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A REMARK ON FOUR-DIMENSIONAL ALMOST KÄHLER-EINSTEIN MANIFOLDS WITH NEGATIVE SCALAR CURVATURE

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Concerning the Goldberg conjecture, we will prove a result obtained by applying the result of Iton in terms of L^2-norm of the scalar curvature.

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1. Introduction. An almost Hermitian manifold M is called an almost Kähler manifold if the corresponding Kähler form is a closed 2-form. It is well known that an almost Kähler manifold with integrable almost-complex structure is Kählerian. Concerning the integrability of almost Kähler manifold, the following conjecture by Goldberg is known (see [2]).

Conjecture 1.1. *A compact almost Kähler-Einstein manifold is Kählerian.*

Sekigawa [8] proved that the conjecture is true if the scalar curvature τ of M is nonnegative. But the conjecture is still open in the case where τ is negative. Recently, applying the Seiberg-Witten theory, Itoh [4] obtained the following integrability condition for certain almost Kähler-Einstein 4-manifolds in terms of the L^2-norm of the scalar curvature.

Theorem 1.2 [4]. *Let M be a four-dimensional compact almost Kähler-Einstein manifold with negative scalar curvature. If M satisfies*

$$\int_M \tau^2 dV = 32\pi^2 \left(2\chi(M) + p_1(M)\right), \tag{1.1}$$

then it must be Kähler-Einstein. Here, $\chi(M)$ and $p_1(M)$ are the Euler characteristic and the first Pontrjagin number of M, respectively.

As a corollary, he also proved the following.

Corollary 1.3 [4]. *Let M be a four-dimensional compact almost Kähler-Einstein manifold with negative scalar curvature. If M satisfies*

$$\int_M \tau^2 dV \leq 24 \int_M \left|\left|\mathcal{W}^+\right|\right|^2 dV, \tag{1.2}$$

or, more strictly, if $|\tau| \leq 2\sqrt{6}\|\mathcal{W}^+\|$ at each point of M, then M must be Kähler-Einstein. Here, $\mathcal{W}^+$ is the self-dual Weyl curvature operator of the metric g.

In this paper, concerning the Goldberg conjecture, we will prove a result obtained by using Corollary 1.3 (see Theorem 2.2).

2. Preliminaries and the result. Let $M = (M, J, g)$ be a four-dimensional almost Kähler-Einstein manifold with the almost-complex structure J and the Hermitian metric g. We denote by Ω the Kähler form of M defined by $\Omega(X,Y) = g(X,JY)$ for $X,Y \in \mathfrak{X}(M)$, the set of all smooth vector fields on M. We assume that M is oriented by the volume form $dV = \Omega^2/2$. We denote by ∇, R, ρ, and τ the Riemannian connection, the curvature tensor, the Ricci tensor, and the scalar curvature of M, respectively. We assume that the curvature tensor is defined by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for $X,Y,Z \in \mathfrak{X}(M)$. We denote by ρ^* the Ricci $*$-tensor of M defined by

$$\rho^*(x,y) = \frac{1}{2} \text{ trace of } (z \longmapsto R(x,Jy)Jz) \tag{2.1}$$

for $x,y,z \in T_pM$, the tangent space of M at $p \in M$. The Ricci $*$-tensor satisfies $\rho^*(x, y) = \rho^*(Jy, Jx)$ for any $x,y \in T_pM$, $p \in M$. We note that if M is Kählerian, the Ricci tensor and the Ricci $*$-tensor coincide on M. The $*$-scalar curvature τ^* of M is the trace of the linear endomorphism Q^* defined by $g(Q^*x, y) = \rho^*(x,y)$ for $x,y \in T_pM$, $p \in M$. Since $\|\nabla J\|^2 = 2(\tau^* - \tau)$, M is a Kähler manifold if and only if $\tau^* - \tau = 0$ on M. An almost Hermitian manifold M is called a weakly $*$-Einstein manifold if $\rho^* = \lambda^* g$ ($\lambda^* = \tau^*/4$) and a $*$-Einstein if M is weakly $*$-Einstein with constant $*$-scalar curvature. The following identity holds for any four-dimensional almost Hermitian Einstein manifold:

$$\frac{1}{2}\{\rho^*(x,y) + \rho^*(y,x)\} = \frac{\tau^*}{4} g(x,y) \tag{2.2}$$

for $x,y \in T_pM$, $p \in M$.

Now, let $\wedge^2 M$ be the vector bundle of all real 2-forms on M. The bundle $\wedge^2 M$ inherits a natural inner product g and we have an orthogonal decomposition

$$\wedge^2 M = \mathbb{R}\Omega \oplus LM \oplus \wedge_0^{1,1} M, \tag{2.3}$$

where LM (resp., $\wedge_0^{1,1}M$) is the bundle of J-skew-invariant (resp., J-invariant) 2-forms on M perpendicular to Ω. We can identify the subbundle $\mathbb{R}\Omega \oplus LM$ (resp., $\wedge_0^{1,1}M$) with the bundle $\wedge_+^2 M$ (resp., $\wedge_-^2 M$) of self-dual (resp., anti-self-dual) 2-forms on M. Since M is Einstein, it is well known that the curvature operator $\mathcal{R} : \wedge^2 M \to \wedge^2 M$ preserves $\wedge_\pm^2 M$ and that the Weyl curvature operator $\mathcal{W} : \wedge^2 M \to \wedge^2 M$ is given by

$$\mathcal{W}(\iota(X) \wedge \iota(Y)) = \mathcal{R}(\iota(X) \wedge \iota(Y)) - \frac{\tau}{12} \iota(X) \wedge \iota(Y), \tag{2.4}$$

where ι is the duality between the tangent bundle and the cotangent bundle of M by means of the metric g. Let $\{e_1, e_2 = Je_1,\ e_3, e_4 = Je_3\}$ be a (local) unitary frame field and put $e^i = \iota(e_i)$. Then, the Kähler form is represented by $\Omega = -e^1 \wedge e^2 - e^3 \wedge e^4$. Further, we see that

$$\{\Phi, J\Phi\} = \left\{\frac{1}{\sqrt{2}}(e^1\wedge e^3 - e^2\wedge e^4), \frac{1}{\sqrt{2}}(e^1\wedge e^4 + e^2\wedge e^3)\right\},$$
$$\{\Psi_1,\Psi_2,\Psi_3\} = \left\{\frac{1}{\sqrt{2}}(e^1\wedge e^2 - e^3\wedge e^4), \frac{1}{\sqrt{2}}(e^1\wedge e^3 + e^2\wedge e^4), \frac{1}{\sqrt{2}}(e^1\wedge e^4 - e^2\wedge e^3)\right\} \tag{2.5}$$

are (local) orthonormal bases of LM and $\wedge_0^{1,1}M = \wedge_-^2 M$, respectively.

In this paper, for any orthonormal basis (resp., any local orthonormal frame field) $\{e_1,e_2,e_3,e_4\}$ of a point $p\in M$ (resp., on a neighborhood of p), we will adopt the following notational convention:

$$\begin{gathered} J_{ij} = g(Je_i,e_j), \qquad \Gamma_{ijk} = g(\nabla_{e_i}e_j,e_k),\\ R_{ijkl} = g(R(e_i,e_j)e_k,e_l),\dots,R_{\bar i\bar j\bar k\bar l} = g(R(Je_i,Je_j)Je_k,Je_l),\\ \rho_{ij} = \rho(e_i,e_j),\dots,\rho_{\bar i\bar j} = \rho(e_{\bar i},e_{\bar j}),\\ \rho^*_{ij} = \rho^*(e_i,e_j),\dots,\rho^*_{\bar i\bar j} = \rho^*(e_{\bar i},e_{\bar j}),\\ \nabla_i J_{jk} = g((\nabla_{e_i}J)e_j,e_k),\dots,\nabla_{\bar i}J_{\bar j\bar k} = g((\nabla_{e_{\bar i}}J)e_{\bar j},e_{\bar k}), \end{gathered} \tag{2.6}$$

and so on, where the Latin indices run over the range 1, 2, 3, 4. We define functions A, B, C, D, G, and K on M by

$$\begin{aligned} A &= \sum_{i,j,k,l,a=1}^{4}(\nabla_a J_{ij})R_{ijkl}(\nabla_a J_{kl}),\\ B &= \sum_{i,j,k,l,a=1}^{4}(\nabla_a J_{ij})(\nabla_a J_{kl})(\nabla_b J_{ij})(\nabla_b J_{kl}),\\ C &= \sum_{i,j,k,l=1}^{4}R_{ijkl}R_{\bar i\bar j\bar k\bar l}, \qquad D = \sum_{i,j,k,l=1}^{4}(R_{ijkl}-R_{\bar i\bar j\bar k\bar l})^2,\\ G &= \sum_{i,j=1}^{4}(\rho^*_{ij}-\rho^*_{ji})^2, \qquad K = (u-v)^2+4w^2, \end{aligned} \tag{2.7}$$

where $u = g(\mathcal{R}(\Phi),\Phi)$, $v = g(\mathcal{R}(J\Phi),J\Phi)$, and $w = g(\mathcal{R}(\Phi),J\Phi)$. First, we will prove the following.

Lemma 2.1. *The norm of the self-dual Weyl operator $\mathcal{W}^+$ is given by*

$$\|\mathcal{W}^+\|^2 = \frac{1}{16}\left(G+D+(\tau^*)^2-\frac{\tau^2}{3}\right). \tag{2.8}$$

Proof. Let $\{e_1,e_2=Je_1,\ e_3,e_4=Je_3\}$ be any (local) unitary frame field on M and we put $\Omega_0 = -\Omega/\sqrt{2} = (e^1\wedge e^2+e^3\wedge e^4)/\sqrt{2}$, $\Phi = (e^1\wedge e^3 - e^2\wedge e^4)/\sqrt{2}$, and $J\Phi = (e^1\wedge e^4+e^2\wedge e^3)/\sqrt{2}$. Then, $\{\Omega_0,\Phi,J\Phi\}$ is an orthonormal basis of $\wedge_+^2 M$. Thus, we have

$$\begin{aligned} \|\mathcal{W}^+\|^2 = {}& g(\mathcal{W}^+(\Omega_0),\Omega_0)^2 + g(\mathcal{W}^+(\Omega_0),\Phi)^2 + g(\mathcal{W}^+(\Omega_0),J\Phi)^2\\ &+ g(\mathcal{W}^+(\Phi),\Omega_0)^2 + g(\mathcal{W}^+(\Phi),\Phi)^2 + g(\mathcal{W}^+(\Phi),J\Phi)^2\\ &+ g(\mathcal{W}^+(J\Phi),\Omega_0)^2 + g(\mathcal{W}^+(J\Phi),\Phi)^2 + g(\mathcal{W}^+(J\Phi),J\Phi)^2. \end{aligned} \tag{2.9}$$

Taking account of (2.4), we have

$$
\begin{aligned}
g(\mathcal{W}^+(\Omega_0),\Omega_0) &= \frac{1}{2}\Big(-R_{1212}-2R_{1234}-R_{3434}-\frac{\tau}{6}\Big) = \frac{1}{12}(3\tau^*-\tau),\\
g(\mathcal{W}^+(\Omega_0),\Phi) &= \frac{1}{2}(-R_{1213}-R_{1224}-R_{3413}-R_{3424}) = -\frac{1}{2}(\rho^*_{14}-\rho^*_{41}),\\
g(\mathcal{W}^+(\Omega_0),J\Phi) &= \frac{1}{2}(-R_{1214}-R_{1223}-R_{3414}-R_{3423}) = \frac{1}{2}(\rho^*_{13}-\rho^*_{31}),\\
g(\mathcal{W}^+(\Phi),\Phi) &= \frac{1}{2}\Big(-R_{1313}+2R_{1324}-R_{2424}-\frac{\tau}{6}\Big) = -(R_{1313}-R_{1324})-\frac{\tau}{12},\\
g(\mathcal{W}^+(\Phi),J\Phi) &= \frac{1}{2}(-R_{1314}-R_{1323}+R_{2414}+R_{2423}) = -(R_{1314}+R_{1323}),\\
g(\mathcal{W}^+(J\Phi),J\Phi) &= \frac{1}{2}\Big(-R_{1414}-2R_{1423}-R_{2323}-\frac{\tau}{6}\Big) = -(R_{1414}+R_{1423})-\frac{\tau}{12}.
\end{aligned}
\tag{2.10}
$$

Thus, we have

$$
\begin{aligned}
\|\mathcal{W}^+\|^2 &= \frac{1}{12^2}(3\tau^*-\tau)^2+\frac{2\tau^2}{12^2}+\frac{1}{2}(\rho^*_{13}-\rho^*_{31})^2+\frac{1}{2}(\rho^*_{14}-\rho^*_{41})^2\\
&\quad+(R_{1313}-R_{1324})^2+(R_{1314}+R_{1323})^2+(R_{1314}+R_{1323})^2\\
&\quad+(R_{1414}+R_{1423})^2+\frac{\tau}{6}(R_{1313}-R_{1324}+R_{1414}+R_{1423})\\
&= \frac{1}{12^2}(3\tau^*-\tau)^2+\frac{2\tau^2}{12^2}+\frac{G}{8}\\
&\quad+\frac{1}{4}\sum_{i<j,k<l}(R_{ijkl}-R_{ij\bar{k}\bar{l}})^2-\frac{1}{4}\sum_{k<l}(R_{12kl}-R_{12\bar{k}\bar{l}})^2\\
&\quad-\frac{1}{4}\sum_{k<l}(R_{34kl}-R_{34\bar{k}\bar{l}})^2+\frac{\tau}{6}\Big(-\frac{\tau}{4}-R_{1212}-R_{1234}\Big)\\
&= \frac{1}{12^2}(3\tau^*-\tau)^2+\frac{2\tau^2}{12^2}+\frac{G}{8}+\frac{D}{16}-\frac{G}{32}-\frac{G}{32}+\frac{\tau}{6}\Big(-\frac{\tau}{4}+\frac{\tau^*}{4}\Big)\\
&= \frac{D}{16}+\frac{G}{16}+\frac{(\tau^*)^2}{16}-\frac{\tau^2}{48}.
\end{aligned}
\tag{2.11}
$$

The lemma follows. □

Next, we recall the following equalities established in [6]:

$$
\begin{aligned}
A &= \frac{1}{4}B = \frac{(\tau^*-\tau)^2}{2},\\
C &= -2K+\frac{(\tau^*-\tau)^2}{8},\\
G &= 4\|\rho^*\|^2-(\tau^*)^2 = 16\Big\{(\rho^*_{13})^2+(\rho^*_{14})^2\Big\},\\
K &= (u+v)^2+4(w^2-uv) = \frac{(\tau^*-\tau)^2}{16}-4\det\mathcal{R}'_{LM},\\
\|\mathcal{R}_{LM}\|^2 &= \frac{1}{16}D, \qquad \|\mathcal{R}'_{LM}\|^2 = \frac{1}{16}(D-G),
\end{aligned}
\tag{2.12}
$$

where $\mathcal{R}_{LM}$ is the restriction of $\mathcal{R}$ to LM and $\mathcal{R}'_{LM} = P_{LM} \circ \mathcal{R}_{LM}$, the composition of $\mathcal{R}_{LM}$ and the natural projection $P_{LM} : \wedge^2 M \to LM$. We define a vector field $\eta = (\eta_a)$ on M by $\eta_a = \sum_{i,j=1}^{4} (\nabla_a J_{ij}) \rho^*_{\bar{i}\bar{j}}$, then we obtain the following (see [6, (2.23)]):

$$\Delta\tau^* = \frac{G}{2} + 4K + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} - 4\,\mathrm{div}\,\eta. \tag{2.13}$$

Further, from (2.12) and the curvature identity

$$R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}} = 2\sum_{a=1}^{4} (\nabla_a J_{ij}) \nabla_a J_{kl} \tag{2.14}$$

by Gray [3] for almost Kähler manifold, we have

$$\begin{aligned} A &= \frac{1}{2}\sum R_{ijkl}\big(R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}}\big) \\ &= \frac{1}{4}\sum \big(R_{ijkl} - R_{ij\bar{k}\bar{l}}\big)^2 - \frac{1}{4}\sum \big(R_{ijkl} - R_{ij\bar{k}\bar{l}}\big)\big(R_{\bar{i}\bar{j}kl} - R_{\bar{i}\bar{j}\bar{k}\bar{l}}\big) + 2\sum R_{ijkl} R_{i\bar{j}k\bar{l}} \\ &= \frac{D}{4} - \frac{1}{4}\Big\{-16\|\mathcal{R}'_{LM}\|^2 + \sum \big(R_{ij12} + R_{ij34} - R_{\bar{i}\bar{j}12} - R_{\bar{i}\bar{j}34}\big)^2\Big\} + 2C \\ &= \frac{D}{4} + 4\|\mathcal{R}'_{LM}\|^2 - \frac{G}{4} + 2C \\ &= \frac{D}{2} - \frac{G}{2} - 4K + \frac{(\tau^* - \tau)^2}{4}. \end{aligned} \tag{2.15}$$

Thus, from (2.12) and this equality, we obtain

$$\frac{D}{2} - \frac{G}{2} - 4K - \frac{(\tau^* - \tau)^2}{4} = 0. \tag{2.16}$$

Now, we are ready to prove the following.

Theorem 2.2. *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler-Einstein manifold with negative scalar curvature. If M satisfies*

$$\int_M \{G + \tau(\tau^* - \tau)\}\, dV \ge 0, \tag{2.17}$$

or, more strictly, if $\tau^* - \tau \le -G/\tau$ at each point of M, then M is Kähler-Einstein.

Proof. From (2.8), we have

$$24\int_M \|\mathcal{W}^+\|^2 dV - \int_M \tau^2 dV = \frac{3}{2}\int_M \{G + D + (\tau^* - \tau)(\tau^* + \tau)\}\, dV. \tag{2.18}$$

On one hand, from (2.13) and (2.16), we have

$$0 = \int_M \Big\{\frac{G}{2} + 4K + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4}\Big\} dV = \int_M \Big\{\frac{D}{2} + \frac{\tau^*(\tau^* - \tau)}{2}\Big\} dV. \tag{2.19}$$

Thus, from (2.18) and (2.19), we obtain

$$24\int_M \|\mathcal{W}^+\|^2 dV - \int_M \tau^2 dV = \frac{2}{3}\int_M \{G+\tau(\tau^*-\tau)\}dV. \tag{2.20}$$

Therefore, from Corollary 1.3, the assertion of the theorem immediately follows. □

Remark 2.3. The above theorem is concerned with the following facts.

(1) For a compact four-dimensional almost Kähler-Einstein manifold, the function $\tau^* - \tau$ vanishes at some point of M (see [1, 5]).

(2) A four-dimensional compact almost Kähler-Einstein and weakly $*$-Einstein manifold ($G \equiv 0$) is a Kähler manifold (see [7]).

(3) Let M be a four-dimensional compact strictly almost Kähler-Einstein, but not weakly $*$-Einstein manifold. Then, we see that $G > 0$ on $M_0 = \{p \in M \mid \tau^* - \tau > 0\}$, and hence $\tau^* - \tau = 0$ at which $G = 0$ (see [5]).

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