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ON UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED SĂLĂGEAN OPERATOR

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We introduce a class of univalent functions $R^n(\lambda,\alpha)$ defined by a new differential operator $D^n f(z)$, $n \in \mathbb{N}_0 = \{0,1,2,\dots\}$, where $D^0 f(z) = f(z)$, $D^1 f(z) = (1-\lambda)f(z)+\lambda z f'(z) = D_\lambda f(z)$, $\lambda \geq 0$, and $D^n f(z) = D_\lambda(D^{n-1}f(z))$. Inclusion relations, extreme points of $R^n(\lambda,\alpha)$, some convolution properties of functions belonging to $R^n(\lambda,\alpha)$, and other results are given.

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1. Introduction. Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

analytic in the unit disc $\Delta = \{z : |z| < 1\}$.

We denote by $R(\alpha)$ the subclass of A for which $\operatorname{Re} f'(z) > \alpha$ in Δ. For a function f in A, we define the following differential operator:

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda \geqslant 0, \tag{1.3}$$

$$D^n f(z) = D_\lambda\left(D^{n-1} f(z)\right). \tag{1.4}$$

If f is given by (1.1), then from (1.3) and (1.4) we see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left[1+(k-1)\lambda\right]^n a_k z^k. \tag{1.5}$$

When $\lambda = 1$, we get Sălăgean's differential operator [8].

Let $R^n(\lambda,\alpha)$ denote the class of functions $f \in A$ which satisfy the condition

$$\operatorname{Re}\left(D^n f(z)\right)' > \alpha, \quad z \in \Delta, \tag{1.6}$$

for some $0 \leq \alpha \leq 1$, $\lambda \geqslant 0$, and $n \in \mathbb{N}_0 = \{0,1,2,\dots\}$. It is clear that $R^0(\lambda,\alpha) \equiv R(\alpha) \equiv R^n(0,\alpha)$ and that $R^1(\lambda,\alpha) \equiv R(\lambda,\alpha)$, the class of functions $f \in A$ satisfying

$$\operatorname{Re}\left(f'(z) + \lambda z f''(z)\right) > \alpha, \quad z \in \Delta, \tag{1.7}$$

studied by Ponnusamy [5] and others.

The Hadamard product or convolution of two power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is defined as the power series $(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$, $z \in \Delta$.

The object of this paper is to derive several interesting properties of the class $R^n(\lambda,\alpha)$ such as inclusion relations, extreme points, some convolution properties, and other results.

2. Inclusion relations. Theorem 2.3 shows that the functions in $R^n(\lambda,\alpha)$ belong to $R(\alpha)$ and hence are univalent. We need the following lemmas.

Lemma 2.1. *If $p(z)$ is analytic in Δ, $p(0) = 1$ and $\operatorname{Re} p(z) > 1/2$, $z \in \Delta$, then for any function F analytic in Δ, the function $p * F$ takes its values in the convex hull of $F(\Delta)$.*

The assertion of Lemma 2.1 follows by using the Herglotz representation for p. The next lemma is due to Fejér [3].

A sequence $a_0, a_1, \ldots, a_n, \ldots$ of nonnegative numbers is called a *convex null sequence* if $a_n \to 0$ as $n \to \infty$ and

$$a_0 - a_1 \geqslant a_1 - a_2 \geqslant \cdots \geqslant a_n - a_{n+1} \geqslant \cdots \geqslant 0. \tag{2.1}$$

Lemma 2.2. *Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function $p(z) = c_0/2 + \sum_{k=1}^{\infty} c_k z^k$, $z \in \Delta$, is analytic and $\operatorname{Re} p(z) > 0$ in Δ.*

Now we prove the following theorem.

Theorem 2.3.

$$R^{n+1}(\lambda,\alpha) \subset R^n(\lambda,\alpha). \tag{2.2}$$

Proof. Let f belong to $R^{n+1}(\lambda,\alpha)$ and let it be given by (1.1). Then from (1.5), we have

$$\operatorname{Re}\left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k[1+(k-1)\lambda]^{n+1} a_k z^{k-1}\right) > \frac{1}{2}. \tag{2.3}$$

Now

$$\begin{aligned}(D^n f(z))' &= 1 + \sum_{k=2}^{\infty} k[1+(k-1)\lambda]^n a_k z^{k-1} \\ &= \left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k[1+(k-1)\lambda]^{n+1} a_k z^{k-1}\right) \\ &\quad * \left(1 + 2(1-\alpha) \sum_{k=2}^{\infty} \frac{z^{k-1}}{1+(k-1)\lambda}\right).\end{aligned} \tag{2.4}$$

Applying Lemma 2.2, with $c_0 = 1$ and $c_k = 1/(1+k\lambda)$, $k = 1, 2, \ldots$, we get

$$\operatorname{Re}\left(1 + 2(1-\alpha) \sum_{k=2}^{\infty} \frac{z^{k-1}}{[1+(k-1)\lambda]}\right) > \alpha. \tag{2.5}$$

Applying Lemma 2.1 to $(D^n f(z))'$, we get the required result. □

We also have a better result than Theorem 2.3.

Theorem 2.4. *Let $f \in R^{n+1}(\lambda,\alpha)$. Then $f \in R^n(\lambda,\beta)$, where*

$$\beta = \frac{2\lambda^2+(1+3\lambda)\alpha}{(1+\lambda)(1+2\lambda)} \geqslant \alpha. \tag{2.6}$$

Proof. Let $f \in R^{n+1}(\lambda,\alpha)$. It is shown in [9], as an example, that if $\lambda \geqslant 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \frac{z^k}{1+(k-1)\lambda}, \tag{2.7}$$

then

$$\operatorname{Re}\frac{g(z)}{z} > \frac{4\lambda^2+3\lambda+1}{2(1+\lambda)(1+2\lambda)}. \tag{2.8}$$

Hence

$$\operatorname{Re}\left(1+2(1-\alpha)\sum_{k=2}^{\infty}\frac{z^{k-1}}{1+(k-1)\lambda}\right) > \frac{2\lambda^2+(1+3\lambda)\alpha}{(1+\lambda)(1+2\lambda)}. \tag{2.9}$$

Now an application of Lemma 2.1 to $(D^n f(z))'$ in the previous theorem completes the proof. □

Remark 2.5. If we put $n = 1$ in Theorem 2.4, then we have

$$\operatorname{Re}(f'(z)+\lambda z f''(z)) > \alpha \Longrightarrow \operatorname{Re} f'(z) > \frac{2\lambda^2+(1+3\lambda)\alpha}{(1+\lambda)(1+2\lambda)}, \tag{2.10}$$

which is an improvement of the result of Saitoh [7] for $\lambda \geq 1$, where he shows that, for $\lambda > 0$,

$$\operatorname{Re}(f'(z)+\lambda z f''(z)) > \alpha \Longrightarrow \operatorname{Re} f'(z) > \frac{2\alpha+\lambda}{2+\lambda}. \tag{2.11}$$

Using Theorem 2.4 ($(n-m)$ times) we get, after some calculations, the following theorem.

Theorem 2.6. *Let $f \in R^n(\lambda,\alpha)$ and let $n > m \geqslant 0$. Then $f \in R^m(\lambda,\beta)$ if*

$$\beta = \left[\left(\frac{1+3\lambda}{(1+\lambda)(1+2\lambda)}\right)^{n-m}\alpha + \frac{2\lambda^2}{(1+\lambda)(1+2\lambda)}\sum_{k=0}^{n-m-1}\left(\frac{1+3\lambda}{(1+\lambda)(1+2\lambda)}\right)^k\right] \geqslant \alpha. \tag{2.12}$$

If we put $m = 0$ in Theorem 2.6, we obtain the following interesting result.

Corollary 2.7. *Let $f \in R^n(\lambda,\alpha)$. Then $\operatorname{Re} f'(z) > \beta$, where β is given by (2.12) with $m = 0$.*

Remark 2.8. Since D_λ (given by (1.3)) is a linear function of λ, it is clear that

$$R^n(\lambda,\alpha) \subset R^n(\lambda',\alpha), \tag{2.13}$$

where $\lambda > \lambda'$.

The following theorem deals with the partial sum of the functions in $R^n(\lambda,\alpha)$. For the proof we need the following result, due to Ahuja and Jahangiri [2].

Lemma 2.9. *Let* $-1 < t \leqslant S = 4.567802$. *Then*

$$\operatorname{Re}\left(\sum_{k=2}^{m} \frac{z^{k-1}}{k+t-1}\right) > -\frac{1}{1+t}, \quad z \in \Delta. \tag{2.14}$$

Theorem 2.10. *Let* $S_m(z,f)$ *denote the* m*th partial sum of a function* f *in* $R^n(\lambda,\alpha)$. *If* $f \in R^n(\lambda,\alpha)$ *and* $\lambda \geq 1/s = 0.21892$, *then* $S_m(z,f) \in R^{n-1}(\lambda,\beta)$, *where*

$$\beta = \frac{2\alpha+\lambda-1}{\lambda+1}. \tag{2.15}$$

Proof. Let $f \in R^n(\lambda,\alpha)$ and let it be given by (1.1). Then from (1.5) we have

$$\operatorname{Re}\left(1+\sum_{k=2}^{\infty} k[1+(k-1)\lambda]^n a_k z^{k-1}\right) > \alpha \tag{2.16}$$

or

$$\operatorname{Re}\left(1+\frac{2}{\lambda+1}\sum_{k=2}^{\infty} k[1+(k-1)\lambda]^n a_k z^{k-1}\right) > \frac{2\alpha+\lambda-1}{\lambda+1}. \tag{2.17}$$

Now

$$\begin{aligned}(D^{n-1}S_m(z,f))' &= 1+\sum_{k=2}^{m} k[1+(k-1)\lambda]^{n-1} a_k z^{k-1} \\ &= \left(1+\frac{2}{\lambda+1}\sum_{k=2}^{\infty} k[1+(k-1)\lambda]^n a_k z^{k-1}\right) \\ &\quad * \left(1+\frac{\lambda+1}{2\lambda}\sum_{k=2}^{m} \frac{z^{k-1}}{1/\lambda+(k-1)}\right), \quad \lambda > 0.\end{aligned} \tag{2.18}$$

From Lemma 2.9, we see that, for $\lambda \geq 1/s = 0.21892$,

$$\operatorname{Re}\sum_{k=2}^{m} \frac{z^{k-1}}{1/\lambda+(k-1)} > -\frac{\lambda}{\lambda+1}, \tag{2.19}$$

hence

$$\operatorname{Re}\left(1+\frac{\lambda+1}{2\lambda}\sum_{k=2}^{m} \frac{z^{k-1}}{1/\lambda+(k-1)}\right) > \frac{1}{2}, \tag{2.20}$$

and the result follows by application of Lemma 2.1. □

Now we prove the following theorem.

Theorem 2.11. *The set $R^n(\lambda,\alpha)$ is convex.*

Proof. Let the functions

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{ki} z^k \quad (i = 1,2) \tag{2.21}$$

be in the class $R^n(\lambda,\alpha)$. It is sufficient to show that the function $h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$, with μ_1 and μ_2 nonnegative and $\mu_1 + \mu_2 = 1$, is in the class $R^n(\lambda,\alpha)$.

Since

$$h(z) = z + \sum_{k=2}^{\infty} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^k, \tag{2.22}$$

then from (2.4) we have

$$(D^n h(z))' = 1 + \sum_{k=2}^{\infty} k(\mu_1 a_{k1} + \mu_2 a_{k2})[1+(k-1)\lambda]^n z^{k-1}, \tag{2.23}$$

hence

$$\begin{aligned}\operatorname{Re}(D^n h(z))' = \operatorname{Re}&\left(1 + \mu_1 \sum_{k=2}^{\infty} k[1+(k-1)\lambda]^n a_{k1} z^{k-1}\right)\\ &+ \operatorname{Re}\left(1 + \mu_2 \sum_{k=2}^{\infty} k[1+(k-1)\lambda]^n a_{k2} z^{k-1}\right).\end{aligned} \tag{2.24}$$

Since $f_1, f_2 \in R^n(\lambda,\alpha)$, this implies that

$$\operatorname{Re}\left(1 + \mu_i \sum_{k=2}^{\infty} k[1+(k-1)\lambda]^n a_{ki} z^{k-1}\right) > 1 + \mu_i(\alpha - 1) \quad (i = 1,2). \tag{2.25}$$

Using (2.25) in (2.24), we obtain

$$\operatorname{Re}(D^n h(z))' > 1 + \alpha(\mu_1 + \mu_2) - (\mu_1 + \mu_2), \tag{2.26}$$

and since $\mu_1 + \mu_2 = 1$, the theorem is proved. □

Hallenbeck [4] showed that

$$\operatorname{Re} f'(z) > \alpha \Longrightarrow \operatorname{Re}\frac{f(z)}{z} > (2\alpha - 1) + 2(1-\alpha)\log 2. \tag{2.27}$$

Using Theorem 2.3 and (2.27), we obtain the following theorem.

Theorem 2.12. *Let $f \in R^n(\lambda,\alpha)$. Then*

$$\operatorname{Re}\frac{D^n f(z)}{z} > (2\alpha - 1) + 2(1-\alpha)\log 2. \tag{2.28}$$

This result is sharp as can be seen by the function f_x given by (3.1).

3. Extreme points. The extreme points of the closed convex hull of $R(\alpha)$ were determined by Hallenbeck [4]. We denote the closed convex hull of a family F by $\operatorname{clco} F$, and we make use of some results in [4] to determine the extreme points of $R^n(\lambda,\alpha)$.

Theorem 3.1. *The extreme points of $R^n(\lambda,\alpha)$ are*

$$f_x(z) = z + 2(1-\alpha)\sum_{k=2}^{\infty}\frac{x^{k-1}z^k}{k[1+(k-1)\lambda]^n}, \quad |x| = 1,\ z \in \Delta. \tag{3.1}$$

Proof. Since $D^n : f \to D^n f$ is an isomorphism from $R^n(\lambda,\alpha)$ to $R(\alpha)$, it preserves the extreme points and, in [4], it is shown that the extreme points of $R(\alpha)$ are

$$z + 2(1-\alpha)\sum_{k=2}^{\infty}\frac{1}{k}x^{k-1}z^k, \quad |x| = 1,\ z \in \Delta. \tag{3.2}$$

Hence from (1.5), we see that the extreme points of $\operatorname{clco} R^n(\lambda,\alpha)$ are given by (3.1). Since the family $R^n(\lambda,\alpha)$ is convex (Theorem 2.6) and therefore equal to its convex hull, we get the required result. □

As consequences of Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let f belong to $R^n(\lambda,\alpha)$ and let it be given by (1.1). Then*

$$|a_k| \leqslant \frac{2(1-\alpha)}{k[1+(k-1)\lambda]^n}, \quad k \geqslant 2. \tag{3.3}$$

This result is sharp as shown by the function $f_x(z)$ given by (3.1).

Corollary 3.3. *If $f \in R^n(\lambda,\alpha)$, then*

$$\begin{aligned} |f(z)| &\leqslant r + \sum_{k=2}^{\infty}\frac{2(1-\alpha)}{k[1+(k-1)\lambda]^n}r^k, \quad |z| = r,\\ |f'(z)| &\leqslant 1 + \sum_{k=2}^{\infty}\frac{2(1-\alpha)}{[1+(k-1)\lambda]^n}r^{k-1}, \quad |z| = r. \end{aligned} \tag{3.4}$$

This result is sharp as shown by the function $f_x(z)$ given by (3.1) at $z = \bar{x}r$.

4. Convolution properties. Ruscheweyh and Sheil-Small [6] verified the Polya-Schoenberg conjecture and its analogous results, namely, $C * C \subset C$, $C * S^\star \subset S^\star$, and $C * K \subset K$, where C, $S^\star$, and K denote the classes of convex, starlike, and close-to-convex univalent functions, respectively. In the following, we prove the analogue of the Polya-Schoenberg conjecture for the class $R^n(\lambda,\alpha)$.

Theorem 4.1. *Let $f \in R^n(\lambda,\alpha)$ and $g \in C$. Then $f * g \in R^n(\lambda,\alpha)$.*

Proof. It is known that if g is convex univalent in Δ, then

$$\operatorname{Re}\frac{g(z)}{z} > \frac{1}{2}. \tag{4.1}$$

Using convolution properties, we have

$$\operatorname{Re}(D^n(f*g)(z))' = \operatorname{Re}\left((D^n f(z))' * \frac{g(z)}{z}\right), \tag{4.2}$$

and the result follows by application of Lemma 2.1. □

Theorem 4.2. *Let f and g belong to $R^n(\lambda,\alpha)$. Then $f*g \in R^n(\lambda,\beta)$, where*

$$\beta = \frac{\lambda(2\alpha+1)+4\alpha-1}{2(\lambda+1)} \geqslant \alpha. \tag{4.3}$$

Proof. Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in R^n(\lambda,\alpha)$, then

$$\operatorname{Re}\left(1+\sum_{k=2}^{\infty} k[1+(k-1)\lambda]^n b_k z^{k-1}\right) > \alpha. \tag{4.4}$$

Let $c_0 = 1$ and

$$c_k = \frac{\lambda+1}{(k+1)[1+k\lambda]^n}, \quad k \geqslant 1. \tag{4.5}$$

Then $\{c_k\}_{k=0}^{\infty}$ is a convex null sequence. Hence, by Lemma 2.2, we have

$$\operatorname{Re}\left(1+\sum_{k=2}^{\infty} \frac{\lambda+1}{k[1+(k-1)\lambda]^n} z^{k-1}\right) > \frac{1}{2}. \tag{4.6}$$

Now we take the convolution of (4.4) and (4.6) and apply Lemma 2.1 to obtain

$$\operatorname{Re}\left(1+(\lambda+1)\sum_{k=2}^{\infty} b_k z^{k-1}\right) > \alpha \tag{4.7}$$

or

$$\operatorname{Re}\frac{g(z)}{z} = \operatorname{Re}\left(1+\sum_{k=2}^{\infty} b_k z^{k-1}\right) > \frac{\lambda+\alpha}{\lambda+1}. \tag{4.8}$$

Hence

$$\operatorname{Re}\left(\frac{g(z)}{z} - \frac{2\alpha+\lambda-1}{2(\lambda+1)}\right) > \frac{1}{2}. \tag{4.9}$$

Since $f \in R^n(\lambda,\alpha)$, by applying Lemma 2.1, we obtain

$$\operatorname{Re}\left((D^n f(z))' * \left(\frac{g(z)}{z} - \frac{2\alpha+\lambda-1}{2(\lambda+1)}\right)\right) > \alpha \tag{4.10}$$

or

$$\operatorname{Re}\left((D^n f(z))' * \frac{g(z)}{z}\right) > \frac{\lambda(2\alpha+1)+4\alpha-1}{2(\lambda+1)} = \beta, \tag{4.11}$$

and by (4.2), the result follows. □

Remark 4.3. If we put $\lambda = 0$ in Theorem 4.2, we get the corresponding result for functions in $R(\alpha)$, given by Ahuja [1].

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