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A REMARK ON THE EXTENSION OF THE CONCEPT OF INCIDENCE ALGEBRAS TO NONLOCALLY FINITE PARTIALLY ORDERED SETS

BONIFACE I. EKE

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An incidence algebra of a nonlocally finite partially ordered set Q is a very rare concept, perhaps nonexistent. In this note, we will attempt to construct such an algebra.

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1. Introduction. Let P be a partially ordered set (poset) and K a field of characteristic 0. The functions $f : P \times P \to K$, such that $x \nleq y$ implies $f(x,y) = 0$, are called the incidence functions of P over K. The set of such functions is denoted by $\vartheta(K,P)$. P is called locally finite if for every $x,y \in P$ the interval $[x,y] = \{t \in P \mid x \le t \le y\}$ is finite. When P is locally finite, $\vartheta(K,P)$ becomes a K-algebra under a multiplication $(*)$ defined by convolution:

$$f * g(x,y) = \sum_{x \le t \le y} f(x,t)g(t,y), \tag{1.1}$$

and the algebra $\vartheta(K,P)$ is called the incidence algebra of P over K [1, 2].

If P is not locally finite, the expression (1.1) may not make sense. So, one does not often hear of an incidence algebra of a nonlocally finite poset. Our purpose in this note is to show that if Q is any nonlocally finite poset and P is a locally finite poset, we can form a nonlocally finite poset $QS(P)$ for which an incidence algebra $\vartheta(K,QS(P))$ can be constructed.

Moreover, the posets Q and P are both embeddable in $QS(P)$, while the set $\vartheta(K,Q)$ and the algebra $\vartheta(K,P)$ are both embeddable in $\vartheta(K,QS(P))$, and if $|P| \le |Q|$, then $|QS(P)| = |Q|$. Besides, for the fixed posets P and Q, the incidence algebra $\vartheta(K,QS(P))$ is unique up to isomorphism. All these are established in Section 2.

In Section 3, we isolate the auxiliary locally finite poset P and try to deal directly with Q. However, because of the problem still posed by (1.1), we can only construct a sequence of what are called truncated incidence algebras for the nonlocally finite poset Q. For this purpose, we will need an additional hypothesis that Q is well ordered.

2. The construction of $QS(P)$ and $\vartheta(K,QS(P))$. We will assume throughout that P is a locally finite poset, Q a nonlocally finite poset, and K is a field of characteristic 0. Let $QS(P)$ be the Cartesian product $P \times Q$. We will denote the order relation in P by $\le^{(1)}$ and the order relation in Q by $\le^{(2)}$. Then we define an order relation $\le$ in $QS(P)$ by

$(x,r) \le (y,s)$ if and only if $x \le^{(1)} y$ and $r \le^{(2)} s$. It is clear that, with the relation $\le$, $QS(P)$ is a partially ordered set. However, $QS(P)$ is not locally finite.

We will define addition and scalar multiplication on $\vartheta(K, QS(P))$ as in [1]. We now need to define the convolution multiplication shown in (1.1) on $\vartheta(K, QS(P))$ so that it will make sense.

For a fixed $r \in Q$, denote $P \times \{r\}$ by P_r. If (x,r) and (y,s) are any two elements of $QS(P)$, then $(x,r) \in P_r$, while $(y,s) \in P_s$. Moreover, P_r and P_s are locally finite subposets of $QS(P)$. Denote (x,r) by u and (y,s) by v, and let $T = \{t \in P \mid x \le^{(1)} t \le^{(1)} y\}$. Then the set $T(u,v) = (T \times \{r\}) \cup (T \times \{s\}) \subseteq QS(P)$ is finite. Let $J(u,v) = [u,v] \cap T(u,v)$. We define the operation $(*)$ on $\vartheta(K, QS(P))$ by the following: for all elements u and v in $QS(P)$ and for all f, g in $\vartheta(K, QS(P))$,

$$f * g(u,v) = \sum_{z \in J_{(u,v)}} f(u,z) g(z,v). \tag{2.1}$$

Clearly, (2.1) is now well defined. The associativity follows from [1, Proposition 4.1]. Consequently, with (2.1), $\vartheta(K, QS(P))$ is an incidence algebra of $QS(P)$ over K.

P is isomorphic to P_r for each $r \in Q$. Similarly, for each $y \in P$, Q is isomorphic to $Q_y = \{y\} \times Q$. Hence both P and Q are embeddable in $QS(P)$. Moreover, the correspondence $\mu_r : f \mapsto f_r$, where f_r is defined by $f_r(x_r, y_r) = f(x,y)$, is an isomorphism of $\vartheta(K,P)$ onto $\vartheta(K,P_r)$. Consequently, $\vartheta(K,P)$ is embeddable in $\vartheta(K, QS(P))$. By a similar device, we find that $\vartheta(K,Q)$ is also embeddable in $\vartheta(K, QS(P))$. For the uniqueness of $\vartheta(K, QS(P))$, we will prove the following.

Proposition 2.1. *If P' and Q' are any posets such that P is isomorphic to P' and Q is isomorphic to Q', then $\vartheta(K, QS(P))$ is isomorphic to $\vartheta(K, Q'S(P'))$.*

Proof. Let $\sigma : P \to P'$ be an isomorphism while $\theta : Q \to Q'$ is an isomorphism. Define $\eta : QS(P) \to Q'S(P')$ by $\eta(x,r) = (\sigma(x), \theta(r))$. If $\eta(x,r) = \eta(y,s)$, then $(\sigma(x), \theta(r)) = (\sigma(y), \theta(s))$. By the definition of the order relation in $Q'S(P')$, we must have $\sigma(x) = \sigma(y)$ and $\theta(r) = \theta(s)$. Consequently, $x = y$ and $r = s$. Hence, $(x,r) = (y,s)$. This shows that η is injective. Clearly, also η is surjective. Therefore, η is an isomorphism. For each $u \in QS(P)$, denote $\eta(u)$ by u'. Now define $\beta : \vartheta(K, QS(P)) \to \vartheta(K, Q'S(P'))$ by $\beta(f) = f'$, where f' is defined by $f'(u', v') = f(u,v)$ for all $u', v' \in Q'S(P')$. One can directly check that β is also an isomorphism. Hence, the proposition holds. □

We observe that for the locally finite poset P, one could have chosen any nonempty finite subset of Q itself. We will call the algebra $\vartheta(K, QS(P))$ the incidence algebra of Q relative to P.

3. Truncated incidence algebras. Our interest now is to see what we can achieve by isolating the locally finite poset P and dealing directly with Q. However, (1.1) still poses a problem. Nevertheless, following the motivation received from Section 2, what we need is to try to use a finite number of elements of the interval $[r,s]$ at a time, for any two elements r and s of the nonlocally finite poset Q. Then arises the question of how to choose the finite number of elements from $[r,s]$. The formula for choosing such elements is outlined below for the case where Q is well ordered. What makes it possible is the property of a well-ordered set whereby not only does every nonempty subset of such a set have a first element, but also such a first element is unique [3, Theorems 64 and 65, page 76]. First, we show the existence of a well-ordered nonlocally finite poset Q.

Example 3.1. Let $Q = \{1/n \mid n \in \mathbb{N}\} \cup \{O\}$, where $\mathbb{N}$ is the set of natural numbers. Q is a poset subject to the usual relation "$\geq$" (greater than or equal to). Clearly, also Q is well ordered by "$\geq$". However, for any $a \in Q$, $a \neq 0$, the interval $[0,a]$ is infinite. Hence, Q is not locally finite.

Now let W be any well-ordered poset and let $r \leq s \in W$. Set $W_0 = [r,s]$. Let $W_1 = W_0 - \{r\}$. Then, if $W_1 \neq \emptyset$, W_1 has a unique first element t_1. Let $W_2 = w_1 - \{t_1\}$. If $W_2 \neq \emptyset$, then W_2 has a unique first element t_2. In general, $W_i = W_{i-1} - \{t_{i-1}\}$, where $t_{i-1} =$ first element of W_{i-1}, and $t_0 \equiv r$.

For any fixed natural number n, let $T_n(r,s) = \{r,t_1,\dots,t_n,s\}$. Let

$$J_n(r,s) = \begin{cases} [r,s] & \text{if } [r,s] \text{ is finite,} \\ T_n(r,s) & \text{otherwise.} \end{cases} \tag{3.1}$$

Then define the convolution multiplication $*$ on $\vartheta(K,W)$ by the following: for all $r,s \in W$ and for all $f,g \in \vartheta(K,W)$,

$$f * g(r,s) = \sum_{t \in J_n(r,s)} f(r,t)g(t,s). \tag{3.2}$$

Subject to (3.2), $\vartheta(K,W)$ is an incidence algebra. We denote this incidence algebra by $\vartheta_n(K,W)$, and $\vartheta_n(K,W)$ is called a *truncated incidence algebra* of W over K.

It is clear that $T_n(r,s) \subseteq T_{n+1}(r,s)$ for all $n \in \mathbb{N}$. We will call the incidence algebra $\vartheta_{n+1}(K,W)$ a *refinement* of the incidence algebra $\vartheta_n(K,W)$. The sequence $\{\vartheta_n(K,W)\}$ of incidence algebras is finite if and only if W is locally finite.

We now observe that a well-ordered nonlocally finite poset Q is associated with an infinite sequence of truncated incidence algebras, where each is a nontrivial refinement of the one before it. Unifying these algebras to form one incidence algebra of Q over K remains an open problem.

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Boniface I. Eke: Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA

E-mail address: beke@morgan.edu