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COUNTING OCCURRENCES OF 132 IN AN EVEN PERMUTATION

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We study the generating function for the number of even (or odd) permutations on n letters containing exactly $r \geq 0$ occurrences of a 132 pattern. It is shown that finding this function for a given r amounts to a routine check of all permutations in $\mathfrak{S}_{2r}$.

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1. Introduction. Let $[n] = \{1,2,\dots,n\}$ and let $\mathfrak{S}_n$ denote the set of all permutations of $[n]$. We will view permutations in $\mathfrak{S}_n$ as words with n distinct letters in $[n]$. A pattern is a permutation $\sigma \in \mathfrak{S}_k$, and an occurrence of σ in a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ is a subsequence of π that is order equivalent to σ. For example, an occurrence of 213 is a subsequence $\pi_i\pi_j\pi_k$ $(1 \leq i < j < k \leq n)$ of π such that $\pi_j < \pi_i < \pi_k$. We denote by $\tau(\pi)$ the number of occurrences of τ in π, and we denote by $s_\sigma^r(n)$ the number of permutations $\pi \in \mathfrak{S}_n$ such that $\sigma(\pi) = r$.

In the last decade much attention has been paid to the problem of finding the numbers $s_\sigma^r(n)$ for a fixed $r \geq 0$ and a given pattern τ (see [1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 16, 17, 18, 19]). Most of the authors consider only the case $r = 0$, thus studying permutations *avoiding* a given pattern. Only a few papers consider the case $r > 0$, usually restricting themselves to patterns of length 3. Using two simple involutions (*reverse* and *complement*) on $\mathfrak{S}_n$, it is immediate that, with respect to being equidistributed, the six patterns of length three fall into the two classes $\{123,321\}$ and $\{132,213,231,312\}$. Noonan [13] proved that

$$s_{123}^1(n) = \frac{3}{n}\binom{2n}{n-3}. \tag{1.1}$$

Noonan and Zeilberger [14] suggested a general approach to the problem; they gave another proof of Noonan's result, and conjectured that

$$\begin{gathered} s_{123}^2(n) = \frac{59n^2+117n+100}{2n(2n-1)(n+5)}\binom{2n}{n-4}, \\ s_{132}^1(n) = \binom{2n-3}{n-3}. \end{gathered} \tag{1.2}$$

The second conjecture was proved by Bóna in [6] and the first conjecture was proved by Fulmek [8]. Noonan and Zeilberger conjectured that $s_\sigma^r(n)$ is P-recursive in n for any r and τ. It was proved by Bóna [4] for $\sigma = 132$. Mansour and Vainshtein [11] suggested a new approach to this problem in the case $\sigma = 132$, which allows one to get an explicit expression for $s_{132}^r(n)$ for any given r. More precisely, they presented an algorithm that computes the generating function $\sum_{n\geq 0} s_{132}^r(n)x^n$ for any $r \geq 0$.

Let π be any permutation. The number of *inversions* of π is given by $i_\pi = |\{(i,j) : \pi_i > \pi_j,\ i < j\}|$. The *signature* of π is given by $\operatorname{sign}(\pi) = (-1)^{i_\pi}$. We say π is an *even permutation* (resp., *odd permutation*) if $\operatorname{sign}(\pi) = 1$ (resp., $\operatorname{sign}(\pi) = -1$). We denote by E_n (resp., O_n) the set of all even (resp., odd) permutations in $\mathfrak{S}_n$. Clearly, $|E_n| = |O_n| = (1/2)n!$ for all $n \geq 2$.

We denote by $e_\sigma^r(n)$ (resp., $o_\sigma^r(n)$) the number of even (resp., odd) permutations $\pi \in E_n$ (resp., $\pi \in O_n$) such that $\sigma(\pi) = r$.

Apparently, for the first time the relation between even (odd) permutations and pattern-avoidance problem was suggested by Simion and Schmidt in [16] for $\sigma \in \mathfrak{S}_3$. In particular, Simion and Schmidt [16] proved that

$$\begin{aligned} e_{132}^0(n) &= \frac{1}{2(n+1)}\binom{2n}{n} + \frac{1}{n+1}\binom{n-1}{\frac{n-1}{2}}, \\ o_{132}^0(n) &= \frac{1}{2(n+1)}\binom{2n}{n} - \frac{1}{n+1}\binom{n-1}{\frac{n-1}{2}}, \end{aligned} \tag{1.3}$$

where $\binom{n-1}{(n-1)/2} = 0$ and n is an even number.

In this note, as a consequence of [12], we suggest a new approach to this problem in the case of even (or odd) permutations where $\sigma = 132$, which allows one to get an explicit expression for $e_{132}^r(n)$ for any given r. More precisely, we present an algorithm that computes the generating functions $E_r(x) = \sum_{n\geq 0} e_{132}^r(n)x^n$ and $O_r(x) = \sum_{n\geq 0} o_{132}^r(n)x^n$ for any $r \geq 0$. To get the result for a given r, the algorithm performs certain routine checks for each element of the symmetric group $\mathfrak{S}_{2r}$. The algorithm has been implemented in C, and yields explicit results for $0 \leq r \leq 6$.

2. Definitions and preliminary results. Recall the definitions (*kernel permutation*, *kernel cell decomposition*, *feasible cells*, *shapes*, *kernel shapes*, and *cells*) and the notations (s *the size of the kernel*, c *the capacity of the kernel*, and f *the number of the feasible cells in the kernel cell decomposition*) which are given in [12]. In this section we describe how the cell decomposition approach (see [12]) can be determined by the generating function for the number of even permutations which contain the pattern 132 exactly r times.

Let π be any permutation with a kernel permutation ρ, and assume that the feasible cells of the kernel cell decomposition associated with ρ are ordered linearly according to $\prec$, $C^1, C^2, \ldots, C^{f(\rho)}$ (see [12, Lemma 3]). Let d_j be the size of C^j. For example, let $\pi = 67382451$ with kernel permutation $\rho = 1423$, then $d_1 = 2$, $d_2 = 1$, $d_3 = 0$, and $d_4 = 1$ (see Figure 2.1).

We denote by $l_j(\rho)$ the number of the entries of ρ that lie to the north-west from C^j or lie to the south-east from C^j. For example, let $\rho = 1423$, as on Figure 2.1, then $l_1(\rho) = 3$, $l_2(\rho) = 2$, $l_3(\rho) = 3$, and $l_4(\rho) = 4$. Clearly, $l_1(\rho) = s(\rho) - 1$ and $l_{f(\rho)} = s(\rho)$ for any nonempty kernel permutation ρ. Define $\operatorname{sign}(C) = (-1)^{21(C)}$ for any cell C.

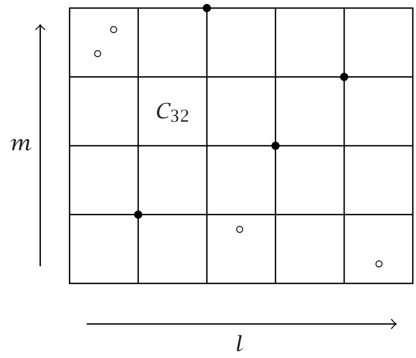

FIGURE 2.1. Kernel cell decomposition for $\pi = 67382451 \in \mathfrak{S}(1423)$.

LEMMA 2.1. *For any permutation π with a kernel permutation ρ,*

$$\operatorname{sign}(\pi) = (-1)^{(\sum_{1\le i\le j\le f(\rho)} d_i d_j + \sum_{j=1}^{f(\rho)} d_j l_j(\rho))} \cdot \operatorname{sign}(\rho) \cdot \prod_{j=1}^{f(\rho)} \operatorname{sign}(C^j). \tag{2.1}$$

PROOF. To verify this formula, we count the number of occurrences of the pattern 21 in π. There are four possibilities for an occurrence of 21 in π. The first possibility is an occurrence in one of the cells C^j, so in this case there are $\sum_{j=1}^{f(\rho)} 21(C^j)$ occurrences. The second possibility is an occurrence in $\ker\pi$, so there are $21(\rho)$ occurrences. The third possibility is an occurrence of two elements, of which the first belongs to $\ker\pi$ and the second belongs to C^i, so there are $\sum_{j=1}^{f(\rho)} d_j l_j(\rho)$ occurrences (see [12, Lemmas 4 and 5]). The fourth possibility is an occurrence of two elements, of which the first belongs to C^i and the second belongs to C^j where $i < j$ (see [12, Lemmas 4 and 5]), so there are $\sum_{1\le i<j\le f(\rho)} d_i d_j$ occurrences. Therefore,

$$\operatorname{sign}(\pi) = (-1)^{\sum_{j=1}^{f(\rho)} 21(C^j)} (-1)^{21(\rho)} (-1)^{\sum_{j=1}^{f(\rho)} d_j l_j(\rho)} (-1)^{\sum_{1\le i<j\le f(\rho)} d_i d_j}, \tag{2.2}$$

equivalently, $\operatorname{sign}(\pi) = (-1)^{(\sum_{1\le i\le j\le f(\rho)} d_i d_j + \sum_{j=1}^{f(\rho)} d_j l_j(\rho))} \cdot \operatorname{sign}(\rho) \cdot \prod_{j=1}^{f(\rho)} \operatorname{sign}(C^j)$. $\square$

We say that the vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a *binary vector* if $v_i \in \{0,1\}$ for all i, $1 \le i \le n$. We denote the set of all binary vectors of length n by $\mathcal{B}^n$. For any $\mathbf{v} \in \mathcal{B}^n$, we define $|\mathbf{v}| = v_1 + v_2 + \cdots + v_n$. For example, $\mathcal{B}^2 = \{(0,0),(0,1),(1,0),(1,1)\}$ and $|(1,1,0,0,1)| = 3$.

For any kernel permutation ρ and for $a = \pm 1$, we denote by $X_a(\rho)$ (resp., Y_a^ρ) the set of all the binary vectors $\mathbf{v} \in \mathcal{B}^{f(\rho)}$ such that $(-1)^{|\mathbf{v}|+s(\rho)} = a$ (resp., $(-1)^{|\mathbf{v}|} = a$). For any $\mathbf{v} \in \mathcal{B}^{f(\rho)}$, we define

$$z_\rho(\mathbf{v}) = (-1)^{\sum_{1\le i<j\le f(\rho)} v_i v_j + \sum_{j=1}^{f(\rho)} l_j(\rho) v_j} \operatorname{sign}(\rho). \tag{2.3}$$

Letting ρ be any kernel permutation and $\mathbf{v} = (v_1, v_2, \ldots, v_{f(\rho)}) \in \mathcal{B}^{f(\rho)}$, we denote by $\mathfrak{S}(\rho;\mathbf{v})$ the set of all permutations of all sizes with kernel permutation ρ such that the corresponding cells C^j satisfy $(-1)^{d_j} = (-1)^{v_j}$; in such a context $\mathbf{v}$ is called a *length argument vector* of ρ. By definitions, the following result holds immediately.

Lemma 2.2. *For any kernel permutation ρ,*

$$\mathfrak{S}(\rho) = \bigcup_{\mathbf{v}\in\mathcal{B}^{f(\rho)}} \mathfrak{S}(\rho;\mathbf{v}). \tag{2.4}$$

Letting ρ be any kernel permutation and letting

$$\mathbf{v} = (v_1, v_2, \dots, v_{f(\rho)}), \qquad \mathbf{u} = (u_1, u_2, \dots, u_{f(\rho)}) \in \mathcal{B}^{f(\rho)}, \tag{2.5}$$

we denote by $\mathfrak{S}(\rho;\mathbf{v},\mathbf{u})$ the set of all permutations in $\mathfrak{S}(\rho;\mathbf{v})$ such that the corresponding cells C^j satisfy $\operatorname{sign}(C^j) = 1$ if and only if $u_j = 0$; in such a context $\mathbf{u}$ is called a *signature argument vector* of ρ. By Lemma 2.2, the following result holds immediately.

Lemma 2.3. *For any kernel permutation ρ,*

$$\mathfrak{S}(\rho) = \bigcup_{\mathbf{v}\in\mathcal{B}^{f(\rho)}} \mathfrak{S}(\rho;\mathbf{v}) = \bigcup_{\mathbf{v}\in\mathcal{B}^{f(\rho)}} \bigcup_{\mathbf{u}\in\mathcal{B}^{f(\rho)}} \mathfrak{S}(\rho;\mathbf{v},\mathbf{u}). \tag{2.6}$$

For any $a, b \in \{0,1\}$ we define

$$H_r(a,b) = \begin{cases} \frac{1}{2}\left(E_r(x) + (-1)^a E_r(-x)\right), & \text{if } b = 0, \\ \frac{1}{2}\left(O_r(x) + (-1)^a O_r(-x)\right), & \text{if } b = 1. \end{cases} \tag{2.7}$$

By definitions, the following result holds immediately.

Lemma 2.4. *Let $a, b \in \{0,1\}$. Then the generating function for all permutations π such that $132(\pi) = r$, $(-1)^{|\pi|} = (-1)^a$, and $\operatorname{sign}(\pi) = (-1)^b$ is given by $H_r(a,b)$, where $|\pi|$ denotes the length of the permutation π.*

3. Main theorem. The main result of this note can be formulated as follows. Denote by K the set of all kernel permutations, and by K_t the set of all kernel shapes for permutations in $\mathfrak{S}_t$. Letting ρ be any kernel permutation, for any $a, b \in \{0,1\}$ and any $r_1, \dots, r_{f(\rho)}$, we define

$$L^{\rho}_{r_1,\dots,r_{f(\rho)}}(a,b) = \sum_{\mathbf{v}\in X^{\rho}_{(-1)^a}} \sum_{\mathbf{u}\in Y^{\rho}_{(-1)^b z_\rho(\mathbf{v})}} \prod_{j=1}^{f(\rho)} H_{r_j}(v_j, u_j). \tag{3.1}$$

Theorem 3.1. *Let $r \geq 1$. For any $a, b \in \{0,1\}$,*

$$H_r(a,b) = \sum_{\rho\in K_{2r+1}} \sum_{r_1+\cdots+r_{f(\rho)}=r-c(\rho),\, r_j\geq 0} L^{\rho}_{r_1,\dots,r_{f(\rho)}}(a,b). \tag{3.2}$$

Proof. We fix a kernel permutation $\rho \in K_{2r+1}$, a length argument vector $\mathbf{v} = (v_1, \dots, v_{f(\rho)}) \in X_{(-1)^a}(\rho)$, and a signature argument vector $\mathbf{u} = (u_1, \dots, u_{f(\rho)}) \in Y^{f(\rho)}_{(-1)^b z_\rho(\mathbf{v})}$.

Recall that the kernel ρ of any π contains exactly $c(\rho)$ occurrences of 132. The remaining $r-c(\rho)$ occurrences of 132 are distributed among the feasible cells of the kernel cell decomposition of π. By [12, Theorem 2], each occurrence of 132 belongs entirely to one feasible cell, and the occurrences of 132 in different cells do not influence one another.

Let π be any permutation such that $132(\pi) = r$, $\operatorname{sign}(\pi) = (-1)^b$, and $(-1)^{|\pi|} = (-1)^a$, together with a kernel permutation ρ, length argument vector $\mathbf{v}$, and signature argument vector $\mathbf{u}$. Then by Lemma 2.3, the cells C^j satisfy the following conditions:

(1) $v_j = 0$ if and only if d_j is an even number,
(2) $u_j = 0$ if and only if $\operatorname{sign}(C^j) = 1$,
(3) $(-1)^{v_1+\cdots+v_{f(\rho)}+s(\rho)} = (-1)^a$,
(4) $(-1)^{u_1+\cdots+u_{f(\rho)}} z_\rho(\mathbf{v}) = (-1)^b$.

Therefore, by Lemma 2.4, this contribution gives

$$x^{s(\rho)} \sum_{r_1+\cdots+r_{f(\rho)}=r-c(\rho),\, r_j\ge 0} \; \prod_{j=1}^{f(\rho)} H_{r_j}(v_j,u_j). \tag{3.3}$$

Hence by Lemma 2.3 and [12, Theorem 1], summing over all the kernel permutations $\rho \in K_{2r+1}$, length argument vectors $\mathbf{v} \in X_{(-1)^a}(\rho)$, and signature argument vectors $\mathbf{u} \in Y^{f(\rho)}_{(-1)^b z_\rho(\mathbf{v})}$, then we get the desired result. $\square$

Theorem 3.1 provides a finite algorithm for finding $E_r(x)$ and $O_r(x)$ for any given $r \ge 0$, since we only have to consider all permutations in $\mathfrak{S}_{2r+1}$ and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following theorem.

Theorem 3.2. *The only kernel permutation of capacity $r \ge 1$ and size $2r+1$ is*

$$\rho = 2r-1, 2r+1, 2r-3, 2r, \ldots, 2r-2j-3, 2r-2j, \ldots, 1, 4, 2. \tag{3.4}$$

Its parameters are given by $s(\rho) = 2r+1$, $c(\rho) = r$, $f(\rho) = r+2$, $\operatorname{sign}(\rho) = -1$, and $z_\rho(v_1,\ldots,v_{r+2}) = (-1)^{(1+v_{r+2}+\sum_{1\le i<j\le r+2} v_i v_j)}$.

Proof. The first part of the theorem holds by [12, Proposition]. Besides, by using the form of ρ we get $s(\rho) = 2r+1$, $c(\rho) = r$, $f(\rho) = r+2$, $\operatorname{sign}(\rho) = -1$, and $l_j(\rho) = 2r$ for all $j = 1,2,\ldots,r+1$ and $l_{r+2}(\rho) = 2r+1$. Therefore, $z_\rho(v_1,\ldots,v_{r+2}) = (-1)^{(1+v_{r+2}+\sum_{1\le i<j\le r+2} v_i v_j)}$. $\square$

By this theorem, it suffices to search only permutations in $\mathfrak{S}_{2r}$. Below we present several explicit calculations.

3.1. The case $r = 0$. We start from the case $r = 0$. Observe that Theorem 3.1 remains valid for $r = 0$, provided that the left-hand side of (3.2) for $a = b = 0$ is replaced by $H_r(0,0) - 1 = (1/2)(E_r(x) + E_r(-x)) - 1$; subtracting 1 here accounts for the empty permutation. So, we begin with finding kernel shapes for all permutations in $\mathfrak{S}_1$.

The only shape obtained is $\rho_1 = 1$, and it is easy to see that $s(\rho_1) = 1$, $c(\rho_1) = 0$, $f(\rho_1) = 2$,

$$\begin{gathered} X_1(\rho_1) = Y_{-1} = \{(1,0),(0,1)\}, \qquad X_{-1}(\rho_1) = Y_1 = \{(0,0),(1,1)\}, \\ z_{\rho_1}(0,0) = z_{\rho_1}(1,0) = z_{\rho_1}(1,1) = -z_{\rho_1}(0,1) = 1. \end{gathered} \tag{3.5}$$

Therefore, (3.2) for $a = b = 0$ gives

$$\begin{aligned} \frac{1}{2}(E_0(x)+E_0(-x))-1 = {} & xH_0(1,0)H_0(0,0)+xH_0(1,1)H_0(0,1) \\ & +xH_0(1,0)H_0(0,1)+xH_0(1,1)H_0(0,0); \end{aligned} \tag{3.6}$$

(3.2) for $a = 1$ and $b = 0$ gives

$$\frac{1}{2}(E_0(x)-E_0(-x)) = xH_0^2(0,0)+xH_0^2(0,1)+xH_0^2(1,0)+xH_0^2(1,1); \tag{3.7}$$

(3.2) for $a = 0$ and $b = 1$ gives

$$\begin{aligned} \frac{1}{2}(O_0(x)+O_0(-x)) = {} & xH_0(1,1)H_0(0,0)+xH_0(1,0)H_0(0,1) \\ & +xH_0(0,0)H_0(1,0)+xH_0(0,1)H_0(1,1); \end{aligned} \tag{3.8}$$

and (3.2) for $a = b = 1$ gives

$$\frac{1}{2}(O_0(x)-O_0(-x)) = 2xH_0(0,1)H_0(0,0)+2xH_0(1,1)H_0(1,0). \tag{3.9}$$

Our present aim is to find explicitly $E_0(x)$ and $O_0(x)$, thus we need the following notation. We define

$$M_r(x) = E_r(x)-O_r(x), \qquad F_r(x) = E_r(x)+O_r(x) \tag{3.10}$$

for all $r \geq 0$. Clearly,

$$\begin{aligned} H_r(0,0)-H_r(0,1) &= \frac{1}{2}(M_r(x)+M_r(-x)), \\ H_r(0,0)+H_r(0,1) &= \frac{1}{2}(F_r(x)+F_r(-x)), \\ H_r(1,0)-H_r(1,1) &= \frac{1}{2}(M_r(x)-M_r(-x)), \\ H_r(1,0)+H_r(1,1) &= \frac{1}{2}(F_r(x)-F_r(-x)), \end{aligned} \tag{3.11}$$

for all $r \geq 0$. Therefore, by subtracting (resp., adding) (3.8) and (3.6), and by subtracting (resp., adding) (3.9) and (3.7), we get

$$\begin{gathered} M_0(x)+M_0(-x) = 2, \\ M_0(x)-M_0(-x) = x(M_0^2(x)+M_0^2(-x)), \\ F_0(x)+F_0(-x) = 2+x(F_0^2(x)-F_0^2(-x)), \\ F_0(x)-F_0(-x) = x(F_0^2(x)+F_0^2(-x)). \end{gathered} \tag{3.12}$$

Hence,

$$M_0(x) = 1 + \frac{1-\sqrt{1-4x^2}}{2x}, \qquad F_0(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{3.13}$$

Theorem 3.3. (i) *The generating function for the number of even permutations avoiding* 132 *is given by (see* [16]*)*

$$E_0(x) = \frac{1}{2}\left(\frac{1-\sqrt{1-4x}}{2x} + 1 + \frac{1-\sqrt{1-4x^2}}{2x}\right). \tag{3.14}$$

(ii) *The generating function for the number of odd permutations avoiding* 132 *is given by (see* [16]*)*

$$O_0(x) = \frac{1}{2}\left(\frac{1-\sqrt{1-4x}}{2x} - 1 - \frac{1-\sqrt{1-4x^2}}{2x}\right). \tag{3.15}$$

(iii) *The generating function for the number of permutations avoiding* 132 *is given by (see* [9]*)*

$$F_0(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{3.16}$$

3.2. The case $r = 1$. Since permutations in $\mathfrak{S}_2$ do not exhibit kernel shapes distinct from ρ_1, the only possible new shape is the exceptional one, $\rho_2 = 132$. Calculation of the parameters of ρ_2 gives $s(\rho_2) = 3$, $c(\rho_2) = 1$, $f(\rho_2) = 3$,

$$\begin{gathered} X_1(\rho_2) = Y_{-1} = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}, \\ X_{-1}(\rho_2) = Y_1 = \{(0,0,0),(1,1,0),(1,0,1),(1,1,0)\}, \\ z_{\rho_2}(0,0,0) = z_{\rho_2}(1,0,0) = z_{\rho_2}(0,1,0) = -z_{\rho_2}(1,1,0) = 1, \\ -z_{\rho_2}(0,0,1) = z_{\rho_2}(1,0,1) = z_{\rho_2}(0,1,1) = z_{\rho_2}(1,1,1) = 1. \end{gathered} \tag{3.17}$$

Therefore, by Theorem 3.1, we have

$$\begin{aligned} &2(H_1(0,0) - H_1(0,1)) \\ &\quad = M_1(x) + M_1(-x) = \frac{x^3}{2}(M_0(-x) - M_0(x))(M_0^2(-x) + M_0^2(x)), \\ &2(H_1(1,0) - H_1(1,1)) \\ &\quad = M_1(x) - M_1(-x) \\ &\quad = 2x(M_0(x)M_1(x) + M_0(-x)M_1(-x)) \\ &\qquad - \frac{x^3}{2}(M_0(-x) + M_0(x))(M_0^2(-x) + M_0^2(x)). \end{aligned} \tag{3.18}$$

Using the expression for $M_0(x)$ (see the case $r = 0$) we get

$$M_1(x) = \frac{1}{2}(-1 + 3x + 2x^2) + \frac{1 - 3x - 4x^2 + 4x^3}{2}(1 - 4x^2)^{-1/2}. \tag{3.19}$$

Similarly, considering the expressions for $H_1(0,0)+H_1(0,1)$ and $H_1(1,0)+H_1(1,1)$ we get

$$F_1(x)=\frac{1}{2}(x-1)+\frac{1-3x}{2}(1-4x)^{-1/2}. \tag{3.20}$$

Theorem 3.4. (i) *The generating function for the number of even permutations containing* 132 *exactly once is given by*

$$\begin{aligned}E_1(x)=&-\frac{1}{2}\left(1-2x-x^2\right)\\&+\frac{1-3x}{4}(1-4x)^{-1/2}+\frac{1-3x-4x^2+4x^3}{4}\left(1-4x^2\right)^{-1/2}.\end{aligned} \tag{3.21}$$

(ii) *The generating function for the number of odd permutations containing* 132 *exactly once is given by*

$$O_1(x)=-\frac{1}{2}\left(x+x^2\right)+\frac{1-3x}{4}(1-4x)^{-1/2}-\frac{1-3x-4x^2+4x^3}{4}\left(1-4x^2\right)^{-1/2}. \tag{3.22}$$

(iii) *The generating function for the number of permutations containing* 132 *exactly once is given by (see* [6]*)*

$$F_1(x)=\frac{1}{2}(x-1)+\frac{1-3x}{2}(1-4x)^{-1/2}. \tag{3.23}$$

3.3. The case $r=2$. We have to check the kernel shapes of permutations in $\mathfrak{S}_4$. Exhaustive search adds four new shapes to the previous list; these are 1243, 1342, 1423, and 2143; besides, there is the exceptional $35142\in\mathfrak{S}_5$. Calculation of the parameters s, c, f, z, X_a, Y_a is straightforward, and we get the following theorem.

Theorem 3.5. (i) *The generating function for the number of even permutations containing* 132 *exactly twice is given by*

$$\begin{aligned}E_2(x)=&\frac{1}{2}x\left(x^3+3x^2-4x-1\right)+\frac{1}{4}\left(2x^4-4x^3+29x^2-15x+2\right)(1-4x)^{-3/2}\\&-\frac{1}{4}\left(16x^7-48x^6-76x^5+64x^4+36x^3-21x^2-5x+2\right)\left(1-4x^2\right)^{-3/2}.\end{aligned} \tag{3.24}$$

(ii) *The generating function for the number of odd permutations containing* 132 *exactly once is given by*

$$\begin{aligned}O_2(x)=&-\frac{1}{2}\left(x^4+3x^3-5x^2-4x+2\right)\\&+\frac{1}{4}\left(2x^4-4x^3+29x^2-15x+2\right)(1-4x)^{-3/2}\\&+\frac{1}{4}\left(16x^7-48x^6-76x^5+64x^4+36x^3-21x^2-5x+2\right)\left(1-4x^2\right)^{-3/2}.\end{aligned} \tag{3.25}$$

(iii) *The generating function for the number of permutations containing* 132 *exactly twice is given by (see* [12]*)*

$$F_2(x)=\frac{1}{2}\left(x^2+3x-2\right)+\frac{1}{2}\left(2x^4-4x^3+29x^2-15x+2\right)(1-4x)^{-3/2}. \tag{3.26}$$

3.4. The cases $r = 3,4,5,6$. Let $r = 3,4,5,6$; exhaustive search in $\mathfrak{S}_6$, $\mathfrak{S}_8$, $\mathfrak{S}_{10}$, and $\mathfrak{S}_{12}$ reveals 20, 104, 503, and 2576 new nonexceptional kernel shapes, respectively, and we get the following theorem.

Theorem 3.6. *Let $r = 3,4,5,6$, then*

$$\begin{aligned} M_r(x) &= \frac{1}{2}\left(A_r(x)+B_r(x)(1-4x^2)^{-r+1/2}\right), \\ F_r(x) &= \frac{1}{2}\left(C_r(x)+D_r(x)(1-4x)^{-r+1/2}\right), \end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
A_3(x) &= 2x^6+10x^5-24x^4-30x^3+23x^2+7x-2, \\
A_4(x) &= 2x^8+14x^7-46x^6-90x^5+117x^4+85x^3-42x^2-8x+1, \\
A_5(x) &= 2x^{10}+18x^9-76x^8-198x^7+360x^6+440x^5-355x^4 \\
&\quad -171x^3+62x^2+10x-2, \\
A_6(x) &= 256x^{13}-446x^{12}-618x^{11}+194x^{10}-140x^9+798x^8 \\
&\quad +1404x^7-1702x^6-1430x^5+815x^4+302x^3-88x^2-15x+4, \\
B_3(x) &= 64x^{11}-320x^{10}-800x^9+1216x^8+1124x^7-972x^6 \\
&\quad -524x^5+312x^4+100x^3-43x^2-7x+2, \\
B_4(x) &= -256x^{15}+1792x^{14}+6112x^{13}-13120x^{12}-19840x^{11} \\
&\quad +22224x^{10}+19054x^9-14780x^8-8328x^7+4772x^6 \\
&\quad +1840x^5-775x^4-197x^3+56x^2+8x-1, \\
B_5(x) &= 1024x^{19}-9216x^{18}-40064x^{17}+111744x^{16}+228896x^{15} \\
&\quad -343264x^{14}-404056x^{13}+398712x^{12}+321058x^{11} \\
&\quad -234686x^{10}-137468x^9+78480x^8+33896x^7 \\
&\quad -15400x^6-4780x^5+1723x^4+351x^3-98x^2-10x+2, \\
B_6(x) &= 524288x^{24}+1175552x^{23}-1593344x^{22}-2324992x^{21} \\
&\quad +1162752x^{20}+298112x^{19}+2696448x^{18}+4856864x^{17} \\
&\quad -7020288x^{16}-7464568x^{15}+6981056x^{14}+5445696x^{13} \\
&\quad -3868942x^{12}-2335450x^{11}+1324884x^{10}+627306x^9 \\
&\quad -290536x^8-106510x^7+40772x^6+11046x^5-3543x^4 \\
&\quad -632x^3+176x^2+15x-4, \\
C_3(x) &= 2x^3-5x^2+7x-2, \\
C_4(x) &= 5x^4-7x^3+2x^2+8x-3,
\end{aligned}$$

$$\begin{aligned}
C_5(x) &= 14x^5-17x^4+x^3-16x^2+14x-2,\\
C_6(x) &= 42x^6-44x^5+5x^4+4x^3-20x^2+19x-4,\\
D_3(x) &= -22x^6-106x^5+292x^4-302x^3+135x^2-27x+2,\\
D_4(x) &= 2x^9+218x^8+1074x^7-1754x^6+388x^5+1087x^4,\\
D_5(x) &= -50x^{11}-2568x^{10}-10826x^9+16252x^8-12466x^7+16184x^6\\
&\quad -16480x^5+9191x^4-2893x^3+520x^2-50x+2,\\
D_6(x) &= 4x^{14}+820x^{13}+32824x^{12}+112328x^{11}-205530x^{10}+141294x^9\\
&\quad -30562x^8-67602x^7+104256x^6-74090x^5+30839x^4-7902x^3\\
&\quad +1230x^2-107x+4.
\end{aligned} \tag{3.28}$$

Moreover, for $r=3,4,5,6$*,*

$$\begin{aligned}
E_r(x) &= \frac{1}{4}\left(A_r(x)+C_r(x)+D_r(x)(1-4x)^{-r+1/2}+B_r(x)(1-4x^2)^{-r+1/2}\right),\\
O_r(x) &= \frac{1}{4}\left(A_r(x)-C_r(x)+D_r(x)(1-4x)^{-r+1/2}-B_r(x)(1-4x^2)^{-r+1/2}\right).
\end{aligned} \tag{3.29}$$

4. Further results and open questions. First of all, we simplify the expression

$$L^{\rho}_{r_1,\dots,r_{f(\rho)}}(a,0)-L^{\rho}_{r_1,\dots,r_{f(\rho)}}(a,1), \tag{4.1}$$

where $a=0,1$, $r_j\ge 0$ for all j.

Lemma 4.1. *Let* $\mathbf{v}\in\{0,1\}^n$ *be any vector and let* $a\in\{1,-1\}$*. Then*

$$\sum_{\mathbf{x}\in Y_a}\prod_{j=1}^{n}H_r(v_j,x_j)-\sum_{\mathbf{y}\in Y_{-a}}\prod_{j=1}^{n}H_r(v_j,y_j)=a\prod_{j=1}^{n}g_r(j), \tag{4.2}$$

where $g_r(j)=H_r(v_j,0)-H_r(v_j,1)=(1/2)(M_r(x)+(-1)^{v_j}M_r(-x))$ *for all* j*.*

Proof. For any two vectors $u,v\in\mathcal{B}^n$, define $uv=1$ if $u_i=v_i$ for all $i\ne j$ and $u_j+v_j=1$, otherwise $uv=0$. Using the standard reflected Gray code, with each of the standard Gray-code vectors reflected left-for-right (for more details, see [20]), we get that there exists an arrangement of the binary vectors of $\mathcal{B}^n$, say $(0,\dots,0)=\mathbf{u}^1,\mathbf{u}^2,\dots,\mathbf{u}^{2^n}$, such that the first 2^m vectors in the sequence, say $\mathbf{v}^1,\dots,\mathbf{v}^{2^m}$, when restricted to their first m coordinates, satisfy that $\mathbf{v}^j\mathbf{v}^{j+1}=1$ for all j. In such a context this arrangement is called *Gray-code arrangement.*

Now we are ready to prove the lemma. Without loss of generality we can assume that $(0,0,\dots,0)\in Y_a^n$ (which means $a=1$); otherwise it is enough to replace a by $-a$. Let $\mathbf{x}^1,\dots,\mathbf{x}^{2^n}$ be all the vectors of $\mathcal{B}^n$ with the Gray-code arrangement. Thus, using $(0,0,\dots,0)\in Y_a^n$, we get that $\mathbf{x}^{2i-1}\in Y_a^n$ and $\mathbf{x}^{2i}\in Y_{-a}^n$ for all $i=1,2,\dots,2^{n-1}$. Therefore, for all $i = 1, 2, \ldots, 2^{n-1}$,

$$\begin{aligned}\sum_{i=1}^{2^{n-1}} &\left(\prod_{j=1}^{n} H_r\left(v_j, \mathbf{x}_j^{2i-1}\right) - \prod_{j=1}^{n} H_r\left(v_j, \mathbf{x}_j^{2i}\right) \right) \\ &= \sum_{i=1}^{2^{n-1}} \left((-1)^{i-1} g_r(1) \prod_{j=2}^{n} H_r\left(v_j, \mathbf{x}_j^{2i-1}\right) \right) \\ &= g_r(1) \sum_{i=1}^{2^{n-2}} \left(\prod_{j=1}^{n-1} H_r\left(\tilde{v}_j, \mathbf{y}_j^{2i-1}\right) - \prod_{j=1}^{n-1} H_r\left(\tilde{v}_j, \mathbf{y}_j^{2i}\right) \right),\end{aligned} \tag{4.3}$$

where $\mathbf{y}^p = (\mathbf{x}_2^{2p}, \ldots, \mathbf{x}_n^{2p})$ for all $p = 1, 2, \ldots, 2^{n-1}$, and $\tilde{v} = (v_2, v_3, \ldots, v_n)$. The Gray-code arrangement for $\mathbf{x}^1, \ldots, \mathbf{x}^{2^n}$ implies that the vectors $\mathbf{y}^1, \ldots, \mathbf{y}^{2^{n-1}}$ are arranged as Gray-code arrangement in $\mathcal{B}^{n-1}$. Hence, by induction on n (by definitions, the lemma holds for $n = 1$), we get that the expression equals $a \prod_{j=1}^{n} g_r(j)$. □

As a remark, the vector $(0, \ldots, 0) \in Y^{\rho}_{z_\rho(\mathbf{v})}$ if and only if $z_\rho(\mathbf{v}) = 1$ for any kernel permutation ρ and vector $\mathbf{v}$. Therefore, by Theorem 3.1 and Lemma 4.1 we get the following theorem.

Theorem 4.2. *Let $a \in \{0, 1\}$ and $r \geq 0$. Then*

$$\begin{aligned}&\frac{1}{2}\left(M_r(x) + (-1)^a M_r(-x)\right) - \delta_{r+a,0} \\ &= \sum_{\rho \in K_{2r+1}} x^{s(\rho)} \sum_{r_1, \ldots, r_{f(\rho)} = r - c(\rho)} \left(\sum_{\mathbf{v} \in X_{(-1)^a}(\rho)} 2^{-f(\rho)} z_\rho(\mathbf{v}) \prod_{j=1}^{f(\rho)} \left(M_{r_j}(x) + (-1)^{v_j} M_{r_j}(-x)\right) \right).\end{aligned} \tag{4.4}$$

As a remark, the above theorem yields two equations (for $a = 0$ and $a = 1$) that are linear on $M_r(x)$ and $M_r(-x)$. So, Theorem 4.2 provides a finite algorithm for finding $M_r(x)$ for any given $r \geq 0$, since we have to consider all permutations in $\mathfrak{S}_{2r+1}$ and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following proposition which holds immediately by Theorems 3.2 and 4.2.

Proposition 4.3. *Let $r \geq 1$, $a \in \{0, 1\}$, and*

$$\rho = 2r-1, 2r+1, 2r-3, 2r, \ldots, 2r-2j-3, 2r-2j, \ldots, 1, 4, 2. \tag{4.5}$$

Then the expression

$$x^{s(\rho)} \sum_{r_1, \ldots, r_{f(\rho)} = r - c(\rho),\, r_j \geq 0} \left(\sum_{\mathbf{v} \in X_{(-1)^a}(\rho)} 2^{-f(\rho)} z_\rho(\mathbf{v}) \prod_{j=1}^{f(\rho)} \left(M_{r_j}(x) + (-1)^{v_j} M_{r_j}(-x)\right) \right) \tag{4.6}$$

is given by

$$\sum_{j=a}^{[(r+2)/2]}(-1)^{j-a+1}2^{-r-2}\binom{r+2}{2j+1-a}x^{2r+1}\left(M_0(x)-M_0(-x)\right)^j\left(M_0(x)+M_0(-x)\right)^{r+2-j}. \tag{4.7}$$

By this proposition, it is sufficient to search only permutations in $\mathfrak{S}_{2r}$. Besides, using Theorem 4.2 and the case $r=0$, together with induction on r, we get the following result.

Theorem 4.4. *$M_r(x)$ is a rational function on x and $\sqrt{1-4x^2}$ for any $r\geq 0$.*

In view of our explicit results, we have an even stronger conjecture.

Conjecture 4.5. *For any $r\geq 1$, there exist polynomials $A_r(x)$, $B_r(x)$, $C_r(x)$, and $D_r(x)$ with integer coefficients such that*

$$\begin{aligned} E_r(x) &= \frac{1}{4}\left(A_r(x)+B_r(x)\right)+\frac{1}{4}C_r(x)(1-4x)^{-r+1/2}+\frac{1}{4}D_r(x)\left(1-4x^2\right)^{-r+1/2}, \\ O_r(x) &= \frac{1}{4}\left(A_r(x)-B_r(x)\right)+\frac{1}{4}C_r(x)(1-4x)^{-r+1/2}-\frac{1}{4}D_r(x)\left(1-4x^2\right)^{-r+1/2}. \end{aligned} \tag{4.8}$$

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