ (1-\Delta\alpha L-\alpha_l)K' &= -\frac{(\Delta u_2)^2}{\Delta p\Delta v_2}H_2'. \end{aligned} \tag{4.10}$$

Adding the two equations of (4.10) and after integration, one obtains the relation

$$K = -\frac{(\Delta u_1)^2}{\Delta p\Delta v_1}H_1 + \frac{(\Delta u_2)^2}{\Delta p\Delta v_2}H_2, \tag{4.11}$$

which implies the jump relation

$$\frac{(\Delta u_1)^2}{\Delta v_1} + \frac{(\Delta u_2)^2}{\Delta v_2} = -\Delta p. \tag{4.12}$$

As a result we can consider the following theorem.

4.3. Theorem. Equation (4.7) is a solution of (4.6) if and only if the following relations hold

$$\begin{aligned}
M_1 &= H_1, \qquad c - u_{1l} = -v_{1l}\frac{\Delta u_1}{\Delta v_1},\\
M_2 &= H_2, \qquad c - u_{2l} = -v_{2l}\frac{\Delta u_2}{\Delta v_2},\\
K &= -\frac{(\Delta u_1)^2}{\Delta p \Delta v_1}H_1 + \frac{(\Delta u_2)^2}{\Delta p \Delta v_2}H_2,\\
(\Delta\alpha L + \alpha_l)K' &= -\frac{(\Delta u_1)^2}{\Delta p \Delta v_1}H_1'.
\end{aligned} \tag{4.13}$$

The last equation of (4.13) has the product LK' which yields an ambiguity in the framework of classical theory of distributions, but now in the framework of Colombeau's algebra we have

$$LK' = A\delta, \tag{4.14}$$

where A is a real number and therefore there is not any ambiguity in this new framework.

5. Cosmological models with signature change

5.1. Einstein equations for singular hypersurfaces. Assume two space-times manifolds M^+ and M^- with space-like or time-like boundaries Σ^+ and Σ^-. We want to glue these two space-time manifolds together. Coordinates on the two space-time manifolds are defined independently as x_+^μ and x_-^μ, and the metrics are denoted by $g^+_{\alpha\beta}(x_+^\mu)$ and $g^-_{\alpha\beta}(x_-^\mu)$. The induced metrics on the boundaries are called $g^+_{ij}(\xi_+^k)$ and $g^-_{ij}(\xi_-^k)$, where $\xi_\pm^k$ are intrinsic coordinates on $\Sigma^\pm$, respectively. Now to paste the manifolds together, we demand that the boundaries be isometric, having the same coordinates $\xi_-^k = \xi_+^k = \xi^k$. The identification $\Sigma_- = \Sigma_+ =: \Sigma$ gives the single-glued manifold $M = M_+ \bigcup M_-$. There are two methods of handling singular hypersurface Σ in general relativity. The most used method of Darmois-Israel, based on the Gauss-Codazzi decomposition of space-time, is handicapped through the junction conditions which make the formalism unhandy. For our purposes the distributional approach is the most suitable one [7]. In this formalism the whole space-time manifold, including the singular hypersurface, is treated with a unified metric without bothering about the junction conditions along the hypersurface. These conditions are shown to be automatically fulfilled as part of the field equations. In the distributional approach one chooses special coordinates which are continuous along the singular hypersurface to avoid nonlinear operations of distributions. Here, using Colombeau's algebra, which allows for nonlinear operations of distributions, we generalize the distributional method to the special case of signature changing cosmological models.

Consider as a simple model universe a space-time with the following Friedmann-Robertson-Walker FRW metric containing a step-like lapse function

$$ds^2 = -f(t)dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right), \tag{5.1}$$

where

$$f(t) = \theta(t) - \theta(-t), \tag{5.2}$$

$$a^2(t) = a_+^2(t)\theta(t) + a_-^2(t)\theta(-t). \tag{5.3}$$

We assume $[a] = a_+ - a_- = 0$ to achieve continuity of the metric on the surface of signature change. Note that we have assumed for simplicity $k_+ = k_- = k$. This metric describes a signature changing space-time with the singular surface $t = 0$. It describes a Riemannian space for $t < 0$ and a Lorentzian space-time for $t > 0$. We choose

$$\theta(t)|_{t=0} = \tau \quad \text{with } \tau > \frac{1}{2}. \tag{5.4}$$

Since $\theta(-t) = 1 - \theta(t)$, we have $\theta(-t) = 1 - \tau$ and

$$f(t)\big|_{t=0} = 2\tau - 1. \tag{5.5}$$

This value gives us the correct change of sign in going from $t < 0$ to $t > 0$. This "regularization" of $f(t)$ at $t = 0$ allows us to use operations such as $f(t)^{-1}$, $f^2(t)$, and $|f(t)|^{-1}$.

In what follows we consider $f(t)$ to be the regularized function $\tilde{f}_\epsilon$, defined according to Colombeau's algebra. Now, we are prepared to calculate the dynamics of the signature changing hypersurface in the line of distributional procedure [7]. First we calculate the relevant components of the Einstein tensor for the metric (5.1):

$$G_{tt} = -\frac{-3kf^3}{f^2a^2} - \frac{3f^2(\dot{a}^2)}{f^2a^4}, \tag{5.6}$$

$$G_{rr} = \frac{1}{1-kr^2}\left(\frac{2a\ddot{a}f}{f^2} - \frac{a\dot{a}\dot{f}}{f^2} + \frac{f(a\dot{a})^2}{a^2f^2} + \frac{kf^2}{f^2}\right), \tag{5.7}$$

$$G_{\theta\theta} = r^2\left(\frac{2a\ddot{a}f}{f^2} - \frac{a\dot{a}\dot{f}}{f^2} + \frac{f(a\dot{a})^2}{a^2f^2} + \frac{kf^2}{f^2}\right), \tag{5.8}$$

$$G_{\varphi\varphi} = r^2\sin^2\theta\left(\frac{2a\ddot{a}f}{f^2} - \frac{a\dot{a}\dot{f}}{f^2} + \frac{f(a\dot{a})^2}{a^2f^2} + \frac{kf^2}{f^2}\right). \tag{5.9}$$

According to the standard calculus of distributions, we have

$$\dot{f}(t) = \dot{\theta}(t) - \dot{\theta}(-t) = \delta(t) + \delta(-t) = 2\delta(t), \tag{5.10}$$

taking into account $\delta(-t) = \delta(t)$. Now, using Colombeau's algebra we can write (based on (4.14))

$$\theta(t)\delta(t) \approx \frac{1}{2}\delta(t). \tag{5.11}$$

Therefore we may write

$$f(t)\delta(t) = \theta(t)\delta(t) - \theta(-t)\delta(t) \approx \frac{1}{2}\delta(t) - \frac{1}{2}\delta(t) \approx 0. \tag{5.12}$$

In evaluating (5.6)–(5.9) we should take care of the following property of association. Having

$$AB \approx AC, \tag{5.13}$$

we are not allowed to conclude

$$B \approx C. \tag{5.14}$$

Since the time derivative of any discontinuous function F is given by

$$\dot{F} = \dot{F_+}\theta(t) + \dot{F_-}\theta(-t) + [F]\delta(t), \tag{5.15}$$

using the relations (5.6)–(5.9) we obtain for the singular parts of these equations

$$\begin{aligned}
\hat{G}_t^t &\approx 0,\\
\hat{G}_r^r &\approx \left(\frac{f[\dot{a}]}{f^2 a} - \frac{[\dot{a}]}{f^2 a}\right)\delta(t),\\
\hat{G}_\theta^\theta &\approx \left(\frac{f[\dot{a}]}{f^2 a} - \frac{[\dot{a}]}{f^2 a}\right)\delta(t),\\
\hat{G}_\varphi^\varphi &\approx \left(\frac{f[\dot{a}]}{f^2 a} - \frac{[\dot{a}]}{f^2 a}\right)\delta(t),
\end{aligned} \tag{5.16}$$

where multiplication of the distribution $\delta(t)$ with the generalised functions $1/f^2$ and f/f^2 is defined as in (2.3).

This is a rigorous calculation concerning the question of vanishing the left-hand side of the Einstein equation on the surface of signature change. Our calculation based on Coloumbeau's algebra shows definitely that there are nonvanishing terms on the left-hand side of the field equations related to the signature change surface. Now we have to look at the energy-momentum tensor on the right-hand side, its possible interpretation and consequences for the dynamics of the signature change surface.

According to [7] the complete energy-momentum tensor (with any kind of matter content) can be written as

$$T_{\mu\nu} = \theta(t)T_{\mu\nu}^+ + \theta(-t)T_{\mu\nu}^- + CS_{\mu\nu}\delta(t), \tag{5.17}$$

where $T_{\mu\nu}^\pm$ are energy-momentum tensors corresponding to Lorentzian and Euclidean regions, respectively, and C is a constant which can be obtained by taking the following pill box integration defining $S_{\mu\nu}$:

$$S_{\mu\nu} = \lim_{\Sigma\to 0}\int_{-\Sigma}^{\Sigma}\left(T_{\mu\nu} - g_{\mu\nu}\frac{\Lambda}{\kappa}\right)dn = \frac{1}{\kappa}\lim_{\Sigma\to 0}\int_{-\Sigma}^{\Sigma} G_{\mu\nu}\,dn. \tag{5.18}$$

Since

$$\begin{aligned}
\hat{T}_{\mu\nu} &= CS_{\mu\nu}\delta(\Phi(x)),\\
\int \hat{T}_{\mu\nu}\,dn &= CS_{\mu\nu}\int \delta(\Phi(x))\,dn = CS_{\mu\nu}\left|\frac{dn}{d\Phi}\right|,
\end{aligned} \tag{5.19}$$

we find

$$C=\left|\frac{d\Phi}{dn}\right|=\left|n^{\mu}\partial_{\mu}\Phi\right|, \tag{5.20}$$

where $\Phi = t = 0$ defines the singular surface Σ. The vector n_μ is normal to the surface Φ and n measures the distance along it. Using the metric (5.1), we obtain

$$C=\frac{1}{|f(t)|}. \tag{5.21}$$

The distributional part of the Einstein equation reads as follows:

$$\hat{G}_{\mu\nu}=\kappa\hat{T}_{\mu\nu}. \tag{5.22}$$

Using (5.16), (5.21), and (5.22), we obtain

$$\begin{aligned}
0 &\approx \frac{\kappa}{|f(t)|}S_t^t\delta(t),\\
\hat{G}_r^r &\approx \left(\frac{f[\dot{a}]}{f^2a}-\frac{[\dot{a}]}{f^2a}\right)\delta(t)\approx\frac{\kappa}{|f(t)|}S_r^r\delta(t),\\
\hat{G}_\theta^\theta &\approx \left(\frac{f[\dot{a}]}{f^2a}-\frac{[\dot{a}]}{f^2a}\right)\delta(t)\approx\frac{\kappa}{|f(t)|}S_\theta^\theta\delta(t),\\
\hat{G}_\varphi^\varphi &\approx \left(\frac{f[\dot{a}]}{f^2a}-\frac{[\dot{a}]}{f^2a}\right)\delta(t)\approx\frac{\kappa}{|f(t)|}S_\varphi^\varphi\delta(t).
\end{aligned} \tag{5.23}$$

Now using (2.3), we must define the multiplication of δ-distribution with the discontinuous functions $1/|f|$ and $1/f^2$. To this end we consider them as Colombeau's regularized functions:

$$\begin{aligned}
\tilde{G}_{1\epsilon}(t) &:= \delta_\epsilon(t)\left(\frac{1}{|f(t)|}\right)_\epsilon,\\
\tilde{G}_{2\epsilon}(t) &:= \delta_\epsilon(t)\left(\frac{1}{f^2(t)}\right)_\epsilon.
\end{aligned} \tag{5.24}$$

Now according to (2.3), these two multiplications are defined as follows:

$$\begin{aligned}
\left(\delta(t)\frac{1}{|f(t)|},\Psi\right) &:= \lim_{\epsilon\to0}\int\tilde{G}_{1\epsilon}(t)\Psi(t)dt,\\
\left(\delta(t)\frac{1}{f^2(t)},\Psi\right) &:= \lim_{\epsilon\to0}\int\tilde{G}_{2\epsilon}(t)\Psi(t)dt,
\end{aligned} \tag{5.25}$$

for any test function Ψ. Now we argue that $\tilde{G}_{1\epsilon}$ and $\tilde{G}_{2\epsilon}$ are associates in Colombeau's sense, that is,

$$\lim_{\epsilon\to0}\int\left(\tilde{G}_{1\epsilon}(t)-\tilde{G}_{2\epsilon}(t)\right)\Psi(t)dt=0. \tag{5.26}$$

This is correct for any test function Ψ because, although $\tilde{G}_{1\epsilon}$ and $\tilde{G}_{2\epsilon}$ are divergent at a common point, the difference in their "microscopic structure" at that point tends to zero for $\epsilon \to 0$. Therefore, we obtain from (5.23) the final form of the energy-momentum tensor of the singular surface, or the dynamics of, Σ:

$$\begin{aligned} S_t^t &\approx 0, \\ \kappa S_r^r &\approx \left(\frac{f[\dot{a}]}{a} - \frac{[\dot{a}]}{a}\right), \\ \kappa S_\theta^\theta &\approx \left(\frac{f[\dot{a}]}{a} - \frac{[\dot{a}]}{a}\right), \\ \kappa S_\varphi^\varphi &\approx \left(\frac{f[\dot{a}]}{a} - \frac{[\dot{a}]}{a}\right). \end{aligned} \tag{5.27}$$

Therefore the "energy-momentum" tensor of the singular hypersurface is

$$\begin{aligned} \kappa S_\mu^\nu &= \operatorname{diag}\left(0, \frac{f[\dot{a}]}{a} - \frac{[\dot{a}]}{a}, \frac{f[\dot{a}]}{a} - \frac{[\dot{a}]}{a}, \frac{f[\dot{a}]}{a} - \frac{[\dot{a}]}{a}\right) \\ &= \operatorname{diag}\left(0, 2[H_0](\tau - 1), 2[H_0](\tau - 1), 2[H_0](\tau - 1)\right), \end{aligned} \tag{5.28}$$

where we have used (5.5). In this equation all quantities are to be taken at $t = 0$, and H_0 is defined as

$$H_0 = \frac{\dot{a}}{a}\bigg|_{t=0}, \tag{5.29}$$

which is the familiar Hubble constant at the signature change surface. This is our nontrivial and nonexpected result. One may question the validity of Coloumbeau's algebra, although it sounds physically well motivated and based on good physical intuition. The above result shows that within this algebra it is not reasonable to assume that the energy momentum tensor at the singular hypersurface of signature change is vanishing, as is usually assumed in the literature. If we assume that $[\dot{a}] = 0$ (as is usually assumed in the literature), then (5.28) will give $S_\mu^\nu = 0$, but this is not necessary in general. Therefore the condition $[\dot{a}] = 0$ is not compulsory on the singular surface.

We have seen that the requirement of signature change leads to a very specific and nonvanishing form for the $S_{\mu\nu}$. Since the nonvanishing terms of $S_{\mu\nu}$ are related to the extrinsic curvature of the signature change surface, they tell us how it is embedded in the space-time. Therefore one should not be bothered about its matter interpretation. This form of the energy-momentum tensor of the singular hypersurface we have obtained set limits to the possible space-times emerged after signature change. As an example, we will consider in the next section the possibility of the emergence of de Sitter space-time after signature change.

5.2. Junction conditions for de Sitter manifold in the line of differential geometry. According to the Hartle-Hawking proposal, the universe after signature change should be a de Sitter universe (inflationary phase). We assume that the space-time after signature change is a de Sitter one. Consider now the following de Sitter metric with appropriate lapse function $f(t)$ in order to contain signature change at $t = 0$. The $t = \text{const}$ sections of this metric are surfaces of constant curvature $k = 1$:

$$ds^2 = -f(t)dt^2 + a^2(t)\left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\theta^2\right)\right), \tag{5.30}$$

where $f(t)$ is defined as in (5.2) and

$$a^2(t) = \alpha_+^2 \cosh^2(\alpha_+^{-1}t)\theta(t) + \alpha_-^2 \cos^2(\alpha_-^{-1}t)\theta(-t). \tag{5.31}$$

Since $[a] = 0$, we will have $\alpha_+ = \alpha_- := R$. Now, the Euclidean sector can be interpreted as S^4 with S^3 sections defined by $t = \text{const}$. The boundary of the Euclidean sector, defined by $t = 0$, is an S^3 having the radius $R = \alpha_- = H_0^{-1}$ which is the maximum value of $\alpha_- \cos(\alpha_-^{-1}t)$.

In the Lorentzian sector, the cosmological constant is given by $\Lambda = 3\alpha_+^{-2} = 3H_0^2$. The $t = \text{const}$ surfaces are S^3 with radius $\alpha_+ \cosh(\alpha_+^{-1}t)$ having the minimum value $R = \alpha_+ = H_0^{-1}$. Therefore, the following relation between the cosmological constant and the radius of the boundary is obtained:

$$\Lambda = \frac{3}{R^2}. \tag{5.32}$$

Following the same procedure as for the metric (5.1) and again using Colombeau's algebra, or simply using (5.28), we find for the elements of energy-momentum tensor of the hypersurface

$$\kappa S_\mu^\nu = \operatorname{diag}(0, \Pi, \Pi, \Pi), \tag{5.33}$$

where Π is defined as

$$\Pi = \left(\frac{2}{R}\tanh(R^{-1}t) - \frac{2}{R}\tan(R^{-1}t)\right)\Big|_{(t=0)(\tau-1)} = 0. \tag{5.34}$$

We therefore conclude that given the de Sitter metric in the form of (5.6), the energy-momentum tensor of the hypersurface of signature change defined by $t = 0$ vanishes. This is a familiar result that the previous authors usually assume from the beginning, but we obtain it as a special case depending on the form of the metric of space-time.

It may be useful to look at the Darmois-Israel approach. There, we have the following relation between the energy-momentum tensor of the singular hypersurface and the jump of the extrinsic curvature:

$$\kappa S_i^j = [K_i^j] - h_i^j[K], \tag{5.35}$$

where h_i^j is the three-metric of the singular hypersurface. The extrinsic curvature is defined as

$$K_{ij} = e_i^\mu e_j^\nu \nabla_\mu n_\nu, \tag{5.36}$$

where e_i, the mutually normal unit 4-vectors in signature changing surface Φ, are defined as

$$e_i^\mu = \frac{\partial x^\mu}{\partial \xi^i}, \quad i = 1, 2, 3. \tag{5.37}$$

ξ^i are coordinates adopted to the signature changing surface Σ and ∇_μ denotes the covariant derivative with respect to the 4-geometry. We then find for the nonvanishing components of extrinsic curvature in Lorentzian sector (with $f(t) = +1$)

$$K_i^{+i} = -\frac{1}{\alpha_+}\tanh(\alpha_+^{-1}t), \quad i = 1,2,3. \tag{5.38}$$

The corresponding components in the Euclidean sector are (with $f(t) = -1$)

$$K_i^{-i} = \frac{1}{\alpha_-}\tan(\alpha_-^{-1}t), \quad i = 1,2,3. \tag{5.39}$$

Now we obtain for the jump of the extrinsic curvature on the signature change surface

$$[K_i^i] \equiv (K_i^{+i} - K_i^{-i})|_{t=0} = \left(-\frac{1}{R}\tanh(R^{-1}t) - \frac{1}{R}\tan(R^{-1}t)\right)\Big|_{t=0} = 0, \quad i = 1,2,3. \tag{5.40}$$

Within the Darmois-Israel approach to signature change it is used to assume that the energy-momentum tensor of the singular hypersurface vanishes. Therefore, given the above result, the junction condition (5.35) is satisfied and it is concluded that the matching is possible. In contrast to this within the distributional approach, using Coloumbeau's algebra, we obtain in general a nonvanishing expression for the energy-momentum tensor $S_{\mu\nu}$ and no explicit junction condition. The Einstein equations written for the whole manifold imply the junction conditions. Only in the special case of the metric (5.30) the matching at $t = 0$ leads to $S_{\mu\nu} = 0$. One could require a matching along other sections corresponding to a nonmaximum radius of the Euclidean sector or a nonminimum radius of the Lorenzian sector. In this case the energy-momentum tensor of the singular surface may not be vanishing any more, and has to be checked in each case.

6. Conclusions. Since the classical theory of distributions is unable to handle nonlinear operations on distributions, Colombeau's theory of generalized functions gives a reasonable framework to do such nonlinear operations and as a result this theory can be used to remove ambiguities of classical theory. In this paper, we have used this new theory to solve the partial differential equations of two systems: a system of elasticity and a system of multifluid flow. Although physically this problem must have unambiguous travelling waves solutions, classical theory of distributions for such solutions encounter some ambiguities such as infinite number of jump conditions. Since jump conditions have most of the information on the boundary surface, infinite number of jump conditions physically is not acceptable, so one have to resolve this ambiguity. These ambiguities in the classical theory of distributions are unavoidable, but in new theory of generalized functions there are mathematically consistent way to remove these ambiguities. In the line of removing these ambiguities we arrived at the important result between Heaviside step function and Dirac δ-function, relation (4.14) with $A = 1/2$. One can do some numerical calculations based on these analytical considerations to find deeper insight in this subject, which is the title of our forthcoming paper.

On the other hand, based on the important result (4.14), one can consider the formalism of singular hypersurfaces in general relativity in the framework of distributional approach. We have considered such a problem for the case of cosmological models with signature change and we have found the dynamics of this hypersurface in Colombeau's algebra. As we have shown, using Coloumbeau's algebra we could show that the energy-momentum tensor of the hypersurface of signature change does not vanish in general, and its space components are proportional to the jump of the derivative of the scale factor, or to the jump of the Hubble parameter. For the special case of de Sitter space-time we have shown that this jump vanishes, and the matching along the $t = 0$ hypersurface, corresponding to the equator of the Euclidean sector, is possible. This is in agreement with the previous results based on the Darmois-Israel approach. One could try to do the matching along other sections or other metrics and compare the results with that of the Darmois-Israel approach. This will be done in a forthcoming paper.

References

[1] H. Balasin, *Colombeau's generalized functions on arbitrary manifolds*, preprint, 1996, http://arXiv.org/abs/gr-qc/9610017.

[2] C. J. S. Clarke, J. A. Vickers, and J. P. Wilson, *Generalized functions and distributional curvature of cosmic strings*, Classical Quantum Gravity **13** (1996), no. 9, 2485–2498.

[3] D. Hartley, R. W. Tucker, P. A. Tuckey, and T. Dray, *Tensor distributions on signature-changing space-times*, preprint, 1997, http://arXiv.org/abs/gr-qc/9701046.

[4] R. P. Kanwal, *Generalized Functions: Theory and Technique*, Mathematics in Science and Engineering, vol. 171, Academic Press, Florida, 1983.

[5] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, Pergamon Press, London, 1986.

[6] R. Mansouri and K. Nozari, *A new distributional approach to signature change*, Gen. Relativity Gravitation **32** (2000), no. 2, 253–269.

[7] K. Nozari and R. Mansouri, *The methods of gluing manifolds in general relativity*, J. Math. Phys. **43** (2002), no. 3, 1519–1535.

[8] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*, Pitman Research Notes in Mathematics Series, vol. 259, Longman Scientific & Technical, Harlow, 1992.

[9] J. P. Wilson, *Distributional curvature of time-dependent cosmic strings*, preprint, 1997, http:// arXiv.org/abs/gr-qc/9705032.

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