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CONVERGENCY OF THE FUZZY VECTORS IN THE PSEUDO-FUZZY VECTOR SPACE OVER $F_p^1(1)$

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In 2003, we considered the pseudo-fuzzy vector space SFR over $F_p^1(1)$. Here, we further discuss the convergency of the fuzzy vectors in SFR.

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1. Introduction. In this paper, we discuss the convergency of the fuzzy space over $F_p^1(1)$ (see [4]). In [4, Section 2], we stated the pseudo-fuzzy vector space SFR over $F_p^1(1)$ as follows: for two points $P=(x^{(1)},x^{(2)},\dots,x^{(n)})$ and $Q=(y^{(1)},y^{(2)},\dots,y^{(n)})$ on $\mathbb{R}^n$, we have the crisp vector $\overrightarrow{PQ}=(y^{(1)}-x^{(1)},y^{(2)}-x^{(2)},\dots,y^{(n)}-x^{(n)})$ in a pseudo-fuzzy vector space $F_p^n(1)=\{(a^{(1)},a^{(2)},\dots,a^{(n)})_1\,\forall(a^{(1)},a^{(2)},\dots,a^{(n)})\in\mathbb{R}^n\}$.

There is a one-to-one onto mapping $P=(x^{(1)},x^{(2)},\dots,x^{(n)})\leftrightarrow\widetilde{P}=(x^{(1)},x^{(2)},\dots,x^{(n)})_1$. Therefore, for the crisp vector $\overrightarrow{PQ}$, we can define the fuzzy vector $\overrightarrow{\widetilde{P}\widetilde{Q}}=(y^{(1)}-x^{(1)},y^{(2)}-x^{(2)},\dots,y^{(n)}-x^{(n)})_1=\widetilde{Q}\ominus\widetilde{P}$.

Let the family of the fuzzy sets on $\mathbb{R}^n$ satisfying the definitions of *convex* and *normal* be F_c. Obviously, $F_p^n(1)\subset F_c$. Next, we extend the fuzzy vector $\overrightarrow{\widetilde{P}\widetilde{Q}}=\widetilde{Q}\ominus\widetilde{P}$ to F_c, and $\widetilde{X},\widetilde{Y}\in F_c$, and define the fuzzy vector $\overrightarrow{\widetilde{X}\widetilde{Y}}=\widetilde{Y}\ominus\widetilde{X}$. Let SFR $=\{\overrightarrow{\widetilde{X}\widetilde{Y}}\,\forall\widetilde{X},\widetilde{Y}\in F_c\}$. Then we have the pseudo-fuzzy vector space over $F_p^n(1)$ $(=a_1\,\forall a\in\mathbb{R})$. In Section 3, we will discuss the convergency of the fuzzy vectors in SFR.

2. Preparation. In [4], we discussed the pseudo-fuzzy vector space SFR over $F_p^1(1)$. In order to discuss the convergence of the fuzzy vectors in SFR, we need to know some definitions.

Definition 2.1. (1°) The fuzzy set $\widetilde{A}$ on $\mathbb{R}=(-\infty,\infty)$ is convex if and only if every ordinary set $A(\alpha)=\{x\mid\mu_{\widetilde{A}}(x)\ge\alpha\ \forall\alpha\in[0,1]\}$ is convex, and hence $A(\alpha)$ is a closed interval of $\mathbb{R}$.

(2°) The fuzzy set $\widetilde{A}$ on $\mathbb{R}$ is normal if and only if $\bigvee_{x\in\mathbb{R}}\mu_{\widetilde{A}}(x)=1$.

Next, we extend this definition to $\mathbb{R}^n$ by saying that the membership function of the fuzzy set $\widetilde{D}$ on $\mathbb{R}^n$ is $\mu_{\widetilde{D}}(x^{(1)},x^{(2)},\dots,x^{(n)})\in[0,1]$ for all $(x^{(1)},x^{(2)},\dots,x^{(n)})\in\mathbb{R}^n$.

Definition 2.2. The α-cut $(0\le\alpha\le1)$ of the fuzzy set $\widetilde{D}$ on $\mathbb{R}^n$ is defined by $D(\alpha)=\{(x^{(1)},x^{(2)},\dots,x^{(n)})\mid\mu_{\widetilde{D}}(x^{(1)},x^{(2)},\dots,x^{(n)})\ge\alpha\}$.

Definition 2.3. (1°) The fuzzy set $\widetilde{D}$ on $\mathbb{R}^n$ is convex if and only if every ordinary set $D(\alpha)=\{(x^{(1)},x^{(2)},\dots,x^{(n)})\mid\mu_{\widetilde{D}}(x^{(1)},x^{(2)},\dots,x^{(n)})\ge\alpha\ \forall\alpha\in[0,1]\}$ is a convex closed subset of $\mathbb{R}^n$.

(2°) The fuzzy set $\tilde{D}$ is normal if and only if $\bigvee_{(x^{(1)},x^{(2)},\dots,x^{(n)})\in\mathbb{R}^n}\mu_{\tilde{D}}(x^{(1)},x^{(2)},\dots,x^{(n)})=1$.

Let the family of the fuzzy sets on $\mathbb{R}^n$ satisfying Definition 2.3 (1°), (2°) be F_c.

Definition 2.4 (Pu and Liu [3]). The fuzzy set a_α $(0\le\alpha\le 1)$ on $\mathbb{R}$ is called a level α fuzzy point on $\mathbb{R}$ if its membership function $\mu_{a_\alpha}(x)$ is

$$\mu_{a_\alpha}(x)=\begin{cases}\alpha, & x=a,\\ 0, & x\ne a.\end{cases}\tag{2.1}$$

Let the family of all level α fuzzy points on $\mathbb{R}$ be $F_p^{(1)}(\alpha)=\{a_\alpha\,\forall\,\alpha\in\mathbb{R}\}$, $0\le\alpha\le 1$.

Definition 2.5. The fuzzy set $(a^{(1)},a^{(2)},\dots,a^{(n)})_\alpha$ $(0\le\alpha\le 1)$ is called a level α fuzzy point on $\mathbb{R}^n$ if its membership function is

$$\begin{aligned}&\mu_{(a^{(1)},a^{(2)},\dots,a^{(n)})_\alpha}(x^{(1)},x^{(2)},\dots,x^{(n)})\\&\quad=\begin{cases}\alpha, & \text{if } (x^{(1)},x^{(2)},\dots,x^{(n)})=(a^{(1)},a^{(2)},\dots,a^{(n)}),\\ 0, & \text{elsewhere.}\end{cases}\end{aligned}\tag{2.2}$$

Let the family of all level α fuzzy points on $\mathbb{R}^n$ be

$$\begin{gathered}F_p^{(n)}(\alpha)=\{(a^{(1)},a^{(2)},\dots,a^{(n)})_\alpha\,\forall\,(a^{(1)},a^{(2)},\dots,a^{(n)})\in\mathbb{R}^n\},\quad 0\le\alpha\le 1,\\ F_p^{(n)}=\bigcup_{0\le\alpha\le 1}F_p^{(n)}(\alpha).\end{gathered}\tag{2.3}$$

For each $a_\alpha\in F_p^1(\alpha)$, regard $a_\alpha=(a,a,\dots,a)_\alpha$ as a special level α fuzzy point on $\mathbb{R}^n$ degenerated from a level α fuzzy point $(a^{(1)},a^{(2)},\dots,a^{(n)})$ with $a^{(1)}=a^{(2)}=\cdots=a^{(n)}=a$. Hence, we have the following expression:

$$\begin{aligned}\mu_{(a,a,\dots,a)_\alpha}(x^{(1)},x^{(2)},\dots,x^{(n)})&=\begin{cases}\alpha, & (x^{(1)},x^{(2)},\dots,x^{(n)})=(a,a,\dots,a),\\ 0, & (x^{(1)},x^{(2)},\dots,x^{(n)})\ne(a,a,\dots,a),\end{cases}\\&=\mu_{a_\alpha}(x^{(1)},x^{(2)},\dots,x^{(n)}).\end{aligned}\tag{2.4}$$

Definition 2.6. For $D\subset\mathbb{R}^n$, call D_α, $0\le\alpha\le 1$, a level α fuzzy domain on $\mathbb{R}^n$ if its membership function is

$$\mu_{D_\alpha}(x^{(1)},x^{(2)},\dots,x^{(n)})=\begin{cases}\alpha, & \text{if } (x^{(1)},x^{(2)},\dots,x^{(n)})\in D,\\ 0, & \text{if } (x^{(1)},x^{(2)},\dots,x^{(n)})\notin D.\end{cases}\tag{2.5}$$

Let the family of all the level α fuzzy domains on $\mathbb{R}^n$ be $FD^*=\{E_\alpha\,\forall\,E\subset\mathbb{R}^n\}$, and let the family of all subsets of $\mathbb{R}^n$ be $\mathcal{P}(\mathbb{R}^n)=\{E\,\forall\,E\subset\mathbb{R}^n\}$.

Then there is a one-to-one mapping η between $\mathcal{P}(\mathbb{R}^n)$ and FD^*:

$$E \in \mathcal{P}(\mathbb{R}^n) \longleftrightarrow \eta(E) = E_\alpha \in FD^*, \qquad \eta^{(-1)}(E_\alpha) = E, \quad \alpha \in [0,1]. \tag{2.6}$$

Since $\widetilde{D} \in F_c$, the α-cut $D(\alpha)$ $(0 \le \alpha \le 1)$ of $\widetilde{D}$ can be mapped to $D(\alpha)_\alpha$.

Thus, we have the following decomposition principle:

$$\forall \widetilde{D} \in F_c, \quad \widetilde{D} = \bigcup\nolimits_{\alpha\in[0,1]} D(\alpha)_\alpha. \tag{2.7}$$

From Kaufmann and Gupta [2], we have for $D, E \subset \mathbb{R}^n$, $k \in \mathbb{R}$,

$$\begin{aligned} D(+)E = \{&(x^{(1)}+y^{(1)}, x^{(2)}+y^{(2)}, \dots, x^{(n)}+y^{(n)}) \\ &\forall (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in D,\ (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in E\}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} D(-)E = \{&(x^{(1)}-y^{(1)}, x^{(2)}-y^{(2)}, \dots, x^{(n)}-y^{(n)}) \\ &\forall (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in D,\ (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in E\}, \end{aligned} \tag{2.9}$$

$$k(\cdot)D = \{(kx^{(1)}, kx^{(2)}, \dots, kx^{(n)}) \forall (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in D\}. \tag{2.10}$$

From (2.6), (2.7), (2.8), (2.9), (2.10), and the definition of the α-cut, we have that

(i) the α-cut of $\widetilde{D}(+)\widetilde{E}$ is $D(\alpha)+E(\alpha)$,

$$\widetilde{D} \oplus \widetilde{E} = \bigcup_{0\le\alpha\le1} \left(D(\alpha)(+)E(\alpha)\right)_\alpha, \tag{2.11}$$

(ii) the α-cut of $\widetilde{D}(-)\widetilde{E}$ is $D(\alpha)-E(\alpha)$,

$$\widetilde{D} \ominus \widetilde{E} = \bigcup_{0\le\alpha\le1} \left(D(\alpha)(-)E(\alpha)\right)_\alpha, \tag{2.12}$$

(iii) the α-cut of $k_1 \odot wtD$ is $k(\cdot)D(\alpha)$,

$$k_1 \odot \widetilde{D} = \bigcup_{0\le\alpha\le1} \left(k(\cdot)D(\alpha)\right)_\alpha. \tag{2.13}$$

In the crisp case on $\mathbb{R}^n$, we can consider the n-dimensional vector space E^n over $\mathbb{R}$.

Let $P = (p^{(1)}, p^{(2)}, \dots, p^{(n)})$, $Q = (q^{(1)}, q^{(2)}, \dots, q^{(n)})$, $A = (a^{(1)}, a^{(2)}, \dots, a^{(n)})$, $B = (b^{(1)}, b^{(2)}, \dots, b^{(n)}) \in \mathbb{R}^n$; $k \in \mathbb{R}$.

Define the crisp vectors $\overrightarrow{PQ}$, $\overrightarrow{AB}+\overrightarrow{PQ}$, and $k\cdot\overrightarrow{PQ}$ as follows:

$$\overrightarrow{PQ} = (q^{(1)}-p^{(1)}, q^{(2)}-p^{(2)}, \dots, q^{(n)}-p^{(n)}) = Q(-)P, \tag{2.14}$$

$$\begin{aligned} \overrightarrow{AB}+\overrightarrow{PQ} = (&b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)}, b^{(2)}+q^{(2)}-a^{(2)}-p^{(2)}, \\ &\dots, b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)}), \end{aligned} \tag{2.15}$$

$$k\cdot\overrightarrow{PQ} = (kq^{(1)}-kp^{(1)}, kq^{(2)}-kp^{(2)}, \dots, kq^{(n)}-kp^{(n)}). \tag{2.16}$$

Let $O = (0,0,\dots,0) \in \mathbb{R}^n$, $\overrightarrow{OP} = (p^{(1)},p^{(2)},\dots,p^{(n)})$, $\overrightarrow{OO} = (0,0,\dots,0)$, and let $E^n = \{\overrightarrow{PQ} = (q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)})\,\forall P,Q \in \mathbb{R}^n\}$. This is an n-dimensional vector space over $\mathbb{R}$. There is a one-to-one onto mapping between the point $(a^{(1)},a^{(2)},\dots,a^n)$ on $\mathbb{R}^n$ and the level 1 fuzzy point $(a^{(1)},a^{(2)},\dots,a^n)_1$ on $F_p^n(1)$:

$$\begin{aligned}\rho : (a^{(1)},a^{(2)},\dots,a^{(n)}) \in \mathbb{R}^n &\longleftrightarrow \rho(a^{(1)},a^{(2)},\dots,a^{(n)})\\ &= (a^{(1)},a^{(2)},\dots,a^{(n)})_1 \in F_p^n(1).\end{aligned} \tag{2.17}$$

Let $\tilde{P} = (p^{(1)},p^{(2)},\dots,p^{(n)})_1$, $\tilde{Q} = (q^{(1)},q^{(2)},\dots,q^{(n)})_1 \in F_p^n(1)$. From (2.14) and (2.17), we have the following definition:

$$\overrightarrow{\tilde{P}\tilde{Q}} = (q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)})_1 = \tilde{Q}\ominus\tilde{P}. \tag{2.18}$$

Let $FE^n = \{\overrightarrow{\tilde{P}\tilde{Q}}\,\forall \tilde{P},\tilde{Q} \in F_p^n(1)\}$. From (2.14) and (2.18), we have the one-to-one onto mappings

$$\begin{aligned}\overrightarrow{PQ} &= (q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)})\\ &\longleftrightarrow \rho(\overrightarrow{PQ}) = (q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)})_1\\ &= \overrightarrow{\tilde{P}\tilde{Q}} \in FE^n,\\ \overrightarrow{AB}+\overrightarrow{PQ} &= (b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)}-a^{(2)}-p^{(2)},\\ &\qquad \dots,b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)})\\ &\longleftrightarrow (b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)}-a^{(2)}-p^{(2)},\\ &\qquad \dots,b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)})_1\\ &= \overrightarrow{\tilde{A}\tilde{B}}\oplus\overrightarrow{\tilde{P}\tilde{Q}},\\ k\cdot\overrightarrow{PQ} &= (kq^{(1)}-kp^{(1)},kq^{(2)}-kp^{(2)},\dots,kq^{(n)}-kp^{(n)})\\ &\longleftrightarrow (kq^{(1)}-kp^{(1)},kq^{(2)}-kp^{(2)},\dots,kq^{(n)}-kp^{(n)})_1\\ &= k_1\odot\overrightarrow{\tilde{P}\tilde{Q}}.\end{aligned} \tag{2.19}$$

Therefore, $FE^n = \{\overrightarrow{\tilde{P}\tilde{Q}}\,\forall \tilde{P},\tilde{Q} \in F_p^n(1)\}$ is a vector space over $F_p^n(1)$ in fuzzy sense.

In [4], we further extend FE^n as follows. For $\tilde{X},\tilde{Y} \in F_c$, define $\overrightarrow{\tilde{X}\tilde{Y}} = \tilde{Y}\ominus\tilde{X}$ and call $\overrightarrow{\tilde{X}\tilde{Y}}$ a fuzzy vector. Let SFR $= \{\overrightarrow{\tilde{X}\tilde{Y}}\,\forall \tilde{X},\tilde{Y} \in F_c\}$. In [4], we proved that the following properties hold.

Property 2.7. For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{W}\tilde{Z}} \in \text{SFR}$,

$$\overrightarrow{\tilde{X}\tilde{Y}} = \overrightarrow{\tilde{W}\tilde{Z}} \Longleftrightarrow \tilde{Y} \ominus \tilde{X} = \tilde{Z} \ominus \tilde{W}. \tag{2.20}$$

Property 2.8. For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{W}\tilde{Z}} \in \text{SFR}$, $k \in \mathbb{R}$,

(1°) $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{W}\tilde{Z}} = \overrightarrow{\tilde{A}\tilde{B}}$, where $\tilde{A} = \tilde{X} \oplus \tilde{W}$, $\tilde{B} = \tilde{Y} \oplus \tilde{Z}$;

(2°) $k_1 \odot \overrightarrow{\tilde{X}\tilde{Y}} = \overrightarrow{\tilde{C}\tilde{D}}$, where $\tilde{C} = k_1 \odot \tilde{X}$, $\tilde{D} = k_1 \odot \tilde{Y}$.

Property 2.9. For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{W}\tilde{Z}}, \overrightarrow{\tilde{U}\tilde{V}} \in \text{SFR}$, $k, t \in \mathbb{R}$,

(1°) $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{W}\tilde{Z}} = \overrightarrow{\tilde{W}\tilde{Z}} \oplus \overrightarrow{\tilde{X}\tilde{Y}}$;

(2°) $(\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{W}\tilde{Z}}) \oplus \overrightarrow{\tilde{U}\tilde{V}} = \overrightarrow{\tilde{X}\tilde{Y}} \oplus (\overrightarrow{\tilde{W}\tilde{Z}} \oplus \overrightarrow{\tilde{U}\tilde{V}})$;

(3°) $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{O}\tilde{O}} = \overrightarrow{\tilde{X}\tilde{Y}}$, where $\overrightarrow{\tilde{O}\tilde{O}} = (0, 0, \dots, 0)_1$;

(4°) $k_1 \odot (t_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}) = (kt)_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}$;

(5°) $k_1 \odot (\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{W}\tilde{Z}}) = (k_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}) \oplus (k_1 \odot \overrightarrow{\tilde{W}\tilde{Z}})$;

(6°) $1 \odot \overrightarrow{\tilde{X}\tilde{Y}} = \overrightarrow{\tilde{X}\tilde{Y}}$.

In SFR, the following do not hold.

(7°) For $\overrightarrow{\tilde{X}\tilde{Y}} \in \text{SFR}$ and $\overrightarrow{\tilde{X}\tilde{Y}} \neq \overrightarrow{\tilde{O}\tilde{O}}$, there exists $\overrightarrow{\tilde{W}\tilde{Z}}$ $(\neq \overrightarrow{\tilde{O}\tilde{O}}) \in \text{SFR}$ such that $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{W}\tilde{Z}} = \overrightarrow{\tilde{O}\tilde{O}}$;

(8°) $(k+t)_1 \odot \overrightarrow{\tilde{X}\tilde{Y}} = (k_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}) \oplus (t_1 \odot \overrightarrow{\tilde{X}\tilde{Y}})$.

From Property 2.9, we know that SFR satisfies all the conditions that the vector space required, except (7°) and (8°). Therefore, in [4], we called SFR a pseudo-fuzzy vector space over $F_p^1(1)$.

Example 2.10 (a moving station carrying a missile on it)**.** This car left from point $P = (2, 5)$ passing through point $Q = (4, 6)$, arrived at $R = (8, 9)$, and aiming at the target $Z = (100, 200)$. As we can see, the missile usually falls in the vicinity of Z, say $\tilde{Z}$, instead of hitting at Z exactly.

Let the membership function of $\tilde{Z}$ be

$$\mu_{\tilde{Z}}(x^{(1)}, x^{(2)}) = \begin{cases} \dfrac{1}{25}(25 - (x^{(1)} - 100)^2 - (x^{(2)} - 200)^2), \\ \qquad\qquad \text{if } (x^{(1)} - 100)^2 + (x^{(2)} - 200)^2) \leq 25, \\ 0, \qquad\qquad \text{elsewhere}. \end{cases} \tag{2.21}$$

Consider the level 1 fuzzy points $\tilde{P} = (2, 5)_1$, $\tilde{Q} = (4, 6)_1$, and $\tilde{R} = (8, 9)_1$. We have the fuzzy routes

$$\tilde{P} \longrightarrow \tilde{Q} \longrightarrow \tilde{R} \longrightarrow \tilde{Z} \tag{2.22}$$

and hence the fuzzy vectors $\overrightarrow{\tilde{P}\tilde{Q}} = (2,1)_1$, $\overrightarrow{\tilde{Q}\tilde{R}} = (4,3)_1$, $\overrightarrow{\tilde{R}\tilde{Z}} = \tilde{Z} \ominus \tilde{R}$, and $\overrightarrow{\tilde{P}\tilde{Z}} = \tilde{Z} \ominus \tilde{P}$. By extension theory, the membership function of $\overrightarrow{\tilde{R}\tilde{Z}} = \tilde{Z} \ominus \tilde{R}$ is

$$\begin{aligned} \mu_{\overrightarrow{\tilde{R}\tilde{Z}}}(z^{(1)}, z^{(2)}) &= \sup_{z^{(j)} = v^{(j)} - u^{(j)}, j=1,2} \mu_{\tilde{R}}(u^{(1)}, u^{(2)}) \wedge \mu_{\tilde{Z}}(v^{(1)}, v^{(2)}) \\ &= \mu_{\tilde{Z}}(z^{(1)} + 8, z^{(2)} + 9) \\ &= \begin{cases} \dfrac{1}{25}\left(25 - (z^{(1)} - 92)^2 - (z^{(2)} - 191)^2\right), \\ \qquad\qquad \text{if } (z^{(1)} - 92)^2 + (z^{(2)} - 191)^2 \le 25, \\ 0, \qquad\quad \text{elsewhere.} \end{cases} \end{aligned} \tag{2.23}$$

Similarly,

$$\mu_{\overrightarrow{\tilde{P}\tilde{Z}}}(z^{(1)}, z^{(2)}) = \begin{cases} \dfrac{1}{25}\left(25 - (z^{(1)} - 98)^2 - (z^{(2)} - 195)^2\right), \\ \qquad\qquad \text{if } (z^{(1)} - 98)^2 + (z^{(2)} - 195)^2 \le 25, \\ 0, \qquad\quad \text{elsewhere.} \end{cases} \tag{2.24}$$

Let $S = (98, 202)$. It is clear that $(98, 202)$ is within the circle of center $(100, 200)$ and radius 5. The crisp vector which starts with the point $P = (2, 5)$ and ends at $S = (98, 202)$ is $\overrightarrow{PS} = (96, 197)$. Its grade of membership in $\overrightarrow{\tilde{P}\tilde{Z}}$ from (2.23) is $\mu_{\overrightarrow{\tilde{P}\tilde{Z}}}(96, 197) = (1/25)(25 - 2^2 - 2^2) = 0.68$, that is, the grade of membership of the fuzzy vector $\overrightarrow{\tilde{P}\tilde{Z}}$ for the crisp vector $\overrightarrow{PS}$ is 0.68. Let the aim be $T = (100, 200)$. The crisp vector beginning at $P = (2, 5)$ and aiming at $T = (100, 200)$ is $\overrightarrow{PT} = (98, 195)$. Its grade of membership in $\overrightarrow{\tilde{P}\tilde{Z}}$, again from (2.23), is $\mu_{\overrightarrow{\tilde{P}\tilde{Z}}}(98, 195) = (1/25)(25 - 0^2 - 0^2) = 1$, that is, the grade of membership of the fuzzy vector $\overrightarrow{\tilde{P}\tilde{Z}}$ for the crisp vector $\overrightarrow{PT}$ is 1.

Example 2.11. In a shooting practice, let $C((10, 30), 1 + 1/m) = \{(x, y) \mid (x - 10)^2 + (y - 30)^2 \le (1 + 1/m)^2\}$, always shooting at $(1, 2)$ and aiming at $Z = (10, 30)$. At the first time, the bullet was falling in $C((10, 30), 2 (= 1 + 1))$. At the second time, it was falling in $C((10, 30), 1 + 1/2)$. At the mth time, it was falling in $C((10, 30), 1 + 1/m)$. In other words, the bullet was more and more closer to $C((10, 30), 1)$, that is, more and more accurate.

Let the fuzzy aim be $\tilde{Z}_m$, its membership function is

$$\mu_{\tilde{Z}_m} = \begin{cases} \dfrac{1}{(1 + 1/m)^2}\left[\left(1 + \dfrac{1}{m}\right)^2 - (x - 10)^2 - (y - 30)^2\right], \\ \qquad\qquad \text{if } (x - 10)^2 + (y - 30)^2 \le \left(1 + \dfrac{1}{m}\right)^2, \\ 0, \qquad\quad \text{elsewhere.} \end{cases} \tag{2.25}$$

Thus, we have the mth fuzzy vector $\overrightarrow{\tilde{Q}\tilde{Z}_m}$, $m = 1, 2, \ldots$, where $\tilde{Q} = (1, 2)_1$. In the next section, we will discuss the convergency of the fuzzy vectors in SFR and find out the limit fuzzy vector $\lim_{n \to \infty} \overrightarrow{\tilde{Q}\tilde{Z}_m}$.

3. The convergency of the vectors in SFR**.** Before we try to investigate the convergency of the fuzzy vectors in SFR, we first define the following open set in $\mathbb{R}^n$ and discuss some properties (Properties 3.4, 3.7, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, and 3.17). Let

$$\begin{aligned} & O\left(\left(a^{(1,1)},a^{(1,2)}\right),\ldots,\left(a^{(n,1)},a^{(n,2)}\right)\right) \\ & \quad = \left\{\left(x^{(1)},x^{(2)},\ldots,x^{(n)}\right) \mid a^{(j,1)} < x^{(j)} < a^{(j,2)},\ j=1,2,\ldots,n\right\}. \end{aligned} \tag{3.1}$$

From (2.8), (2.9), and (2.10), we have

$$\begin{aligned} & O\left(\left(a^{(1,1)},a^{(1,2)}\right),\ldots,\left(a^{(n,1)},a^{(n,2)}\right)\right)(+)O\left(\left(b^{(1,1)},b^{(1,2)}\right),\ldots,\left(b^{(n,1)},b^{(n,2)}\right)\right) \\ & \quad = \left\{\left(z^{(1)},z^{(2)},\ldots,z^{(n)}\right) \mid z^{(j)} = x^{(j)}+y^{(j)},\ a^{(j,1)} < x^{(j)} < a^{(j,2)}, \right. \\ & \qquad \left. b^{(j,1)} < y^{(j)} < b^{(j,2)},\ j=1,2,\ldots,n\right\} \\ & \quad = O\left(\left(a^{(1,1)}+b^{(1,1)},a^{(1,2)}+b^{(1,2)}\right),\ldots,\left(a^{(n,1)}+b^{(n,1)},a^{(n,2)}+b^{(n,2)}\right)\right), \end{aligned} \tag{3.2}$$

$$\begin{aligned} & O\left(\left(a^{(1,1)},a^{(1,2)}\right),\ldots,\left(a^{(n,1)},a^{(n,2)}\right)\right)(-)O\left(\left(b^{(1,1)},b^{(1,2)}\right),\ldots,\left(b^{(n,1)},b^{(n,2)}\right)\right) \\ & \quad = \left\{\left(z^{(1)},z^{(2)},\ldots,z^{(n)}\right) \mid z^{(j)} = x^{(j)}-y^{(j)},\ a^{(j,1)} < x^{(j)} < a^{(j,2)}, \right. \\ & \qquad \left. b^{(j,1)} < y^{(j)} < b^{(j,2)},\ j=1,2,\ldots,n\right\} \\ & \quad = O\left(\left(a^{(1,1)}-b^{(1,1)},a^{(1,2)}-b^{(1,2)}\right),\ldots,\left(a^{(n,1)}-b^{(n,1)},a^{(n,2)}-b^{(n,2)}\right)\right). \end{aligned} \tag{3.3}$$

If $k > 0$,

$$\begin{aligned} & k(\cdot)O\left(\left(a^{(1,1)},a^{(1,2)}\right),\ldots,\left(a^{(n,1)},a^{(n,2)}\right)\right) \\ & \quad = \left\{\left(z^{(1)},z^{(2)},\ldots,z^{(n)}\right) \mid z^{(j)} = kx^{(j)}, a^{(j,1)} < x^{(j)} < a^{(j,2)},\ j=1,2,\ldots,n\right\} \\ & \quad = O\left(\left(ka^{(1,1)},ka^{(1,2)}\right),\ldots,\left(ka^{(n,1)},ka^{(n,2)}\right)\right). \end{aligned} \tag{3.4}$$

If $k < 0$,

$$\begin{aligned} & k(\cdot)O\left(\left(a^{(1,1)},a^{(1,2)}\right),\ldots,\left(a^{(n,1)},a^{(n,2)}\right)\right) \\ & \quad = \left\{\left(z^{(1)},z^{(2)},\ldots,z^{(n)}\right) \mid z^{(j)} = kx^{(j)},\ a^{(j,1)} < x^{(j)} < a^{(j,2)},\ j=1,2,\ldots,n\right\} \\ & \quad = O\left(\left(ka^{(1,2)},ka^{(1,1)}\right),\ldots,\left(ka^{(n,2)},ka^{(n,1)}\right)\right). \end{aligned} \tag{3.5}$$

Let $\mathcal{B} = \{O((a^{(1,1)},a^{(1,2)}),\ldots,(a^{(n,1)},a^{(n,2)}))_\alpha \forall a^{(j,1)} < a^{(j,2)},\ a^{(j,1)},a^{(j,2)} \in \mathbb{R},\ j = 1,2,\ldots,n;\ 0 \le \alpha \le 1\}$.

Let $\mathcal{B}^*$ be the family of fuzzy sets in $\mathcal{B}$ or any arbitrary unions of these fuzzy sets.

Remark 3.1. Any intersection of two fuzzy sets in $\mathcal{B}$ belongs to $\mathcal{B}$, and when two fuzzy sets in $\mathcal{B}$ have no intersection, we call their intersection $\varnothing$.

From (2.3), let $F = F_p^n \cup F_c \cup \mathcal{B}^*$. In order to consider the problem of convergency, we first consider the topology for F.

Definition 3.2. $\tilde{Q} \in F$ is an open fuzzy set if and only if for each $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{Q}$, there exists $\tilde{O} \in \mathcal{B}$ such that $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{O} \subset \tilde{Q}$.

Let T_F be the family of all open fuzzy sets satisfying Definition 3.2. Obviously, $\mathcal{B}^* \subset T_F$.

Definition 3.3 (Chang [1]). T is a family of fuzzy sets in the space X satisfying the following:
(1°) $\varnothing, X \in T$,
(2°) $\tilde{A}, \tilde{B} \in T$, then $\tilde{A} \cap \tilde{B} \in T$,
(3°) $\tilde{A}_j \in T$, $j \in I$ (any index set), then $\bigcup_{j \in I} \tilde{A}_j \in T$.
T is called a fuzzy topology for X and (X, T) is called a fuzzy topological space (abbreviated as FTS).

Property 3.4. T_F is a fuzzy topology for $\mathbb{R}^n$, $(\mathbb{R}^n, T_F)$ are fuzzy topological sets in $\mathbb{R}^n$ that are restricted in F.

Proof. (1°) It is obvious that $\mathbb{R}^n \in T_F$. Definition 3.3(1°) is fulfilled.

(2°) For $\tilde{D}, \tilde{E} \in T_F$ and $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{D} \cap \tilde{E}$, we have $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{D}$ and $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{E}$. From Definition 3.2, there exist $\tilde{I}, \tilde{J} \in \mathcal{B}$ such that $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{I} \subset \tilde{D}$ and $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{J} \subset \tilde{E}$. Therefore, $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{I} \cap \tilde{J}$. Hence, $\tilde{I} \cap \tilde{J} \subset \tilde{D} \cap \tilde{E}$. Thus, $\tilde{D} \cap \tilde{E} \in T_F$. Definition 3.3(2°) is fulfilled.

(3°) For $\tilde{D}_j \in T_F$, $j \in I$, and each $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \bigcup_{j \in I} \tilde{D}_j$, there exists $m \in I$ such that $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{D}_m$. By Definition 3.2, there is a $\tilde{J} \in \mathcal{B}$ such that $(x^{(1)}, x^{(2)}, \dots, x^{(n)})_\alpha \subset \tilde{J} \subset \tilde{D}_m \subset \bigcup_{j \in I} \tilde{D}_j \subset T_F$. Thus, Definition 3.3(3°) is fulfilled. □

Hence, from Definition 3.3, T_F is a fuzzy topology for $\mathbb{R}^n$ and $(\mathbb{R}^n, T_F)$ is a fuzzy topological space, that is, if we set $X = \mathbb{R}^n$, $T = T_F$ in Definition 3.3, then the definition holds. Therefore, Definitions 3.5, 3.6 and Property 3.7 can all be applied.

Definition 3.5 (Chang [1, Definition 2.3]). A fuzzy set $\tilde{U}$ in an FTS (X, T) is a neighborhood of a fuzzy set $\tilde{A}$ if and only if there exists a fuzzy set $\tilde{O} \in T$ such that $\tilde{A} \subset \tilde{O} \subset \tilde{U}$.

Definition 3.6 (Chang [1, Definition 3]). If a sequence of fuzzy sets $\{\tilde{A}_n,\ n = 1, 2, \dots\}$ is in an FTS (X, T), then this sequence converges to a fuzzy set $\tilde{A}$ if and only if it is eventually contained in each neighborhood of $\tilde{A}$ (i.e., if $\tilde{B}$ is any neighborhood of $\tilde{A}$, there is a positive integer m such that whenever $n \geq m$, $\tilde{A}_n \subset \tilde{B}$).

Property 3.7. $\{\tilde{A}_n\}$ are increasing fuzzy sets, $\tilde{A}_1 \subset \tilde{A}_2 \subset \cdots \subset \tilde{A}$, and

$$\lim_{n \to \infty} \mu_{\tilde{A}_n}(x^{(1)}, x^{(2)}, \dots, x^{(n)}) = \mu_{\tilde{A}}(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \tag{3.6}$$

for all $(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in \mathbb{R}^n$. Then the sequence $\{\tilde{A}_n,\ n = 1, 2, \dots\}$ converges to $\tilde{A}$, denoted by $\lim_{n \to \infty} \tilde{A}_n = \tilde{A}$.

Proof. The proof follows from Definition 3.6 easily. □

Definition 3.8. $\bigcup_{\alpha\in[0,1]} O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\dots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_\alpha (\in T_F)$ is a neighborhood of $\widetilde{D}=\bigcup_{\alpha\in[0,1]} D(\alpha)_\alpha \in F_c$ if and only if for each $\alpha\in[0,1]$, there exists $O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\dots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_\alpha \in \mathcal{B}$ such that $D(\alpha)_\alpha \subset O((a^{(1,1)}(\alpha), a^{(1,2)}(\alpha)),\dots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_\alpha$.

Definition 3.9. In F_c, the sequence of fuzzy sets $\widetilde{D}_k=\bigcup_{\alpha\in[0,1]} D_k(\alpha)_\alpha$, $k=1,2,\dots$, converges to $\widetilde{D}=\bigcup_{\alpha\in[0,1]} D(\alpha)_\alpha$, $k=1,2,\dots$ $(\in F_\alpha)$ if and only if for each neighborhood $\bigcup_{\alpha\in[0,1]} O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\dots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_\alpha$ of $\widetilde{D}$, there exists a natural number m such that whenever $k\ge m$, $D_k(\alpha)_\alpha \subset O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\dots,(a^{(n,1)}(\alpha), a^{(n,2)}(\alpha)))_\alpha$, denoted by $\lim_{k\to\infty}\widetilde{D}_k=\widetilde{D}$.

Since $D\subset\mathbb{R}^n$ and D_α $(\in FD^*)$ is a one-to-one onto mapping, from Definition 3.9, we can get the following property.

Property 3.10. In F_c, the sequence of fuzzy sets $\widetilde{D}_k=\bigcup_{\alpha\in[0,1]} D_k(\alpha)_\alpha, k=1,2,\dots$, converges to $\widetilde{D}=\bigcup_{\alpha\in[0,1]} D(\alpha)_\alpha$ if and only if for each $\alpha\in[0,1]$ and every neighborhood $O((a^{(1,1)},a^{(1,2)}),\dots,(a^{(n,1)},a^{(n,2)}))_\alpha$ of $D(\alpha)_\alpha$, there exists a natural number m such that whenever $k\ge m$, $D_k(\alpha)_\alpha \subset O((a^{(1,1)},a^{(1,2)}),\dots,(a^{(n,1)},a^{(n,2)}))$ if and only if for each $\alpha\in[0,1]$ and every neighborhood $O((a^{(1,1)},a^{(1,2)}),\dots,(a^{(n,1)},a^{(n,2)}))_\alpha$ of $D(\alpha)$, there exists m such that whenever $k\ge m$, $D_k(\alpha)_\alpha\subset O((a^{(1,1)},a^{(1,2)}),\dots,(a^{(n,1)}, a^{(n,2)}))$.

The convergency of fuzzy vectors needs the following property.

Property 3.11. For each $\alpha\in[0,1]$, the α-cuts $D_k(\alpha)$, $E_k(\alpha)$, $k=1,2,\dots,m$, of $\widetilde{D}_k$, $\widetilde{E}_k$ in F_c satisfy the following:

(1°) $(D_k(\alpha)(+)E_k(\alpha))_\alpha = D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha$,

(2°) $(D_k(\alpha)(-)E_k(\alpha))_\alpha = D_k(\alpha)_\alpha \ominus E_k(\alpha)_\alpha$,

(3°) each α-cut of $\bigcup_{k=1}^m[\widetilde{D}_k\oplus\widetilde{E}_k]$ is $\bigcup_{k=1}^m[\widetilde{D}_k(\alpha)(+)\widetilde{E}_k(\alpha)]$,

(3°-1) $(\bigcup_{k=1}^m(D_k(\alpha)(+)E_k(\alpha)))_\alpha=\bigcup_{k=1}^m(D_k(\alpha)(+)E_k(\alpha))_\alpha=\bigcup_{k=1}^m(D_k(\alpha)_\alpha\oplus E_k(\alpha)_\alpha)=(\bigcup_{k=1}^m D_k(\alpha)_\alpha)\oplus(\bigcup_{k=1}^m D_k(\alpha)_\alpha)$,

(3°-2) $\bigcup_{k=1}^m(\widetilde{D}_k\oplus\widetilde{E}_k)=(\bigcup_{k=1}^m\widetilde{D}_k)\oplus(\bigcup_{k=1}^m\widetilde{E}_k)$,

(4°) the α-cut of $\bigcup_{k=1}^m(\widetilde{D}_k\ominus\widetilde{E}_k)$ is $\bigcup_{k=1}^m[D_k(\alpha)(-)E_k(\alpha)]$,

(4°-1) $(\bigcup_{k=1}^m(D_k(\alpha)(-)E_k(\alpha)))_\alpha=\bigcup_{k=1}^m(D_k(\alpha)(-)E_k(\alpha))_\alpha=\bigcup_{k=1}^m(D_k(\alpha)_\alpha\ominus E_k(\alpha)_\alpha)=(\bigcup_{k=1}^m D_k(\alpha)_\alpha)\ominus(\bigcup_{k=1}^m D_k(\alpha)_\alpha)$,

(4°-2) $\bigcup_{k=1}^m(\widetilde{D}_k\ominus\widetilde{E}_k)=(\bigcup_{k=1}^m D_k)\ominus(\bigcup_{k=1}^m E_k)$.

Proof. By extension principle (1°)

$$\begin{aligned}
&\mu_{D_k(\alpha)_\alpha\oplus E_k(\alpha)_\alpha}(z^{(1)},z^{(2)},\dots,z^{(n)})\\
&\quad=\sup_{\substack{z^{(j)}=x^{(j)}+y^{(j)})\\ j=1,2,\dots,n}}\mu_{D_k(\alpha)_\alpha}(x^{(1)},x^{(2)},\dots,x^{(n)})\\
&\qquad\wedge\mu_{E_k(\alpha)_\alpha}(y^{(1)},y^{(2)},\dots,y^{(n)})\\
&\quad=\sup_{(x^{(1)},x^{(2)},\dots,x^{(n)})}\mu_{D_k(\alpha)_\alpha}(x^{(1)},x^{(2)},\dots,x^{(n)})\\
&\qquad\wedge\mu_{E_k(\alpha)_\alpha}(z^{(1)}-x^{(1)},z^{(2)}-x^{(2)},\dots,z^{(n)}-x^{(n)})
\end{aligned}$$

$$\begin{aligned}
&= \alpha, \quad \text{if } (x^{(1)},x^{(2)},\dots,x^{(n)}) \in D_k(\alpha), \\
&\qquad\quad (z^{(1)}-x^{(1)},z^{(2)}-x^{(2)},\dots,z^{(n)}-x^{(n)}) \in E_k(\alpha), \\
&= \alpha, \quad \text{if } (z^{(1)},z^{(2)},\dots,z^{(n)}) \in D_k(\alpha)(+)E_k(\alpha), \\
&= \mu_{(D_k(\alpha)+E_k(\alpha))_\alpha}(z^{(1)},z^{(2)},\dots,z^{(n)}) \quad \forall (x^{(1)},x^{(2)},\dots,x^{(n)}) \in \mathbb{R}^n.
\end{aligned} \tag{3.7}$$

(2°) The proof is similar to that of (1°).

(3°) Let $\widetilde{S}_k = \widetilde{D}_k \oplus \widetilde{E}_k$; from (2.11), we have

$$\bigcup_{k=1}^{m} \widetilde{S}_k = \bigcup_{k=1}^{m} \bigcup_{\alpha\in[0,1]} (D_k(\alpha)(+)E_k(\alpha))_\alpha = \bigcup_{\alpha\in[0,1]} \bigcup_{k=1}^{m} (D_k(\alpha)(+)E_k(\alpha))_\alpha. \tag{3.8}$$

Therefore, the α-cut of $\bigcup_{k=1}^{m}(\widetilde{D}_k \oplus \widetilde{E}_k) = \bigcup_{k=1}^{m} \widetilde{S}_k$ is $\bigcup_{k=1}^{m} S_k(\alpha) = \bigcup_{k=1}^{m} (D_k(\alpha)(+)E_k(\alpha))$.

(3°-1) For each $\alpha \in [0,1]$, the subset $\bigcup_{k=1}^{m} S_k(\alpha)$ of $\mathbb{R}^n$ corresponds to the fuzzy set $\bigcup_{k=1}^{m} S_k(\alpha)_\alpha = \bigcup_{k=1}^{m}(D_k(\alpha)(+)E_k(\alpha))_\alpha$. We first prove

$$\left(\bigcup_{k=1}^{m} S_k(\alpha)\right)_\alpha = \bigcup_{k=1}^{m} S_k(\alpha)_\alpha. \tag{3.9}$$

We have

$$\begin{aligned}
&\mu_{\bigcup_{k=1}^{m} S_k(\alpha)_\alpha}(z^{(1)},z^{(2)},\dots,z^{(n)}) \\
&\quad = \bigvee_{k=1}^{m} \mu_{S_k(\alpha)_\alpha}(z^{(1)},z^{(2)},\dots,z^{(n)}) \\
&\quad = \alpha, \quad \text{if } (z^{(1)},z^{(2)},\dots,z^{(n)}) \in S_k(\alpha) \text{ for some } k \in \{1,2,\dots,m\}, \\
&\quad = \alpha, \quad \text{if } (z^{(1)},z^{(2)},\dots,z^{(n)}) \in \bigcup_{k=1}^{m} S_k(\alpha), \\
&\quad = \mu_{(\bigcup_{k=1}^{m} S_k(\alpha))_\alpha}(z^{(1)},z^{(2)},\dots,z^{(n)}) \quad \forall (z^{(1)},z^{(2)},\dots,z^{(n)}) \in \mathbb{R}^n.
\end{aligned} \tag{3.10}$$

Therefore, $(\bigcup_{k=1}^{m} S_k(\alpha))_\alpha = \bigcup_{k=1}^{m} S_k(\alpha)_\alpha$. Hence

$$\left(\bigcup_{k=1}^{m} (D_k(\alpha)(+)E_k(\alpha))\right)_\alpha = \bigcup_{k=1}^{m} (D_k(\alpha)(+)E_k(\alpha))_\alpha. \tag{3.11}$$

For each $\alpha \in [0,1]$ and each k, (1°) holds. Therefore,

$$\bigcup_{k=1}^{m} (D_k(\alpha)(+)E_k(\alpha))_\alpha = \bigcup_{k=1}^{m} (D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha). \tag{3.12}$$

Finally, we will prove

$$\bigcup_{k=1}^{m} (D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha) = \bigcup_{k=1}^{m} (D_k(\alpha)_\alpha) \oplus \bigcup_{k=1}^{m} (E_k(\alpha)_\alpha). \tag{3.13}$$

We have

$$\begin{aligned}
&\mu_{\bigcup_{k=1}^m (D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha)}(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \\
&\quad = \bigvee_{k=1}^m \mu_{D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha}(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \\
&\quad = \bigvee_{k=1}^m \sup_{\substack{z^{(j)} = x^{(j)} + y^{(j)} \\ j=1,2,\ldots,n}} \mu_{D_k(\alpha)_\alpha}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \\
&\qquad \wedge \mu_{E_k(\alpha)_\alpha}(y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \\
&\quad = \bigvee_{k=1}^m \bigvee_{(y^{(1)}, y^{(2)}, \ldots, y^{(n)})} [\mu_{D_k(\alpha)_\alpha}(z^{(1)} - y^{(1)}, z^{(2)} - y^{(2)}, \ldots, z^{(n)} - y^{(n)}) \\
&\qquad\qquad \wedge \mu_{E_k(\alpha)_\alpha}(y^{(1)}, y^{(2)}, \ldots, y^{(n)})] \\
&\quad = \bigvee_{(y^{(1)}, y^{(2)}, \ldots, y^{(n)})} [\mu_{\bigcup_{k=1}^m D_k(\alpha)_\alpha}(z^{(1)} - y^{(1)}, z^{(2)} - y^{(2)}, \ldots, z^{(n)} - y^{(n)}) \\
&\qquad\qquad \wedge \mu_{\bigcup_{k=1}^m E_k(\alpha)_\alpha}(y^{(1)}, y^{(2)}, \ldots, y^{(n)})] \\
&\quad = \mu_{(\bigcup_{k=1}^m D_k(\alpha)_\alpha) \oplus (\bigcup_{k=1}^m E_k(\alpha)_\alpha)}(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \quad \forall (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in \mathbb{R}^n.
\end{aligned} \tag{3.14}$$

(3°-2) By decomposition theorem and (3°-1), we have

$$\begin{aligned}
\bigcup_{k=1}^m (\widetilde{D}_k \oplus \widetilde{E}_k) &= \bigcup_{k=1}^m \bigcup_{\alpha \in [0,1]} (D_k(\alpha)(+)E_k(\alpha))_\alpha \\
&= \bigcup_{\alpha \in [0,1]} \bigcup_{k=1}^m (D_k(\alpha)(+)E_k(\alpha))\alpha \\
&= \bigcup_{\alpha \in [0,1]} \left[\left(\bigcup_{k=1}^m D_k(\alpha)_\alpha \right) \oplus \left(\bigcup_{k=1}^m E_k(\alpha)_\alpha \right) \right].
\end{aligned} \tag{3.15}$$

Let $\widetilde{A} = \bigcup_{k=1}^m \widetilde{D}_k$, $\widetilde{B} = \bigcup_{k=1}^m \widetilde{E}_k$. From (3.9),

$$A(\alpha)_\alpha = \bigcup_{k=1}^m \widetilde{D}_k(\alpha)_\alpha, \quad B(\alpha)_\alpha = \bigcup_{k=1}^m \widetilde{E}_k(\alpha)_\alpha, \quad \forall \alpha \in [0,1], \tag{3.16}$$

$$\begin{aligned}
\widetilde{A} \oplus \widetilde{B} &= \bigcup_{\alpha \in [0,1]} [A(\alpha)(+)B(\alpha)]_\alpha = \bigcup_{\alpha \in [0,1]} [A(\alpha)_\alpha \oplus B(\alpha)_\alpha] \\
&= \bigcup_{\alpha \in [0,1]} \left[\left(\bigcup_{k=1}^m D_k(\alpha)_\alpha \right) \oplus \left(\bigcup_{k=1}^m E_k(\alpha)_\alpha \right) \right].
\end{aligned} \tag{3.17}$$

From (3.15), (3.17), we have

$$\begin{aligned}\bigcup_{k=1}^{m}(\tilde{D}_k\oplus\tilde{E}_k)&=\bigcup_{\alpha\in[0,1]}\left[\left(\bigcup_{k=1}^{m}D_k(\alpha)_\alpha\right)\oplus\left(\bigcup_{k=1}^{m}E_k(\alpha)_\alpha\right)\right]\\&=\left(\bigcup_{k=1}^{m}\tilde{D}_k\right)\oplus\left(\bigcup_{k=1}^{m}\tilde{E}_k\right).\end{aligned}\tag{3.18}$$

Properties (4°), (4°-1), and (4°-2) can be proved similarly as (3°), (3°-1), and (3°-2). □

Property 3.12. $\tilde{D}_k\in F_c$, $k=1,2,\dots,m$, and $q\neq 0$; then

(1°) the α-cut of $\bigcup_{k=1}^{m}(q_1\odot\tilde{D}_k)$ is $\bigcup_{k=1}^{m}(q(\cdot)D_k(\alpha))$,

(2°) $\bigcup_{k=1}^{m}(q(\odot)D_k(\alpha))_\alpha=q_1\odot(\bigcup_{k=1}^{m}D_k(\alpha)_\alpha)$,

(3°) $\bigcup_{k=1}^{m}(q_1\odot\tilde{D}_k)=q_1\odot(\bigcup_{k=1}^{m}\tilde{D}_k)$.

Proof. The proof goes on the lines of the proof of Property 3.11. □

Property 3.13. $\tilde{D}_m,\tilde{E}_m,\tilde{D},\tilde{E}\in F_c$, $m=1,2,\dots$, and $\lim_{m\to\infty}\tilde{D}_m=\tilde{D}$, $\lim_{m\to\infty}\tilde{E}_m=\tilde{E}$, then

(1°) $\lim_{m\to\infty}(\tilde{D}_m\oplus\tilde{E}_m)=\tilde{D}\oplus\tilde{E}=\lim_{m\to\infty}(\tilde{D}_m)\oplus\lim_{m\to\infty}(\tilde{E}_m)$,

(2°) $\lim_{m\to\infty}(\tilde{D}_m\ominus\tilde{E}_m)=\tilde{D}\ominus\tilde{E}=\lim_{m\to\infty}(\tilde{D}_m)\ominus\lim_{m\to\infty}(\tilde{E}_m)$,

(3°) $\lim_{m\to\infty}(k_1\odot\tilde{D}_m)=k_1\odot\tilde{D}=k_1\odot(\lim_{m\to\infty}(\tilde{D}_m))$, $k\neq 0$.

Proof. (1°) Since $\lim_{m\to\infty}\tilde{D}_m=\tilde{D}$, $\lim_{m\to\infty}\tilde{E}_m=\tilde{E}$, by Property 3.10, for each $\alpha\in[0,1]$ and every neighborhood $O((a^{(1,1)},a^{(1,2)}),\dots,(a^{(n,1)},a^{(n,2)}))$ of $D(\alpha)$, there exists a natural number $m^{(1)}$ such that when $k\geq m^{(1)}$, $D_k(\alpha)\subset O((a^{(1,1)},a^{(1,2)}),\dots,(a^{(n,1)},a^{(n,2)}))$. Also, for every neighborhood $O((b^{(1,1)},b^{(1,2)}),\dots,(b^{(n,1)},b^{(n,2)}))$ of $E(\alpha)$, there exists a natural number $m^{(2)}$ such that when $k\geq m^{(2)}$, $E_k(\alpha)\subset O((b^{(1,1)},b^{(1,2)}),\dots,(b^{(n,1)},b^{(n,2)}))$.

Let $m=\max(m^{(1)},m^{(2)})$. Then, for each $\alpha\in[0,1]$, when $k\geq m$, by (3.2), we have $D_k(\alpha)(+)E_k(\alpha)\subset O((a^{(1,1)}+b^{(1,1)},a^{(1,2)}+b^{(1,2)}),\dots,(a^{(n,1)}+b^{(n,1)},a^{(n,2)}+b^{(n,2)}))$ $(\in T_F)$, and $O((a^{(1,1)}+b^{(1,1)},a^{(1,2)}+b^{(1,2)}),\dots,(a^{(n,1)}+b^{(n,1)},a^{(n,2)}+b^{(n,2)}))$ is the neighborhood of $D(\alpha)(+)E(\alpha)$. By decomposition theorem,

$$\begin{aligned}\tilde{D}_k\oplus\tilde{E}_k&=\bigcup_{\alpha\in[0,1]}[D_k(\alpha)+E_k(\alpha)]_\alpha,\\\tilde{D}\oplus\tilde{E}&=\bigcup_{\alpha\in[0,1]}[D(\alpha)+E(\alpha)]_\alpha.\end{aligned}\tag{3.19}$$

Hence, by Property 3.10, we have $\lim_{m\to\infty}\tilde{D}_m\oplus\tilde{E}_m=\tilde{D}\oplus\tilde{E}$.

Properties (2°) and (3°) can be proved the same way as (1°). □

Property 3.14. $\tilde{D}_k,\tilde{E}_k,\tilde{D},\tilde{E}\in F_c$, $k=1,2,\dots$, and

$$\begin{aligned}&\lim_{m\to\infty}\mu_{\bigcup_{k=1}^{m}\tilde{D}_k}(x^{(1)},x^{(2)},\dots,x^{(n)})=\mu_{\tilde{D}}(x^{(1)},x^{(2)},\dots,x^{(n)}),\\&\lim_{m\to\infty}\mu_{\bigcup_{k=1}^{m}\tilde{E}_k}(x^{(1)},x^{(2)},\dots,x^{(n)})\\&\qquad=\mu_{\tilde{E}}(x^{(1)},x^{(2)},\dots,x^{(n)})\quad\forall(x^{(1)},x^{(2)},\dots,x^{(n)})\in\mathbb{R}^n,\\&\qquad\qquad\mu_{\bigcup_{k=1}^{m}\tilde{D}_k}\subset\tilde{D},\quad\mu_{\bigcup_{k=1}^{m}\tilde{E}_k}\subset\tilde{E},\quad\forall m=1,2,\dots,\end{aligned}\tag{3.20}$$

then

(1°) $\lim_{m\to\infty}\bigcup_{k=1}^{m}(\tilde{D}_k\oplus\tilde{E}_k)=\tilde{D}\oplus\tilde{E}=(\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{D}_k)\oplus(\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{E}_k)$,

(2°) $\lim_{m\to\infty}\bigcup_{k=1}^{m}(\tilde{D}_k\ominus\tilde{E}_k)=\tilde{D}\ominus\tilde{E}=(\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{D}_k)\ominus(\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{E}_k)$,

(3°) when $q\neq 0$, $\lim_{m\to\infty}\bigcup_{k=1}^{m}(q_1\odot\tilde{D}_k)=q_1\odot\tilde{D}$.

Proof. (1°) Since $\tilde{D}_1\subset\tilde{D}_1\cup\tilde{D}_2\subset\cdots\subset\bigcup_{k=1}^{m}\tilde{D}_k\subset\cdots\subseteq\tilde{D}$ and

$$\lim_{m\to\infty}\mu_{\bigcup_{k=1}^{m}\tilde{D}_k}(x^{(1)},x^{(2)},\ldots x^{(n)})=\mu_{\tilde{D}}(x^{(1)},x^{(2)},\ldots,x^{(n)}) \tag{3.21}$$

for all $(x^{(1)},x^{(2)},\ldots,x^{(n)})\in\mathbb{R}^n$, hence, by Property 3.7, we have $\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{D}_k=\tilde{D}$. Similarly, $\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{E}_k=\tilde{E}$. By Property 3.11(3°-2),

$$\bigcup_{k=1}^{m}(\tilde{D}_k\oplus\tilde{E}_k)=\left(\bigcup_{k=1}^{m}\tilde{D}_k\right)\oplus\left(\bigcup_{k=1}^{m}\tilde{E}_k\right). \tag{3.22}$$

From Property 3.13(1°),

$$\lim_{m\to\infty}\bigcup_{k=1}^{m}(\tilde{D}_k\oplus\tilde{E}_k)=\left(\lim_{m\to\infty}\bigcup_{k=1}^{m}(\tilde{D}_k)\right)\oplus\left(\lim_{m\to\infty}\bigcup_{k=1}^{m}(\tilde{E}_k)\right)=\tilde{D}\oplus\tilde{E}, \tag{3.23}$$

and (2°), (3°) can be proved as (1°). □

Next, we will discuss the convergency of the fuzzy vectors in SFR.

Property 3.15. For $\tilde{D}_m,\tilde{E}_m,\tilde{D},\tilde{E}\in F_c$, $m=1,2,\ldots$, $\lim_{m\to\infty}\tilde{D}_m=\tilde{D}$, $\lim_{m\to\infty}\tilde{E}_m=\tilde{E}$, then the fuzzy vectors $\overrightarrow{\tilde{E}_m\tilde{D}_m}$, $m=1,2,\ldots$, converge to the fuzzy vectors $\overrightarrow{\tilde{E}\tilde{D}}$.

Proof. Since $\overrightarrow{\tilde{E}_m\tilde{D}_m}=\tilde{D}_m\ominus\tilde{E}_m$, $\overrightarrow{\tilde{E}\tilde{D}}=\tilde{D}\ominus\tilde{E}$, then, by Property 3.13(2°),

$$\lim_{m\to\infty}\overrightarrow{\tilde{E}_m\tilde{D}_m}=\tilde{D}\ominus\tilde{E}=\overrightarrow{\tilde{E}\tilde{D}}. \tag{3.24}$$

□

Property 3.16. $\tilde{D}_k,\tilde{E}_k,\tilde{D},\tilde{E}\in F_c$, $k=1,2\ldots$; let $\tilde{Q}_m=\bigcup_{k=1}^{m}\tilde{D}_k$, $\tilde{S}_m=\bigcup_{k=1}^{m}\tilde{E}_k$, and let $\lim_{m\to\infty}\mu_{\tilde{Q}_m}(x^{(1)},x^{(2)},\ldots,x^{(n)})=\mu_{\tilde{D}}(x^{(1)},x^{(2)},\ldots,x^{(n)})$ and $\lim_{m\to\infty}\mu_{\tilde{S}_m}(x^{(1)},x^{(2)},\ldots,x^{(n)})=\mu_{\tilde{E}}(x^{(1)},x^{(2)},\ldots,x^{(n)})$ for all $(x^{(1)},x^{(2)},\ldots,x^{(n)})\in\mathbb{R}^n$, and $\tilde{Q}_m\subset\tilde{D}$, $\tilde{S}_m\subset\tilde{E}$. Then the sequence of fuzzy vectors $\overrightarrow{\tilde{S}_m\tilde{Q}_m}$, $m=1,2,\ldots$, converges to the fuzzy vector $\overrightarrow{\tilde{E}\tilde{D}}$.

Proof. Similar to Property 3.14, $\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{D}_k=\tilde{D}$ and $\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{E}_k=\tilde{E}$. By Property 3.13(2°), $\lim_{m\to\infty}\overrightarrow{\tilde{S}_m\tilde{Q}_m}=(\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{D}_k)\ominus(\lim_{m\to\infty}\bigcup_{k=1}^{m}\tilde{E}_k)=\tilde{D}\ominus\tilde{E}=\overrightarrow{\tilde{E}\tilde{D}}$.

For convenience, we denote $(q_1^{(1)}\odot\overrightarrow{\tilde{E}_1\tilde{D}_1})\oplus(q_1^{(2)}\odot\overrightarrow{\tilde{E}_2\tilde{D}_2})\oplus\cdots\oplus(q_1^{(r)}\odot\overrightarrow{\tilde{E}_r\tilde{D}_r})$ by $\sum_{k=1}^{r}\oplus(q_1^{(k)}\odot\overrightarrow{\tilde{E}_k\tilde{D}_k})$. □

Property 3.17. $\tilde{D}_{m,k},\tilde{E}_{m,k},\tilde{D}_k,\tilde{E}_k\in F_c$, $m=1,2,\ldots$, $k=1,2,\ldots,r$, and for each $k\in\{1,2,\ldots,r\}$, $\lim_{m\to\infty}\tilde{D}_{k,m}=\tilde{D}_k$, $\lim_{m\to\infty}\tilde{D}_{k,m}=\tilde{D}_k$, $q^k\neq 0$. The sequence of the fuzzy vectors $\sum_{k=1}^{r}\oplus(q_1^{(k)}\odot\overrightarrow{\tilde{E}_{m,k}\tilde{D}_{m,k}})$, $m=1,2,\ldots$, converges to the fuzzy vector $\sum_{k=1}^{r}\oplus(q_1^{(k)}\odot\overrightarrow{\tilde{E}_k\tilde{D}_k})$.

Proof. Since $\sum_{k=1}^{r}\oplus(q_1^{(k)}\odot\overrightarrow{\tilde{E}_{m,k}\tilde{D}_{m,k}}) = \sum_{k=1}^{r}\oplus(q_1^{(k)}\odot(\tilde{D}_{m,k}\ominus\tilde{E}_{m,k}))$, $m=1,2,\ldots$, for each k, by Property 3.13(2°), $\lim_{m\to\infty}\tilde{D}_{m,k}\ominus\tilde{E}_{m,k} = \tilde{D}_k\ominus\tilde{E}_k$. By Property 3.13(1°), (3°), we have

$$\begin{aligned}\lim_{m\to\infty}&\sum_{k=1}^{r}\oplus(q_1^{(k)}\odot(\tilde{D}_{m,k}\ominus\tilde{E}_{m,k}))\\ &=\sum_{k=1}^{r}\oplus(q_1^{(k)}\odot(\tilde{D}_k\ominus\tilde{E}_k)) = \sum_{k=1}^{r}\oplus(q_1^{(k)}\odot\overrightarrow{\tilde{E}_k\tilde{D}_k}).\end{aligned}\tag{3.25}$$

□

Example 3.18. Consider the fuzzy vectors $\lim_{m\to\infty}\overrightarrow{\tilde{Q}\tilde{Z}_m}$ in Example 2.11. Let

$$\mu_{\tilde{Z}}(x,y)=\begin{cases}1-(x-10)^2-(y-30)^2, & \text{if } (x-10)^2+(y-30)^2\le 1,\\ 0, & \text{elsewhere.}\end{cases}\tag{3.26}$$

We will prove $\lim_{m\to\infty}\tilde{Z}_m=\tilde{Z}$. Since $C((10,30),1+1/m)\subset C((10,30),1+1/(m-1))$ and for any $(x,y)\in\mathbb{R}^2$, the following holds:

$$\begin{aligned}&\frac{1}{(1+1/m)^2}\left[\left(1+\frac{1}{m}\right)^2-(x-10)^2-(y-30)^2\right]\\ &\quad\le\frac{1}{(1+1/(m-1))^2}\left[\left(1+\frac{1}{m-1}\right)^2-(x-10)^2-(y-30)^2\right],\end{aligned}\tag{3.27}$$

therefore, $\mu_{\tilde{Z}_m}(x,y)\le\mu_{\tilde{Z}_{m-1}}(x,y)$ for all $(x,y)\in\mathbb{R}^2$, and hence $\tilde{Z}_1\supset\tilde{Z}_2\supset\cdots\supset\tilde{Z}_m\supset\cdots\supset\tilde{Z}$, and obviously, $\lim_{m\to\infty}\mu_{\tilde{Z}_m}(x,y)=\mu_{\tilde{Z}}(x,y)$ for all $(x,y)\in\mathbb{R}^2$. Let $\tilde{Z}'_m$, $\tilde{Z}'$ be the complement fuzzy sets of $\tilde{Z}_m$, $\tilde{Z}$, respectively. We have $\lim_{m\to\infty}\mu_{\tilde{Z}'_m}(x,y)=\mu_{\tilde{Z}'}(x,y)$ for all $(x,y)\in\mathbb{R}^2$ and $\tilde{Z}'_1\subset\tilde{Z}'_2\subset\cdots\subset\tilde{Z}'_m\subset\cdots\subset\tilde{Z}'$. By Property 3.7, $\lim_{m\to\infty}\tilde{Z}+m'=\tilde{Z}'$. Thus, $\lim_{m\to\infty}\tilde{Z}_m=\tilde{Z}$. Therefore, from Property 3.15, $\lim_{m\to\infty}\overrightarrow{\tilde{Q}\tilde{Z}_m}=\overrightarrow{\tilde{Q}\tilde{Z}}$. Thus, the membership function of $\overrightarrow{\tilde{Q}\tilde{Z}}$ is

$$\begin{aligned}\mu_{\overrightarrow{\tilde{Q}\tilde{Z}}}(x,y)&=\mu_{\tilde{Z}\ominus\tilde{Q}}(x,y)\\ &=\sup_{\substack{x=x^{(1)}-y^{(1)}\\ y=x^{(2)}-y^{(2)}}}\mu_{\tilde{Z}}(x^{(1)},x^{(2)})\wedge\mu_{\tilde{Q}}(y^{(1)},y^{(2)})\\ &=\mu_{\tilde{Z}}(x+1,y+2)\\ &=\begin{cases}1-(x-9)^2-(y-28)^2, & \text{if } (x-9)^2-(y-28)^2\le 1,\\ 0, & \text{elsewhere.}\end{cases}\end{aligned}\tag{3.28}$$

In the crisp case, starting from $Q=(1,2)$, aiming at $Z=(10,30)$, we could have the vector $\overrightarrow{QZ}=(9,28)$. The grade of membership of $\overrightarrow{QZ}$ which belongs to the fuzzy vector $\overrightarrow{\tilde{Q}\tilde{Z}}$ is $\mu_{\overrightarrow{\tilde{Q}\tilde{Z}}}(9,28)=1$, that is, the grade of membership function of the fuzzy vector $\overrightarrow{\tilde{P}\tilde{Z}}$ for the crisp vector $\overrightarrow{PS}$ is 1, and the point $R=(9.5,29.5)$ is in the circle of center $(9,28)$ and radius 1. The crisp vector of Q to $\mathbb{R}$ is $\overrightarrow{QR}=(8.5,27.5)$. The grade of membership function of $\overrightarrow{\tilde{Q}\tilde{\tilde{Z}}}$ is $\mu_{\overrightarrow{\tilde{Q}\tilde{\tilde{Z}}}}(8.5, 27.5) = 0.5$, that is, the grade of membership function of the fuzzy vector $\overrightarrow{\tilde{P}\tilde{\tilde{Z}}}$ for the crisp vector $\overrightarrow{QR}$ is 0.5.

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