VANISHING MOMENTS FOR SCALING VECTORS

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One advantage of scaling vectors over a single scaling function is the compatibility of symmetry and orthogonality. This paper investigates the relationship between symmetry, vanishing moments, orthogonality, and support length for a scaling vector Φ. Some general results on scaling vectors and vanishing moments are developed, as well as some necessary conditions for the symbol entries of a scaling vector with both symmetry and orthogonality. If orthogonal scaling vector Φ has some kind of symmetry and a given number of vanishing moments, we can characterize the type of symmetry for Φ, give some information about the form of the symbol $P(z)$, and place some bounds on the support of each ϕ_i. We then construct an $L^2(\mathbb{R})$ orthogonal, symmetric scaling vector with one vanishing moment having minimal support.

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1. Introduction. In this paper, we will discuss conditions for vanishing moments for *scaling vector* $\Phi = (\phi_1, \dots, \phi_r)^T$ solutions to the *matrix refinement equation* (MRE)

$$\Phi(x) = \sum_{k=0}^{N} C_k \Phi(2x - k) \tag{1.1}$$

with $r \times r$ matrix coefficients C_k. Taking the Fourier transform of this MRE gives

$$\hat{\Phi}(w) = P\left(\frac{w}{2}\right) \hat{\Phi}\left(\frac{w}{2}\right), \tag{1.2}$$

where the 2-*scale symbol* P is the $r \times r$ matrix

$$P(w) = \frac{1}{2} \sum_{k=0}^{N} C_k e^{-ikw} \quad \text{or} \quad P(z) = \frac{1}{2} \sum_{k=0}^{N} C_k z^k \quad \text{with } z = e^{-iw}. \tag{1.3}$$

We will use $p_{ij}(z)$ to denote the i-j entry of P, and we will occasionally interchange z and w for the argument of the symbol P; the context should prevent confusion.

Scaling vectors and solutions to MREs have been studied in [1, 3, 5, 6, 7] and many others. In particular, Heil and Colella [5] showed that the eigenvalues of $\Delta^\infty = \lim_{j \to \infty} P^j(1)$ are crucial to understanding solutions of the MRE. The unconstrained solutions are precisely those which are the compactly supported distributional solutions to the MRE (1.1). For this reason, we will restrict our attention in this paper to unconstrained solutions to (1.1). Indeed, all fundamental solutions of scalar refinement equations are of this type. The vast majority of scaling vector examples in the literature are unconstrained.

Definition 1.1. Solutions to the MRE (1.1) are defined to be *unconstrained* if Δ^∞ exists and is nontrivial: $P(0)$ has 1 for a nondegenerate eigenvalue, and all other eigenvalues have modulus less than 1.

It is also shown in [5] that unconstrained MRE solutions Φ satisfy

$$\hat{\Phi}(1) = \left(\int_{\mathbb{R}} \phi_1, \ldots, \int_{\mathbb{R}} \phi_r \right)^T \neq \mathbf{0}. \tag{1.4}$$

As is standard in the literature, we say the *polynomial accuracy p* of a scaling vector is the maximum integer p for which $x^k \in V_0$ for $k = 0, 1, \ldots, p-1$. A common sufficient condition for the density condition of a multiresolution analysis (MRA) generated by Φ requires polynomial accuracy at least $p \geq 1$ (e.g., [3]). Throughout this paper, all scaling vectors are assumed to have accuracy $p \geq 1$.

The following combines some important results on accuracy and smoothness found in [1, 7] necessary for our discussion.

Theorem 1.2. *Suppose that compactly supported scaling vector Φ generates an MRA. The following are equivalent:*

(1) Φ *provides polynomial accuracy m;*

(2) *the elements of $P(w)$ are trigonometric polynomials and there are vectors $\mathbf{y}^k$ for $k = 0, \ldots, m-1$ satisfying*

$$\sum_{k=0}^{m} \binom{m}{k} (\mathbf{y}^k)^T (2i)^{k-m} (D^{m-k}P)(0) = 2^{-m} (\mathbf{y}^m)^T,$$
$$\sum_{k=0}^{m} \binom{m}{k} (\mathbf{y}^k)^T (2i)^{k-m} (D^{m-k}P)(\pi) = \mathbf{0}^T; \tag{1.5}$$

(3) *the symbol $P(z)$ factors as*

$$P(z) = \frac{1}{2^m} C_{x_0}(z^2) \cdots C_{x_{m-1}}(z^2) P_m(z) C_{x_{m-1}}(z)^{-1} \cdots C_{x_0}(z)^{-1}. \tag{1.6}$$

In this case, if $\inf \gamma_k < m - 1/2$, then $\Phi \in L^2(\mathbb{R})$, where

$$\gamma_k = \frac{1}{k} \log_2 \sup_{\omega} \left\| P_m\left(\frac{\omega}{2}\right) \cdots P_m\left(\frac{\omega}{2^k}\right) \right\|. \tag{1.7}$$

Whether the MRA generated by Φ is orthogonal is determined by the following condition found in [10].

Theorem 1.3. *The scaling vector Φ generates an orthogonal basis if and only if its symbol P satisfies*

$$P(z)P^*\left(\frac{1}{z}\right) + P(-z)P^*\left(\frac{-1}{z}\right) = I_r. \tag{1.8}$$

One advantage to scaling vectors over a single scaling function is the compatibility of symmetry and orthogonality, as seen in [3] and elsewhere. The symmetry condition given below for a scaling vector was proved by Plonka and Strela and appears in [7].

Theorem 1.4. *A scaling vector Φ has component functions that are symmetric or antisymmetric if and only if there exists a matrix $E(z) = \mathrm{diag}(\pm z^{2T_1}, \ldots, \pm z^{2T_r})$ for which*

$$P(z) = E(z^2)P(z^{-1})E^{-1}(z), \tag{1.9}$$

where the T_j are the points of symmetry for the ϕ_j and the $\pm$ sign is $+$ when ϕ_j is symmetric and $-$ when antisymmetric.

When $r = 2$ and $T_1 = 0$, the condition $P(z) = E(z^2)P(z^{-1})E^{-1}(z)$ is equivalent to the following, useful in the sequel:

$$
\begin{aligned}
p_{11}(z) &= p_{11}(z^{-1}), & p_{12}(z) &= \pm z^{-2T_2} p_{12}(z^{-1}), \\
p_{21}(z) &= \pm z^{4T_2} p_{21}(z^{-1}), & p_{22}(z) &= z^{2T_2} p_{22}(z^{-1}).
\end{aligned}
\tag{1.10}
$$

Remark 1.5. We will say that a scaling vector Φ is *symmetric* if all component functions of Φ are either symmetric or antisymmetric.

The following result, needed in the sequel, shows that the length of each polynomial entry in the symbol is unaffected by integer shifts of the scaling functions.

Proposition 1.6. *Let $\Phi = (\phi_1, \phi_2)^T$ be a scaling vector with symbol P. If each ϕ_k is shifted by $S_k \in \mathbb{Z}$, the symbol of the new scaling vector Φ_* is*

$$
P_*(z) = \begin{pmatrix} z^{-S_1} p_{11}(z) & z^{S_2 - 2S_1} p_{12}(z) \\ z^{S_1 - 2S_2} p_{21}(z) & z^{-S_2} p_{22}(z) \end{pmatrix}.
\tag{1.11}
$$

Proof. Note that $\hat{\Phi}_*(z) = T(z)\hat{\Phi}(z)$, where $T(z) = \mathrm{diag}\{z^{-S_1}, z^{-S_2}\}$. Substituting into (1.2), we have

$$
\begin{aligned}
T^{-1}(z^2)\hat{\Phi}_*(z^2) &= P(z)T^{-1}(z)\hat{\Phi}_*(z), \\
\hat{\Phi}_*(z^2) &= T(z^2)P(z)T^{-1}(z)\hat{\Phi}_*(z) \\
&= \begin{pmatrix} z^{-S_1} p_{11}(z) & z^{S_2 - 2S_1} p_{12}(z) \\ z^{S_1 - 2S_2} p_{21}(z) & z^{-S_2} p_{22}(z) \end{pmatrix} \hat{\Phi}_*(z).
\end{aligned}
\tag{1.12}
$$

$\square$

2. Scaling vectors with vanishing moments. The study of vanishing moments for a single scaling function is discussed in [2] and many other papers. In [2], Daubechies shows how to link the symbol's condition for vanishing moments with the condition for polynomial accuracy via Bezout's theorem to obtain a family of coiflets. In the scaling vector situation, the symbol is a matrix, and matters are more complex. In this section, we develop some general results on scaling vectors and vanishing moments, as well as some necessary conditions for the symbol entries of a scaling vector with both symmetry and orthogonality. We first characterize the vanishing moments of Φ in terms of the derivatives of its symbol $D^k P(0)$ according to the following result, first presented in [8].

DEFINITION 2.1. $\Phi = (\phi_1, \ldots, \phi_r)^T$ has $L-1$ *vanishing moments* if

$$\int x^n \phi_i(x)\,dx = 0 \quad \text{for } n = 1, \ldots, L-1 \tag{2.1}$$

for all ϕ_i.

THEOREM 2.2. *The scaling vector $\Phi = (\phi_1, \ldots, \phi_r)^T$ generating an MRA has $L-1$ vanishing moments if and only if $\hat{\Phi}(0)$ is a 0-eigenvector of $D^k P(0)$ for $k = 1, \ldots, L-1$.*

PROOF. As noted in (1.4), $\hat{\Phi}(0) \neq \mathbf{0}$. Differentiating the MRE (1.2) k times with respect to ω yields

$$\hat{\Phi}^{(k)}(\omega) = \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} D^{k-i} P\left(\frac{\omega}{2}\right) \hat{\Phi}^{(i)}\left(\frac{\omega}{2}\right). \tag{2.2}$$

Suppose Φ has $L-1$ vanishing moments. Recall from Fourier analysis that Φ having $L-1$ vanishing moments is equivalent to

$$\hat{\Phi}^{(i)}(0) = \mathbf{0} \tag{2.3}$$

for $i = 1, \ldots, L-1$. Now let $k \in \{1, \ldots, L-1\}$. From the theorem hypothesis, (2.2) at $\omega = 0$ reduces to

$$\mathbf{0} = \frac{1}{2^k} D^k P(0) \hat{\Phi}^{(0)}(0), \tag{2.4}$$

so $\hat{\Phi}(0)$ is a 0-eigenvector of $D^k P(0)$.

Suppose $\hat{\Phi}(0)$ is a 0-eigenvector. We use the principle of strong induction. Define $S = \{n \in \mathbb{N}\colon \text{if } \hat{\Phi}(0) \text{ is a 0-eigenvector of } D^k P(0) \text{ for } k = 1, \ldots, n, \text{ then } \Phi \text{ has } n \text{ vanishing moments}\}$. To see that $1 \in S$, assume $\hat{\Phi}(0)$ is a 0-eigenvector of $DP(0)$. From (2.2) at $\omega = 0$ and $k = 1$, we get

$$\hat{\Phi}^{(1)}(0) = \frac{1}{2^1} DP(0) \hat{\Phi}^{(0)}(0) + P(0) \hat{\Phi}^{(1)}(0) = \frac{1}{2} P(0) \hat{\Phi}^{(1)}(0). \tag{2.5}$$

Thus $P(0)\hat{\Phi}^{(1)}(0) = 2\hat{\Phi}^{(1)}(0)$, so either $\hat{\Phi}^{(1)}(0)$ is a 2-eigenvector of $P(0)$ or $\hat{\Phi}^{(1)}(0) = \mathbf{0}$. However, since we have an MRA, $P(0)$ has spectral radius 1 by [6, Proposition 1.1], which forces $\hat{\Phi}^{(1)}(0) = \mathbf{0}$. Thus $1 \in S$.

Next assume that $\{1, \ldots, k\} \subset S$. To see that $k+1 \in S$, assume that $\hat{\Phi}(0)$ is a 0-eigenvector of $D^i P(0)$ for $i = 1, \ldots, k+1$. Since $\{1, \ldots, k\} \subset S$, Φ has k vanishing moments. From this and (2.2) at $\omega = 0$ with $k+1$, we have

$$\begin{aligned}
\hat{\Phi}^{(k+1)}(0) &= \frac{1}{2^{k+1}} \sum_{i=0}^{k+1} \binom{k+1}{i} D^{k+1-i} P(0) \hat{\Phi}^{(i)}(0) \\
&= \frac{1}{2^{k+1}} P(0) \hat{\Phi}^{(k+1)}(0).
\end{aligned} \tag{2.6}$$

Thus $\hat{\Phi}^{(k+1)}(0)$ is a 2^{k+1}-eigenvector of $P(0)$ or $\hat{\Phi}^{(k+1)}(0) = \mathbf{0}$. Since $P(0)$ has spectral radius 1, $\hat{\Phi}^{(k+1)}(0) = \mathbf{0}$. Thus $k+1 \in S$. $\qquad\square$

We now record some useful results about vanishing moments and symmetric scaling vectors.

PROPOSITION 2.3. *If $f(x)$ is symmetric about some $\beta \neq 0$, and $\int_{\mathbb{R}} x f(x) dx = 0$, then $\int_{\mathbb{R}} f(x) dx = 0$.*

PROOF. Define $g(u) = u f(u + \beta)$ and note that g is odd. Thus $\int_{\mathbb{R}} g(u) du = 0$, so letting $x = u + \beta$, we have $\int_{\mathbb{R}} x f(x) dx = \beta \int_{\mathbb{R}} f(x) dx$ whence $\int_{\mathbb{R}} f(x) dx = 0$.   □

COROLLARY 2.4. *If Φ is a symmetric scaling vector with a vanishing moment, then at least one ϕ_k must be symmetric about 0.*

PROOF. $\hat{\Phi}(0)$ cannot vanish as noted in (1.4). Apply Proposition 2.3.   □

THEOREM 2.5. *Suppose $\Phi = (\phi_1, \ldots, \phi_r)^T$ generates a nontrivial orthonormal basis. Then not all ϕ_i can be symmetric about integers, and not all ϕ_i can be antisymmetric.*

PROOF. That not all ϕ_i can be antisymmetric follows immediately from Corollary 2.4.

To prove that not all ϕ_i can be symmetric, we prove the contrapositive. Begin by shifting each ϕ_i so that it is symmetric about 0, if necessary. Now let $P(z)$ be the symbol of this shifted Φ. By the symmetry Theorem 1.4, $P(z) = P(1/z)$ for all z. Thus the k-k entry of $\theta = P(z)P^*(1/z) + P(-z)P^*(-1/z)$ is

$$\theta_{kk} = \sum_{j=1}^{r} p_{kj}^2(z) + p_{kj}^2(-z). \tag{2.7}$$

Letting n_{kj} denote the degree of the highest-degree nonzero term $a_{n_{kj}} z^{n_{kj}}$ of $p_{kj}(z)$ and $n_k = \max_j(n_{kj})$, we see that θ_{kk} has $2n_k$ degree term

$$\sum_{n_{kj} = n_k} (a_{n_{kj}} z^{n_{kj}})^2 + (a_{n_{kj}}(-z)^{n_{kj}})^2 = 2z^{2n_k} \sum_{n_{kj} = n_k} (a_{n_{kj}})^2 \neq 0. \tag{2.8}$$

Since the basis is nontrivial, $P(z)$ is not constant, so $n_k > 0$ for at least one value of k whence θ_{kk} is not constant for this k value. Thus θ is not the identity matrix, and so by (1.8), Φ does not generate an orthonormal basis.   □

We introduce the notation $\deg L(R) = n - m$ for a polynomial $R(z) = a_m z^m + \cdots + a_n z^n$, where $m \leq n$. The following results are useful for analyzing the symbol $P(z)$ when scaling vector Φ is symmetric.

LEMMA 2.6. *Let polynomial $R(z) = a_m z^m + \cdots + a_n z^n$ satisfy $z^k R(z) = \pm R(z^{-1})$, where $a_m, a_n \neq 0$, $m \leq n$. Then $m = -n - k$ and $\deg L(R) = 2n + k$ is even if and only if k is even.*

PROOF. The condition $z^k R(z) = \pm R(z^{-1})$ can be rewritten as follows:

$$a_m z^{m+k} + \cdots + a_n z^{n+k} = \pm(a_m z^{-m} + \cdots + a_n z^{-n}). \tag{2.9}$$

Then the highest-degree terms of (2.9) must be $a_n z^{n+k} = \pm a_m z^{-m}$ so $n + k = -m$ whence $\deg L(R) = n - m = 2n + k$.   □

PROPOSITION 2.7. *Suppose symmetric* $\Phi = (\phi_1, \phi_2)^T$ *generates a nontrivial orthonormal basis with the* ϕ_k *(anti)symmetric about integers. Then*

$$\mathrm{degL}\,(p_{11}) = \mathrm{degL}\,(p_{12}), \qquad \mathrm{degL}\,(p_{21}) = \mathrm{degL}\,(p_{22}). \tag{2.10}$$

PROOF. By Theorem 2.5, we may assume ϕ_1 is symmetric and ϕ_2 is antisymmetric. Then shift each ϕ_k to be (anti)symmetric about 0. Note that this does not affect $\mathrm{degL}(p_{jk})$ for any j, k by Proposition 1.6. Letting $P(z)$ denote the symbol of this shifted scaling vector, Theorem 1.4 is satisfied with $E(w) = \mathrm{diag}\{1, -1\}$; so $p_{11}(z) = p_{11}(z^{-1})$, $p_{12}(z) = -p_{12}(z^{-1})$, $p_{21}(z) = -p_{21}(z^{-1})$, $p_{22}(z) = p_{22}(z^{-1})$, and so the k-k entry of $\theta = P(z)P^*(1/z) + P(-z)P^*(-1/z)$ is

$$\theta_{kk} = p_{k1}^2(z) + p_{k1}^2(-z) - (p_{k2}^2(z) + p_{k2}^2(-z)). \tag{2.11}$$

Since $\theta_{kk} = 1$ because of the orthogonality, $\deg(p_{k1}^2(z)) = \deg(p_{k2}^2(z))$. Thus $\mathrm{degL}(p_{k1}) = \mathrm{degL}(p_{k2})$. $\qquad\square$

PROPOSITION 2.8. *Suppose symmetric* $\Phi = (\phi_1, \phi_2)^T$ *generates a nontrivial orthonormal basis with* ϕ_1 *symmetric about an integer and* ϕ_2 *symmetric about a half integer. Then*

$$\mathrm{degL}\,(p_{11}) + 1 \leq \mathrm{degL}\,(p_{12}), \qquad \mathrm{degL}\,(p_{21}) + 1 \leq \mathrm{degL}\,(p_{22}). \tag{2.12}$$

PROOF. Shift ϕ_1 and ϕ_2 to be symmetric about 0 and $-1/2$, respectively. Note that this does not affect $\mathrm{degL}(p_{jk})$ for any j, k by Proposition 1.6. Letting $P(z)$ denote the symbol of this shifted scaling vector, Theorem 1.4 is satisfied; so by (1.10), $p_{11}(z) = p_{11}(z^{-1})$, $z^{-1}p_{12}(z) = p_{12}(z^{-1})$, $p_{21}(z) = z^{-2}p_{21}(z^{-1})$, $p_{22}(z) = z^{-1}p_{22}(z^{-1})$, and so the 1-1 entry of $\theta = P(z)P^*(1/z) + P(-z)P^*(-1/z)$ is

$$\theta_{11} = p_{11}^2(z) + p_{11}^2(-z) + z^{-1}p_{12}^2(z) - z^{-1}p_{12}^2(-z). \tag{2.13}$$

Note that the highest-degree terms of $z^{-1}p_{12}^2(z)$ and $z^{-1}p_{12}^2(-z)$ are equal and of odd degree and so $z^{-1}p_{12}^2(z) - z^{-1}p_{12}^2(-z)$ has degree at most $2\deg(p_{12}(z)) - 2$. Since $\theta_{kk} = 1$ because of the orthogonality, $2\deg(p_{12}) - 2 \geq \deg(p_{11}^2) = 2\deg(p_{11})$. Thus $\deg(p_{12}) - 1 \geq \deg(p_{11})$. Using Lemma 2.6, we see that

$$\mathrm{degL}\,(p_{12}) = 2\deg\,(p_{12}) - 1 \geq 2(\deg\,(p_{11}) + 1) - 1 = \mathrm{degL}\,(p_{11}) + 1. \tag{2.14}$$

Proving the second claim is very similar. Using (1.10),

$$\theta_{22} = z^2 p_{21}^2(z) + z^2 p_{21}^2(-z) + z p_{22}^2(z) - z p_{22}^2(-z), \tag{2.15}$$

and noting that $zp_{22}^2(z) - zp_{22}^2(-z)$ has degree at most $2\deg(p_{22})$, we see that $2\deg(p_{22}) \geq 2\deg(p_{21}) + 2$. Using Lemma 2.6, we see that

$$\mathrm{degL}\,(p_{22}) = 2\deg\,(p_{22}) + 1 \geq 2\deg\,(p_{21}) + 3 = \mathrm{degL}\,(p_{21}) + 1. \tag{2.16}$$

$\square$

The following result says that we can shift a function ϕ with vanishing moments without affecting the vanishing moments, provided that $\int_{\mathbb{R}} \phi = 0$. This will simplify some of our proofs.

PROPOSITION 2.9. *If $\int_{\mathbb{R}} x^k f(x)dx = 0$ for $k = 0,1,\ldots,L-1$, then $\int_{\mathbb{R}} x^k f(x-S)dx = 0$ for any S.*

PROOF. Substituting $u = x - S$, apply the binomial theorem and hypothesis: $\int_{\mathbb{R}} x^k f(x-S)dx = \int_{\mathbb{R}} (u+S)^k f(u)du = \sum_{j=0}^{k} \binom{k}{j} S^{k-j} \int_{\mathbb{R}} u^j f(u)du = 0.$ $\square$

3. Support bounds for orthogonal, symmetric scaling vectors with vanishing moments. We now limit discussion to scaling vectors with $r = 2$ component functions.

If orthogonal $\Phi = (\phi_1,\phi_2)^T$ has some kind of symmetry and a given number of vanishing moments, we can characterize the type of symmetry for Φ, give some information about the form of the symbol $P(z)$, and place some bounds on the support of each ϕ_i.

We also demonstrate how to construct an $L^2(\mathbb{R})$ orthogonal symmetric scaling vector with minimal support for one vanishing moment. This support length depends on the type of symmetry.

We first introduce some standard notation about support for scaling vectors needed for our analysis of the interplay between vanishing moments and support. We will say that the *support* of $\Phi = (\phi_1,\ldots,\phi_r)^T$ is $\mathrm{supp}(\Phi) = \cup_i \mathrm{supp}(\phi_i)$, with each $\mathrm{supp}(\phi_i)$ being the convex hull of the support points of ϕ_i, as in [9]. The support length of ϕ_i, denoted by $\mathrm{suppL}(\phi_i)$, is the length of the interval $\mathrm{supp}(\phi_i)$.

A family $\{f_i\}$ of functions on $\mathbb{R}$ is *locally linearly independent (LLI)* if $\sum_i c_i f_i(x) = 0$ on any nontrivial interval I implies that $c_i = 0$ for all i for which $\mathrm{supp}(f_i) \cap I \neq \varnothing$. We say a scaling vector Φ is LLI if the family $\{\phi_i(x-k) : 1 \leq i \leq r,\ k \in \mathbb{Z}\}$ is LLI. Wavelets and the LLI property have been studied together [4, 9], and So and Wang [9] proved the following result about the connection between the symbol $P(z)$ entries' degrees and each $\mathrm{supp}(\phi_i)$. This result allows us to put some bounds on the support length of ϕ_i.

THEOREM 3.1. *Suppose $\Phi = (\phi_1,\phi_2)^T$ is LLI. Then $\mathrm{supp}(\phi_i) = [L_i, R_i]$ for $i = 1,2$, where*

$$
\begin{aligned}
R_1 &= \max\left\{H_{11}, \frac{2H_{12}+H_{21}}{3}, \frac{H_{12}+H_{22}}{2}\right\}, &\quad L_1 &= \min\left\{B_{11}, \frac{2B_{12}+B_{21}}{3}, \frac{B_{12}+B_{22}}{2}\right\}, \\
R_2 &= \max\left\{H_{22}, \frac{2H_{21}+H_{12}}{3}, \frac{H_{11}+H_{21}}{2}\right\}, &\quad L_2 &= \min\left\{B_{22}, \frac{2B_{21}+B_{12}}{3}, \frac{B_{21}+B_{11}}{2}\right\},
\end{aligned}
\tag{3.1}
$$

H_{ij} and B_{ij} being the highest and lowest degrees, respectively, of $p_{ij}(z)$.

COROLLARY 3.2. *Suppose* $\Phi = (\phi_1, \phi_2)^T$ *is LLI. Then*

$$\mathrm{suppL}(\phi_1) \geq \max\left\{ \mathrm{degL}(p_{11}), \frac{2\,\mathrm{degL}(p_{12}) + \mathrm{degL}(p_{21})}{3}, \frac{\mathrm{degL}(p_{12}) + \mathrm{degL}(p_{22})}{2} \right\},$$

$$\mathrm{suppL}(\phi_2) \geq \max\left\{ \mathrm{degL}(p_{22}), \frac{2\,\mathrm{degL}(p_{21}) + \mathrm{degL}(p_{12})}{3}, \frac{\mathrm{degL}(p_{11}) + \mathrm{degL}(p_{21})}{2} \right\}.$$

$$\tag{3.2}$$

PROOF. Since $\mathrm{degL}(p_{ij}) = H_{ij} - B_{ij}$, Theorem 3.1 yields $\mathrm{suppL}(\phi_1) = R_1 - L_1 \geq (1/3)(2H_{12} + H_{21}) - (1/3)(2B_{12} + B_{21}) = (2/3)\,\mathrm{degL}(p_{12}) + (1/3)\,\mathrm{degL}(p_{21})$. This proves that $\mathrm{suppL}(\phi_1) \geq \mathrm{degL}(p_{11}), (1/2)(\mathrm{degL}(p_{12}) + \mathrm{degL}(p_{22}))$ are very similar, which gives us the first claim. The proof of the second claim is the same. $\qquad\square$

The next two lemmas contain useful information about scaling vectors with vanishing moments.

LEMMA 3.3. *Suppose symmetric, orthogonal scaling vector* $\Phi = (\phi_1, \phi_2)^T$ *has accuracy* $p \geq 1$ *and at least one vanishing moment. If* ϕ_1 *is symmetric about 0, then* $\hat{\Phi}(1)$ *must be a multiple of* $(1,0)^T$, $P(1)$ *has a one-dimensional right 1-eigenspace, and* $\mathbf{y}^0$ *of Theorem 1.2 must be a nonzero scalar multiple of* $\hat{\Phi}(1)$.

PROOF. By Proposition 2.3, Corollary 2.4, and Theorem 2.5, ϕ_2 must be either antisymmetric or symmetric about some $\beta \neq 0$. In either case, $\hat{\phi}_2(1) = \int_{\mathbb{R}} \phi_2 = 0$, so $\hat{\Phi}(1)$ must be multiple of $(1,0)^T$ and $P(1)\hat{\Phi}(1) = \hat{\Phi}(1)$; so

$$P(1) = \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}. \tag{3.3}$$

By Definition 1.1, the eigenvalue y must be 1 or less than 1 in magnitude. For the sake of contradiction, suppose that $y = 1$. Then $P(1)$ has two linearly independent eigenvectors $\hat{F}^1(1)$, $\hat{F}^2(1)$ corresponding to two linearly independent scaling vector solutions F^1, F^2 by [5, Theorem 2.3]. Thus one of these, say $F^2 = (f_1^2, f_2^2)^T$, must satisfy $\hat{F}^2(1) = (a,b)^T$ for some $b \neq 0$. But then its second component function $\hat{f}_2^2(1) = \int_{\mathbb{R}} f_2^2 = b \neq 0$. On the other hand, since Φ is symmetric and orthogonal, its symbol P has the form requiring all other scaling vector solutions to be symmetric and orthogonal, by Theorems 1.4 and 1.3, with the first component function symmetric about 0. So, just as with Φ, f_2^2 must be either antisymmetric, or symmetric about some $\beta \neq 0$, and so $\hat{f}_2^2(1) = \int_{\mathbb{R}} f_2^2 = 0$. This is a contradiction; so $|y| < 1$, and $P(1)$ has a one-dimensional right 1-eigenspace.

To see the final claim, multiply the orthogonality condition (1.8), evaluated at $z = 1$, on the left by $(\mathbf{y}^0)^T$ from Theorem 1.2. Recalling that $(\mathbf{y}^0)^T P(1) = (\mathbf{y}^0)^T$ and $(\mathbf{y}^0)^T P(-1) = \mathbf{0}$, this reduces to $(\mathbf{y}^0)^T P^*(1) + \mathbf{0} \cdot P^*(-1) = (\mathbf{y}^0)^T$. Taking transposes gives $P(1)\mathbf{y}^0 = \mathbf{y}^0$; so $\mathbf{y}^0$; is a right 1-eigenvector of $P(1)$. Thus $\mathbf{y}^0$ must be a multiple of $\hat{\Phi}(1)$. $\qquad\square$

LEMMA 3.4. *Suppose* $\Phi = (\phi_1, \phi_2)^T$ *has* $L - 1 \geq 1$ *vanishing moments and* $\hat{\Phi}(1) = A(1,0)^T$ *for some nonzero constant A. Then the entries* p_{11} *and* p_{21} *of symbol* $P(z)$

satisfy

$$p_{11}(z) = 1 + (1-z)^L Q_{11}(z),$$
$$p_{21}(z) = (1-z)^L Q_{21}(z) \tag{3.4}$$

for polynomials Q_{11}, Q_{21}.

PROOF. Since $P(1)\hat{\Phi}(1) = \hat{\Phi}(1)$, we must have $p_{11}(1) = 1$ and $p_{21}(z) = 0$. By Theorem 2.2,

$$D^k P(1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.5}$$

for $k = 1, \ldots, L-1$. Thus $(1-z)^L$ must divide $p_{11}(z) - 1$ and $p_{21}(z)$. ☐

The first symmetric, orthogonal, continuous, compactly supported scaling vector $\Phi = (\phi_1, \phi_2)^T$ in the literature, constructed in [3], had ϕ_1 symmetric about an integer and ϕ_2 symmetric about $1/2$. The next result gives necessary conditions on a scaling vector's symmetry, symbol, and support length if we insist additionally on vanishing moments. In particular, neither ϕ_1 nor ϕ_2 can be antisymmetric.

THEOREM 3.5. *Suppose symmetric, orthogonal, LLI scaling vector* $\Phi = (\phi_1, \phi_2)^T$ *has accuracy* $p \geq 1$ *and* $L - 1 \geq 1$ *vanishing moments. If* ϕ_2 *is symmetric or antisymmetric about* $m + 1/2$ *for integer* m, *then*

(1) ϕ_1 *is symmetric about* 0;

(2) ϕ_2 *is symmetric;*

(3) *if L is even, then* $\operatorname{suppL}(\phi_1) \geq L+1$, $\operatorname{suppL}(\phi_2) \geq L+1$, *and* $P(z)$ *has the form*

$$P(z) = \begin{pmatrix} 1 + (1-z)^L Q_{11}(z) & (1-z^2)(1-z)R_{12}(z) \\ (1-z)^L Q_{21}(z) & (1+z)R_{22}(z) \end{pmatrix}; \tag{3.6}$$

(4) *if L is odd, then* $\operatorname{suppL}(\phi_1) \geq L+2$, $\operatorname{suppL}(\phi_2) \geq L+2$, *and* $P(z)$ *has the form*

$$P(z) = \begin{pmatrix} 1 + (1-z)^{L+1} R_{11}(z) & (1-z^2)(1-z)R_{12}(z) \\ (1-z)^{L+1} R_{21}(z) & (1+z)R_{22}(z) \end{pmatrix}. \tag{3.7}$$

PROOF. By Corollary 2.4, ϕ_1 must be symmetric about 0, thus proving claim (1).

From Lemma 3.3, $\int_{\mathbb{R}} \phi_2 = 0$, so by Propositions 1.6 and 2.9, we may assume WLOG that this integer m is zero.

Suppose ϕ_2 is antisymmetric for the sake of contradiction. From (1.10) with $T_2 = 1/2$, $p_{21}(z) = -z^2 p_{21}(1/z)$ and $p_{22}(z) = z p_{22}(z^{-1})$. Evaluating at $z = \pm 1$, we see $p_{21}(\pm 1) = 0$ and $p_{22}(-1) = 0$. Evaluating the 2-2 position of (1.8) at $z = 1$ simplifies to $(p_{22}(1))^2 = 1$. But $P(1) = \begin{pmatrix} 1 & x \\ 0 & p_{22}(1) \end{pmatrix}$, so Definition 1.1 says $p_{22}(1) = 1$. This says $P(0)$ has a two-dimensional right 1-eigenspace, which is a contradiction of Lemma 3.3. This says that ϕ_2 is in fact symmetric, thus completing the proof of 2.

To prove claims (3) and (4), first note that by Lemma 3.3 and Theorem 1.2, $(\mathbf{y}^0)^T P(1) = (\mathbf{y}^0)^T = A(1,0)^T = A(p_{11}(1), p_{12}(1))$; so we must have $1 - z$ as a factor of $p_{12}(z)$.

Similarly $(\mathbf{y}^0)^T P(-1) = A(p_{11}(-1), p_{12}(-1)) = \mathbf{0}$ says that we must have $1 + z$ as a factor of $p_{12}(z)$.

By Lemma 3.4 and (1.10) with $T_2 = 1/2$, we see

$$
\begin{aligned}
(1-z)^L Q_{11}(z) &= (1-z^{-1})^L Q_{11}(z^{-1}), \qquad z p_{12}(z) = p_{12}(z^{-1}), \\
(1-z)^L Q_{21}(z) &= z^2 (1-z^{-1})^L Q_{21}(z^{-1}), \qquad p_{22}(z) = z p_{22}(z^{-1}).
\end{aligned}
\tag{3.8}
$$

Multiplying the 1-1 and 2-1 positions by z^L and letting $p_{12}(z) = (1-z^2)Q_{12}$, we see that

$$
\begin{aligned}
z^L Q_{11}(z) &= (-1)^L Q_{11}(z^{-1}), \qquad z^3 Q_{12}(z) = -Q_{12}(z^{-1}), \\
z^L Q_{21}(z) &= (-1)^L z^2 Q_{21}(z^{-1}), \qquad p_{22}(-1) = 0;
\end{aligned}
\tag{3.9}
$$

so $1 + z$ divides $p_{22}(z)$ and $Q_{12}(1) = 0$; so $Q_{12}(z) = (1-z)_{12}R(z)$ for some $R_{12}(z)$.

Consider the case when L is even. The form of $P(z)$ has been verified. Using Proposition 2.8, $\deg L(p_{12}) \geq \deg L(p_{11}) + 1 \geq L + 1$ and $\deg L(p_{22}) \geq \deg L(p_{21}) + 1 \geq L + 1$; so by Corollary 3.2, $\mathrm{suppL}(\phi_2), \mathrm{suppL}(\phi_1) \geq L + 1$.

Consider the case when L is odd. In this case, we see from (3.9) that $Q_{11}(1) = Q_{21}(1) = 0$; so $1 - z$ divides both $Q_{11}(z)$ and $Q_{21}(z)$. This proves the form of $P(z)$ and $\deg L(p_{21}), \deg L(p_{11}) \geq L + 1$. Using Proposition 2.8, $\deg L(p_{22}) \geq \deg L(p_{21}) + 1 \geq L + 2$ and $\deg L(p_{12}) \geq \deg L(p_{11}) + 1 \geq L + 2$; so by Corollary 3.2, $\mathrm{suppL}(\phi_2) \geq L + 2$ and $\mathrm{suppL}(\phi_1) \geq (1/2) \deg L(p_{12}) + (1/2) \deg L(p_{22}) \geq L + 2$. □

EXAMPLE 3.6. Set $L = 2$. We use Theorem 3.5 to guide the construction of an $L^2(\mathbb{R})$ orthogonal symmetric scaling vector with minimal support for one vanishing moment. Naturally ϕ_1 is symmetric about 0 and ϕ_2 is symmetric about $1/2$. We begin with $Q_{21}(z)$ being a monomial. By Proposition 2.8, $\deg L(p_{22}) \geq \deg L(p_{21}) + 1 = 3$, so we try $\deg L(R_{22}) = 2$. Brute force with (1.8) and Lemma 2.6 shows that orthogonality cannot be obtained if $\deg L(Q_{11}) < 2$; so we try $\deg L(Q_{11}) = 2$ and thus $\deg L(R_{12}) = 2$ is minimal. We put the unknowns $Q_{11}, Q_{21}, R_{12}, R_{22}$ into form ensuring the symmetry conditions, and solved for the parameters to guarantee orthogonality, with the help of Mathematica:

$$
P(z) = \begin{bmatrix}
1 + \dfrac{(1-z)^2}{z^2}(a_{11} + b_{11}z + a_{11}z^2) & \dfrac{(1-z^2)}{z^3}(1-z)(a_{12} + b_{12}z + a_{12}z^2) \\[2ex]
a_{21}(1-z)^2 & \dfrac{(1+z)}{z}(a_{22} + b_{22}z + a_{22}z^2)
\end{bmatrix},
$$

$$
\begin{aligned}
&a_{11} \approx -0.05176, \; b_{11} \approx 0.14648, \; a_{12} \approx -0.03832, \; b_{12} \approx -0.00337, \\
&a_{21} \approx -0.24986, \; a_{22} \approx -0.185, \; b_{22} \approx 0.16874.
\end{aligned}
\tag{3.10}
$$

Theorem 1.2 was used to verify that the solution was indeed $L^2(\mathbb{R})$. The supports $\mathrm{suppL}(\phi_1) = 4$, $\mathrm{suppL}(\phi_2) = 3$ are minimal for the conditions put on Φ. The plots of ϕ_1 and ϕ_2 are given in Figure 3.1.

We next consider necessary conditions for an orthogonal scaling vector with vanishing moments if both ϕ_1 and ϕ_2 have symmetry centered about integers. In particular,
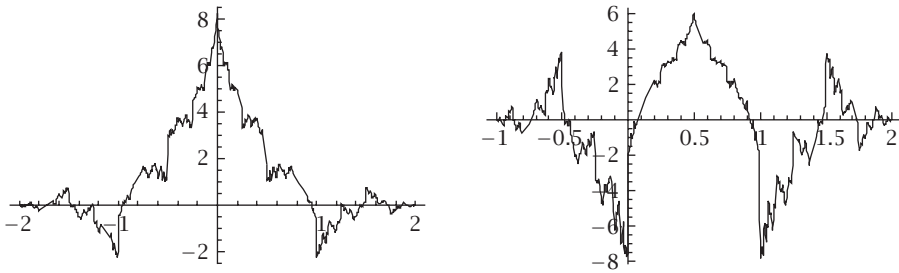
FIGURE 3.1. Scaling functions for Example 3.6.

we find that one scaling function must be symmetric about zero and the other must be antisymmetric.

THEOREM 3.7. *Suppose symmetric, orthogonal, LLI scaling vector* $\Phi = (\phi_1, \phi_2)^T$ *has accuracy* $p \geq 1$ *and* $L - 1 \geq 1$ *vanishing moments. If* ϕ_1 *is symmetric about an integer and* ϕ_2 *is (anti)symmetric about integer, then*

(1) ϕ_2 *must be antisymmetric;*

(2) ϕ_1 *is symmetric about zero;*

(3) *if* L *is even, then* $\mathrm{suppL}(\phi_1) \geq L + 1$, $\mathrm{suppL}(\phi_2) \geq L + 2$, *and the symbol* P *has the form*

$$P(z) = \begin{pmatrix} 1 + (1-z)^L Q_{11}(z) & (1-z^2) Q_{12}(z) \\ (1-z)^L (1-z^2) R_{21}(z) & P_{22}(z) \end{pmatrix}; \tag{3.11}$$

(4) *if* L *is odd, then* $\mathrm{suppL}(\phi_1) \geq L + 1$, $\mathrm{suppL}(\phi_2) \geq L + 1$, *and* $P(z)$ *has the form*

$$P(z) = \begin{pmatrix} 1 + (1-z)^{L+1} Q_{11}(z) & (1-z^2) Q_{12}(z) \\ (1-z)^L (1+z) R_{21}(z) & P_{22}(z) \end{pmatrix}. \tag{3.12}$$

PROOF. By Theorem 2.5, ϕ_2 must be antisymmetric about an integer. Since $\int_{\mathbb{R}} \phi_2 = 0$, by Proposition 1.6, we may assume without loss of generality that this integer is zero. Claim (2) follows from Corollary 2.4.

By Lemma 3.4 and (1.10), we see that

$$\begin{aligned} (1-z)^L Q_{11}(z) &= (1-z^{-1})^L Q_{11}(z^{-1}), & p_{12}(z) &= -p_{12}(z^{-1}), \\ (1-z)^L Q_{21}(z) &= -(1-z^{-1})^L Q_{21}(z^{-1}), & p_{22}(z) &= p_{22}(z^{-1}). \end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned} z^L Q_{11}(z) &= (-1)^L Q_{11}(z^{-1}), \\ z^L Q_{21}(z) &= (-1)^{L+1} Q_{21}(z^{-1}), \\ p_{12}(\pm 1) &= 0 \implies 1 - z^2 \text{ divides } P_{12}(z). \end{aligned} \tag{3.14}$$

Consider the case where L is even. Then $Q_{21}(\pm 1) = 0$, so $1 - z^2$ divides Q_{21}. Thus $(1 - z)^L(1 - z^2)$ divides $P_{21}(z)$, so degL$(P_{21}) \geq L + 2$. From Proposition 2.7, we see degL(P_{22}) is also at least $L + 2$. Thus suppL$(\phi_2) \geq L + 2$.

Now degL(P_{11}) is clearly at least L, so degL$(P_{12}) \geq L$. Then Corollary 3.2 says suppL$(\phi_1) \geq (1/2)$degL$(P_{12}) + (1/2)$degL$(P_{22}) \geq L + 1$.

Consider the case where L is odd. Then $Q_{11}(1) = 0$, so $1 - z$ divides Q_{11}. Thus $(1 - z)^{L+1}$ divides $P_{11}(z) - 1$, so degL$(P_{11}) \geq L + 1$ and thus suppL$(\phi_1) \geq L + 1$. Next, note that $Q_{21}(-1) = 0$, so $(1 - z)^L(1 + z)$ divides P_{21} and thus degL$(P_{21}) \geq L + 1$. Now by Corollary 3.2, suppL$(\phi_2) \geq (1/2)$degL$(P_{11}) + (1/2)$degL$(P_{21}) \geq L + 1$. $\square$

EXAMPLE 3.8. Set $L = 2$. We use this theorem to guide a construction of an $L^2(\mathbb{R})$ LLI, orthogonal, symmetric scaling vector with one vanishing moment and minimal support. Naturally ϕ_1 is symmetric about 0 and ϕ_2 is antisymmetric about $1/2$, suppL$(\phi_1) \geq 3$, suppL$(\phi_2) \geq 4$. We begin with symbol of the form given in Theorem 3.7 part 3. Brute force with (1.8) shows that orthogonality cannot be obtained if degL$(Q_{11}) < 2$, so we try degL$(Q_{11}) = 2$ and thus degL$(Q_{12}) = 2$ is minimal. We put the unknowns Q_{11}, Q_{12}, P_{22}, R_{21} into form ensuring the symmetry conditions, and solved for the parameters to guarantee orthogonality, with the help of Mathematica. Theorem 1.2 was used to verify that the solution was indeed $L^2(\mathbb{R})$. The supports suppL$(\phi_1) = 4$, suppL$(\phi_2) = 4$ are minimal for the conditions required for Φ.

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