IJMMS 2004:45, 2401–2422
PII. S0161171204311105
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# ON SCALAR TYPE SPECTRAL OPERATORS, INFINITE DIFFERENTIABLE AND GEVREY ULTRADIFFERENTIABLE $C_0$-SEMIGROUPS

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Received 6 November 2003

Necessary and sufficient conditions for a scalar type spectral operator in a Banach space to be a generator of an infinite differentiable or a Gevrey ultradifferentiable $C_0$-semigroup are found, the latter formulated exclusively in terms of the operator's spectrum.

2000 Mathematics Subject Classification: 47B40, 47D03, 47B15, 34G10.

**1. Introduction.** Despite what was said in the final remarks to [22], the author did decide to tackle the problems of the generation of *infinite differentiable* and *Gevrey ultradifferentible* $C_0$-semigroups by a *scalar type spectral operator* in a complex Banach space. The more so as, in the former case, the task turned out to be more of a challenge than it seemed initially, the existence of a general characterization of infinite differentiable $C_0$-semigroups [25] (see also [6, 26]) notwithstanding. In the latter case, such characterizations are not to be found in the plethora of the literature on the subject including such authoritative and exhaustive sources as [6, 9, 11, 15, 26, 28, 31].

In [22], the criteria of a scalar type spectral operator in a complex Banach space being a generator of a $C_0$-semigroup and an analytic $C_0$-semigroup were found. In the present paper, necessary and sufficient conditions for a scalar type spectral operator in a complex Banach space to be a generator of an *infinite differentiable* or a *Gevrey ultradifferentiable* $C_0$-semigroup are established. The main purpose is to show that such criteria, as well as those of [22], can be formulated exclusively in terms of the operator's spectrum, without any restrictions on its *resolvent* behavior. This fact distinguishes the case of *scalar type spectral operators* and makes the aforementioned results significantly more transparent and purely qualitative.

## 2. Preliminaries

**2.1. Scalar type spectral operators.** Henceforth, unless otherwise specified, $A$ is a *scalar type spectral operator* in a complex Banach space $X$ with norm $\|\cdot\|$ and $E_A(\cdot)$ is its *spectral measure* (the *resolution of the density*), the operator's spectrum $\sigma(A)$ being the *support* for the latter [2, 5].

Note that, in a Hilbert space, the *scalar type spectral operators* are those similar to the *normal* ones [29].

For such operators, there has been developed an *operational calculus* for Borel measurable functions on $\mathbb{C}$ (on $\sigma(A)$) [2, 5], $F(\cdot)$ being such a function, a new *scalar type* *spectral operator*

$$F(A)=\int_{\mathbb{C}}F(\lambda)\,dE_A(\lambda)=\int_{\sigma(A)}F(\lambda)\,dE_A(\lambda) \tag{2.1}$$

is defined as follows:

$$\begin{aligned} F(A)f &:= \lim_{n\to\infty}F_n(A)f, \quad f\in D(F(A)),\\ D(F(A)) &:= \left\{f\in X \,\middle|\, \lim_{n\to\infty}F_n(A)f \text{ exists}\right\} \end{aligned} \tag{2.2}$$

($D(\cdot)$ is the *domain* of an operator), where

$$F_n(\cdot):=F(\cdot)\chi_{\{\lambda\in\sigma(A)\,|\,|F(\lambda)|\le n\}}(\cdot), \quad n=1,2,\ldots, \tag{2.3}$$

($\chi_\alpha(\cdot)$ is the *characteristic function* of a set $\alpha$), and

$$F_n(A):=\int_{\sigma(A)}F_n(\lambda)\,dE_A(\lambda), \quad n=1,2,\ldots, \tag{2.4}$$

being the integrals of *bounded* Borel measurable functions on $\sigma(A)$, are *bounded scalar type spectral operators* on $X$ defined in the same manner as for *normal operators* (see, e.g., [4, 27]).

The properties of the *spectral measure*, $E_A(\cdot)$, and the *operational calculus* underlying the entire subsequent argument are exhaustively delineated in [2, 5]. We just observe here that, due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded*, that is, there is an $M>0$ such that, for any Borel set $\delta$,

$$\|E_A(\delta)\|\le M, \tag{2.5}$$

see [3].

Observe that, in (2.5), the notation $\|\cdot\|$ was used to designate the norm in the space of bounded linear operators on $X$. We will adhere to this rather common economy of symbols in what follows, adopting the same notation for the norm in the dual space $X^*$ as well.

Due to (2.5), for any $f\in X$ and $g^*\in X^*$ ($X^*$ is the *dual space*), the total variation $v(f,g^*,\cdot)$ of the complex-valued measure $\langle E_A(\cdot)f,g^*\rangle$ ($\langle\cdot,\cdot\rangle$ is the *pairing* between the space $X$ and its dual, $X^*$) is *bounded*. Indeed,

$$\begin{aligned} &v(f,g^*,\sigma(A)) \quad (\delta \text{ being an arbitrary Borel subset of } \sigma(A),\ [3])\\ &\quad\le 4\sup_{\delta\subseteq\sigma(A)}|\langle E_A(\delta)f,g^*\rangle|\le 4\sup_{\delta\subseteq\sigma(A)}\|E_A(\delta)\|\|f\|\|g^*\| \quad (\text{by } (2.5))\\ &\quad\le 4M\|f\|\|g^*\|. \end{aligned} \tag{2.6}$$

For the reader's convenience, we reformulate here [23, Proposition 3.1], heavily relied upon in what follows, which allows to characterize the domains of the Borel measurable functions of a scalar type spectral operator in terms of positive measures (see [23] for a complete proof).

**Proposition 2.1** [23, Proposition 3.1]. *Let $A$ be a scalar type spectral operator in a complex Banach space $X$ and $F(\cdot)$ a complex-valued Borel measurable function on $\mathbb{C}$ (on $\sigma(A)$). Then $f \in D(F(A))$ if and only if*

(i) *for any $g^* \in X^*$,*

$$\int_{\sigma(A)} |F(\lambda)|\,dv(f,g^*,\lambda) < \infty, \tag{2.7}$$

(ii) $\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |F(\lambda)|>n\}} \|F(\lambda)\|\,dv(f,g^*,\lambda) \to 0$ *as* $n\to\infty$.

Observe that, $F(\cdot)$ being an arbitrary Borel measurable function on $\mathbb{C}$ (on $\sigma(A)$), for any $f \in D(F(A))$, $g^* \in X^*$, and arbitrary Borel sets $\delta \subseteq \sigma$,

$$\begin{aligned}
&\int_{\sigma} |F(\lambda)|\,dv(f,g^*,\lambda) \quad \text{(see [3])}\\
&\quad\le 4\sup_{\delta\subseteq\sigma}\left|\int_{\delta} F(\lambda)\,d\langle E_A(\lambda)f,g^*\rangle\right|\\
&\quad= 4\sup_{\delta\subseteq\sigma}\left|\int_{\sigma} \chi_\delta(\lambda)F(\lambda)\,d\langle E_A(\lambda)f,g^*\rangle\right| \quad \text{(by the properties of the } \textit{o.c.}\text{)}\\
&\quad= 4\sup_{\delta\subseteq\sigma}\left|\left\langle\int_{\sigma} \chi_\delta(\lambda)F(\lambda)\,dE_A(\lambda)f,g^*\right\rangle\right| \quad \text{(by the properties of the } \textit{o.c.}\text{)}\\
&\quad= 4\sup_{\delta\subseteq\sigma}|\langle E_A(\delta)E_A(\sigma)F(A)f,g^*\rangle|\\
&\quad\le 4\sup_{\delta\subseteq\sigma}\|E_A(\delta)E_A(\sigma)F(A)f\|\|g^*\|\\
&\quad\le 4\sup_{\delta\subseteq\sigma}\|E_A(\delta)\|\|E_A(\sigma)F(A)f\|\|g^*\| \quad \text{(by (2.5))}\\
&\quad\le 4M\|E_A(\sigma)F(A)f\|\|g^*\|\\
&\quad\le 4M\|E_A(\sigma)\|\|F(A)f\|\|g^*\|.
\end{aligned} \tag{2.8}$$

In particular,

$$\begin{aligned}
&\int_{\sigma(A)} |F(\lambda)|\,dv(f,g^*,\lambda) \quad \text{(by (2.8))}\\
&\quad\le 4M\|E_A(\sigma(A))\|\|F(A)f\|\|g^*\|\\
&\qquad \text{(since } E_A(\sigma(A)) = I \text{ (} I \text{ is the } \textit{identity operator} \text{ on } X\text{))}\\
&\quad\le 4M\|F(A)f\|\|g^*\|.
\end{aligned} \tag{2.9}$$

On account of compactness, the terms *spectral measure* and *operational calculus* for scalar type spectral operators, frequently referred to, will be abbreviated to *s.m.* and *o.c.*, respectively.

Observe also that, as follows directly from the results of [1, 23], if a scalar type spectral operator $A$ generates a $C_0$-semigroup $\{T(t) \mid t \ge 0\}$, the latter is of the form

$$T(t) = e^{tA}, \quad 0 \le t < \infty. \tag{2.10}$$

**2.2. The Gevrey classes of vectors.** Let $A$ be a linear operator in a Banach space $X$ with norm $\|\cdot\|$,

$$C^{\infty}(A) \stackrel{\text{def}}{=} \bigcap_{n=0}^{\infty} D(A^n), \tag{2.11}$$

and $0 \le \beta < \infty$.

The sets of vectors

$$\begin{aligned}\mathscr{E}^{\{\beta\}}(A) &\stackrel{\text{def}}{=} \{f \in C^{\infty}(A) \mid \exists \alpha > 0,\ \exists c > 0 : \|A^n f\| \le c\alpha^n [n!]^{\beta},\ n = 0,1,\dots\},\\ \mathscr{E}^{(\beta)}(A) &\stackrel{\text{def}}{=} \{f \in C^{\infty}(A) \mid \forall \alpha > 0\ \exists c > 0 : \|A^n f\| \le c\alpha^n [n!]^{\beta},\ n = 0,1,\dots\}\end{aligned} \tag{2.12}$$

are called the $\beta$th-order *Gevrey classes* of the operator $A$ of *Roumie's* and *Beurling's types*, respectively.

For $0 \le \beta < \beta' < \infty$,

$$\mathscr{E}^{(\beta)}(A) \subseteq \mathscr{E}^{\{\beta\}}(A) \subseteq \mathscr{E}^{(\beta')}(A) \subseteq \mathscr{E}^{\{\beta'\}}(A) \subseteq C^{\infty}(A). \tag{2.13}$$

In particular, $\mathscr{E}^{\{1\}}(A)$ and $\mathscr{E}^{(1)}(A)$ are the well-known classes of *analytic* and *entire vectors*, correspondingly [10, 24].

Observe that, in the definitions of the Gevrey classes, due to *Stirling's formula*, one can replace $[n!]^{\beta}$ by $n^{\beta n}$.

According to [17], for a *scalar type spectral operator* $A$ in a complex Banach space $X$ and $0 < \beta < \infty$,

$$\begin{aligned}\mathscr{E}^{\{\beta\}}(A) &\supseteq \bigcup_{t>0} D\left(e^{t|A|^{1/\beta}}\right),\\ \mathscr{E}^{(\beta)}(A) &\supseteq \bigcap_{t>0} D\left(e^{t|A|^{1/\beta}}\right),\end{aligned} \tag{2.14}$$

the inclusions becoming equalities provided that the space $X$ is *reflexive*.

**2.3. Gevrey ultradifferentiability.** A smoothness higher than *infinite differentiability* ranging up to *real analyticity* and *entireness* was introduced for numerical functions by Gevrey in 1918 [7] and is naturally extrapolated to functions with values in a Banach space.

Let $I$ be an interval of the *real axis*, $\mathbb{R}$, $C^{\infty}(I,X)$ the set of all $X$-valued functions *strongly infinite differentiable* on $I$, and $0 \le \beta < \infty$.

The sets of vectors

$$\begin{aligned}\mathscr{E}^{\{\beta\}}(I,X) &\stackrel{\text{def}}{=} \Big\{g(\cdot) \in C^{\infty}(I,X) \mid \forall [a,b] \subseteq I\ \exists \alpha > 0,\ \exists c > 0 :\\ &\qquad \max_{a \le t \le b} \|g^{(n)}(t)\| \le c\alpha^n [n!]^{\beta},\ n = 0,1,\dots\Big\},\\ \mathscr{E}^{(\beta)}(I,X) &\stackrel{\text{def}}{=} \Big\{g(\cdot) \in C^{\infty}(I,X) \mid \forall [a,b] \subseteq I,\ \forall \alpha > 0\ \exists c > 0 :\\ &\qquad \max_{a \le t \le b} \|g^{(n)}(t)\| \le c\alpha^n [n!]^{\beta},\ n = 0,1,\dots\Big\}\end{aligned} \tag{2.15}$$

are the *βth-order Gevrey classes of strongly ultradifferentiable functions* of *Roumie's* and *Beurling's types*, respectively (see, e.g., [7, 12, 13, 14]).

Just as above, due to *Stirling's formula*, one can replace $[n!]^\beta$ by $n^{\beta n}$.

For $0 \le \beta < \beta' < \infty$, the following inclusions hold:

$$\mathscr{E}^{(\beta)}(I,X) \subseteq \mathscr{E}^{\{\beta\}}(I,X) \subseteq \mathscr{E}^{(\beta')}(I,X) \subseteq \mathscr{E}^{\{\beta'\}}(I,X). \tag{2.16}$$

In particular, $\mathscr{E}^{\{1\}}(I,X)$ is the class of all *real analytic* on $I$ vector functions (i.e., *analytically continuable* into complex neighborhoods of the interval $I$) and $\mathscr{E}^{(1)}(I,X)$ is the class of all *entire* vector functions (i.e., allowing *entire* continuations) (for numerical functions, see [16]).

Note that it is well known that the Gevrey classes of functions of orders greater than one are *quasianalytic*.

**3. On the strong smoothness of an orbit of a $C_0$-semigroup generated by a scalar type spectral operator.** Let $A$ be a scalar type spectral operator generating a $C_0$-semigroup $\{T(t) \mid t \ge 0\}$.

**Proposition 3.1.** *Let $I$ be a subinterval of $[0,\infty)$ and $0 < \beta < \infty$. Then the restriction of an orbit $T(\cdot)f$, $f \in X$, to $I$*

(i) *belongs to $C^\infty(I,X)$ if and only if $T(t)f \in C^\infty(A)$, for any $t \in I$,*

(ii) *belongs to $\mathscr{E}^{\{\beta\}}(I,X)$ (resp., $\mathscr{E}^{(\beta)}(I,X)$) if and only if $T(t)f \in \mathscr{E}^{\{\beta\}}(A)$ (resp., $\mathscr{E}^{(\beta)}(A)$), for any $t \in I$.*

**Proof**

(i) **"Only if" part.** Assume that the restriction of an orbit $T(\cdot)f$ of the $C_0$-semigroup generated by $A$ to a subinterval $I$ of $[0,\infty)$ belongs to $C^\infty(I,X)$.

Taking into account that $T(\cdot)f$ is a *weak solution* of the evolution equation

$$y'(t) = Ay(t) \tag{3.1}$$

on $[0,\infty)$ [1], we have, for any $g \in D(A^*)$,

$$\langle T'(t)f,g\rangle = \frac{d}{dt}\langle T(t)f,g\rangle = \langle T(t)f,A^*g\rangle, \quad t \in I. \tag{3.2}$$

Whence, by the *closedness* of the operator $A$,

$$T(t)f \in D(A), \quad T'(t)f = AT(t)f, \quad \text{for any } t \in I, \tag{3.3}$$

(see [1, 8] for details).

Let $n > 1$. Then, differentiating (3.3) for an arbitrary fixed $t \in I$, we obtain

$$T''(t)f = \lim_{\Delta t\to 0}\frac{T'(t+\Delta t)f - T'(t)f}{\Delta t} = \lim_{\Delta t\to 0} A\frac{T(t+\Delta t)f - T(t)f}{\Delta t}, \tag{3.4}$$

where the increments $\Delta t$ are such that $t+\Delta \in I$.

Since

$$\lim_{\Delta t\to 0}\frac{T(t+\Delta t)f-T(t)f}{\Delta t}=T'(t)f, \tag{3.5}$$

by the closedness of $A$, we infer that

$$T'(t)f\in D(A),\quad T''(t)f=AT'(t)f,\quad \text{for any } t\in I. \tag{3.6}$$

Thus, (3.3) and (3.6) imply

$$T(t)f\in D(A^2),\quad T''(t)f=A^2T(t)f,\quad \text{for any } t\in I. \tag{3.7}$$

Continuing inductively in this manner, we infer that for any $n=1,2,\dots,$

$$T(t)f\in C^\infty(A),\quad T^{(n)}(t)f=A^nT(t)f,\quad t\in I. \tag{3.8}$$

**"If" part.** Let $T(\cdot)f$ be an orbit of the $C_0$-semigroup generated by $A$ such that

$$T(t)f\in C^\infty(A)\quad \text{for any } t\in I, \tag{3.9}$$

where $I$ is a subinterval of $[0,\infty)$.

Recall that, as was noted in Section 2, the $C_0$-semigroup $\{T(t)\mid t\ge 0\}$ generated by $A$ is of the form (2.10).

The fact that

$$e^{tA}f\in C^\infty(A),\quad t\in I, \tag{3.10}$$

by [23, Proposition 3.1], implies that, for any $g^*\in X^*$,

$$\int_{\sigma(A)}|\lambda|^n e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)<\infty,\quad n=1,2,\dots,\ t\in I. \tag{3.11}$$

Given a natural $n$ and an arbitrary fixed $t\in[0,T)$, we choose a segment $[a,b]\subset[0,T)$ ($a<b$) so that $t$ is its midpoint if $0<t<T$, or $a=0$ if $t=0$. For increments $\Delta t$ such that $a\le t+\Delta t\le b$ and any $g^*\in X^*$, we have

$$\begin{aligned}
&\left|\left\langle\frac{A^{n-1}e^{t+\Delta t}f-A^{n-1}e^tf}{\Delta t}-A^ne^tf,g^*\right\rangle\right| \quad \text{(by (2.10) and the properties of the \textit{o.c.})}\\
&=\left|\left\langle\int_{\sigma(A)}\left[\frac{\lambda^{n-1}e^{(t+\Delta t)\lambda}-\lambda^{n-1}e^{t\lambda}}{\Delta t}-\lambda^ne^{t\lambda}\right]dE_A(\lambda)f,g^*\right\rangle\right|\\
&\qquad\qquad \text{(by the properties of the \textit{o.c.})}\\
&\le\left|\int_{\sigma(A)}\left[\frac{\lambda^{n-1}e^{(t+\Delta t)\lambda}-\lambda^{n-1}e^{t\lambda}}{\Delta t}-\lambda^ne^{t\lambda}\right]d\langle E_A(\lambda)f,g^*\rangle\right|\\
&\le\int_{\sigma(A)}\left|\frac{\lambda^{n-1}e^{(t+\Delta t)\lambda}-\lambda^{n-1}e^{t\lambda}}{\Delta t}-\lambda^ne^{t\lambda}\right|dv(f,g^*,\lambda)\\
&\qquad\qquad \text{(by the \textit{Lebesgue dominated convergence theorem})}\\
&\longrightarrow 0\quad \text{as } \Delta t\longrightarrow 0.
\end{aligned} \tag{3.12}$$

Indeed, for any $\lambda \in \sigma(A)$,

$$\begin{aligned}
&\left| \frac{\lambda^{n-1}e^{(t+\Delta t)\lambda} - \lambda^{n-1}e^{t\lambda}}{\Delta t} - \lambda^k e^{t\lambda} \right| \\
&\quad \le \left| \frac{\lambda^{n-1}e^{(t+\Delta t)\lambda} - \lambda^{n-1}e^{t\lambda}}{\Delta t} \right| + \left| \lambda^n e^{t\lambda} \right| \quad \text{(by the \textit{total change theorem})} \\
&\quad \le \max_{a\le s\le b} \left| \lambda^n e^{s\lambda} \right| + \left| \lambda^n e^{t\lambda} \right| \le 2|\lambda|^n \max_{a\le s\le b} e^{s\operatorname{Re}\lambda} \\
&\quad \le 2\begin{cases} |\lambda|^n e^{a\operatorname{Re}\lambda}, & \text{if } \operatorname{Re}\lambda < 0, \\ |\lambda|^n e^{b\operatorname{Re}\lambda}, & \text{if } \operatorname{Re}\lambda \ge 0, \end{cases} \quad \text{(by (3.11), considering that } a,b \in I) \\
&\quad \in \mathscr{L}^1(\sigma(A), v(f,g^*,\cdot)), \quad n = 1,2,\ldots, \\
&\left| \frac{\lambda^{n-1}e^{(t+\Delta t)\lambda} - \lambda^{n-1}e^{t\lambda}}{\Delta t} - \lambda^n e^{t\lambda} \right| \longrightarrow 0 \quad \text{as } \Delta t \longrightarrow 0.
\end{aligned} \tag{3.13}$$

We have shown that, for any $t \in I$ and an arbitrary $n = 1,2,\ldots,$

$$\frac{A^{n-1}e^{t+\Delta t}f - A^{n-1}e^t f}{\Delta t} \xrightarrow{w} A^n y(t) \longrightarrow 0 \quad \text{as } \Delta t \longrightarrow 0. \tag{3.14}$$

Thus, we have proved that, for any $g^* \in X^*$,

$$\frac{d^n}{dt^n}\langle e^{tA}f, g^*\rangle = \langle A^n e^{tA}f, g^*\rangle, \quad n = 1,2,\ldots, \ t \in I. \tag{3.15}$$

Now, let

$$\Delta_n := \{\lambda \in \mathbb{C} \mid |\lambda| \le n\}. \tag{3.16}$$

We fix an arbitrary natural $k$ and consider the sequence of functions

$$E_A(\Delta_n)A^k e^{tA}f, \quad n = 1,2,\ldots, \ t \in I. \tag{3.17}$$

By the properties of the *o.c.*,

$$\begin{aligned}
&E_A(\Delta_n)A^k e^{tA} \\
&= \int_{\mathbb{C}} \chi_{\Delta_n}(\lambda)\lambda^k e^{t\lambda}\,dE_A(\lambda) \quad \text{(where } \chi_{\Delta_n}(\cdot) \text{ is the \textit{characteristic function} of the set } \Delta_n) \\
&= \int_{\mathbb{C}} [\lambda\chi_{\Delta_n}(\lambda)]^k e^{t\lambda\chi_{\Delta_n}(\lambda)}\,dE_A(\lambda) \\
&= [AE_A(\Delta_n)]^k e^{tAE_A(\Delta_n)}.
\end{aligned} \tag{3.18}$$

Since, by the properties of the *o.c.*, for any natural $n$, the operator $AE_A(\Delta_n)$ is a bounded operator on $X$ ($\|AE_A(\Delta_n)\| \le 4Mn$) [5], the vector function

$$E_A(\Delta_n)A^k e^{tA}f = [AE_A(\Delta_n)]^k e^{tAE_A(\Delta_n)}f, \quad n = 1,2,\ldots, \ t \in I, \tag{3.19}$$

is strongly continuous.

For an arbitrary segment $[a,b] \subseteq I$, we have

$$
\begin{aligned}
&\sup_{a\le t\le b} \|A^k e^{tA} f - E_A(\Delta_n) A^k e^{tA} f\| \quad \text{(by the properties of the } \textit{o.c.}\text{)} \\
&\quad = \sup_{a\le t\le b} \left\| \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n\}} \lambda^k e^{t\lambda} dE_A(\lambda) f \right\| \\
&\qquad\qquad \text{(as follows from the } \textit{Hahn-Banach theorem}\text{)} \\
&\quad = \sup_{a\le t\le b} \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \left| \left\langle \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n\}} \lambda^k e^{t\lambda} dE_A(\lambda) f, g^* \right\rangle \right| \\
&\qquad\qquad \text{(by the properties of the } \textit{o.c.}\text{)} \\
&\quad = \sup_{a\le t\le b} \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \left| \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n\}} \lambda^k e^{t\lambda} d\langle E_A(\lambda) f, g^*\rangle \right| \\
&\quad \le \sup_{a\le t\le b} \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n\}} |\lambda|^k e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\quad \le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \sup_{a\le t\le b} \Bigg[ \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n,\,\operatorname{Re}\lambda\le 0\}} |\lambda|^k e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\qquad\qquad + \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n,\,\operatorname{Re}\lambda> 0\}} |\lambda|^k e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \Bigg] \\
&\quad \le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n,\,\operatorname{Re}\lambda\le 0\}} |\lambda|^k e^{a\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\qquad + \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n,\,\operatorname{Re}\lambda> 0\}} |\lambda|^k e^{b\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\quad \le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n\}} |\lambda|^k e^{a\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\qquad + \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>n\}} |\lambda|^k e^{b\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad \text{(by (2.8))} \\
&\quad \le 4M \Bigg[ \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \|E_A(\{\lambda\in\sigma(A) \mid |\lambda|>n\}) A^k e^{aA} f\| \|g^*\| \\
&\qquad\qquad + \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \|E_A(\{\lambda\in\sigma(A) \mid |\lambda|>n\}) A^k e^{bA} f\| \|g^*\| \Bigg] \\
&\quad \le 4M \left[ \|E_A(\{\lambda\in\sigma(A) \mid |\lambda|>n\}) A^k e^{aA} f\| + \|E_A(\{\lambda\in\sigma(A) \mid |\lambda|>n\}) A^k e^{bA} f\| \right] \\
&\qquad\qquad \text{(by the strong continuity of the } \textit{s.m.}\text{)} \\
&\quad \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{3.20}
$$

Therefore, the vector function

$$
A^k e^{tA} f, \quad k = 1,2,\ldots, \ t \in I, \tag{3.21}
$$

is strongly continuous, being the uniform limit of a sequence of strongly continuous functions on an arbitrary segment $[a,b] \subseteq I$.

We fix an arbitrary $a \in I$ and integrate (3.15) for $n = 1$ between $a$ and an arbitrary $t \in I$. Considering the strong continuity of $Ae^{tA}f$, $t \in I$, we have

$$\langle e^{tA}f - e^{aA}f, g^*\rangle = \left\langle \int_a^t Ae^{sA}f\,ds, g^*\right\rangle, \quad g^* \in X^*. \tag{3.22}$$

Whence, as follows from the *Hahn-Banach theorem*,

$$e^{tA}f - e^{aA}f = \int_a^t Ae^{sA}f\,ds. \tag{3.23}$$

By the strong continuity of $Ae^{tA}f$, $t \in I$, we infer that

$$\frac{d}{dt}e^{tA}f = Ae^{tA}f, \quad t \in I. \tag{3.24}$$

Consequently, by (3.15) for $n = 2$,

$$\frac{d}{dt}\left\langle \frac{d}{dt}e^{tA}f, g^*\right\rangle = \langle A^2e^{tA}f, g^*\rangle, \quad t \in I. \tag{3.25}$$

Whence, analogously,

$$\frac{d^2}{dt^2}e^{tA}f = A^2e^{tA}f, \quad t \in I. \tag{3.26}$$

Continuing inductively in this manner, we infer that, for any natural $n$,

$$\frac{d^n}{dt^n}e^{tA}f = A^ne^{tA}f, \quad t \in I. \tag{3.27}$$

(ii) **"Only if" part.** Assume that an orbit $T(\cdot)f$, $f \in X$, of the $C_0$-semigroup $\{T(t) \mid t \geq 0\}$ generated by $A$ restricted to a subinterval $I \subseteq [0,\infty)$ belongs to $\mathscr{E}^{\{\beta\}}(I,X)$ (resp., $\mathscr{E}^{(\beta)}(I,X)$).

This necessarily implies that $T(\cdot)f \in C^\infty(I,H)$. Whence, by (i),

$$T(t)f \in C^\infty(A), \qquad T^{(n)}(t)f = A^nT(t)f, \quad n = 1,2,\ldots,\ t \in I. \tag{3.28}$$

Furthermore, the fact that the restriction of $y(\cdot)$ to $I$ belongs to the class $\mathscr{E}^{\{\beta\}}(I,X)$ (resp., $\mathscr{E}^{(\beta)}(I,X)$) implies that, for an arbitrary $t \in I$, a certain (any) $\alpha > 0$, and a certain $c > 0$,

$$\|A^nT(t)f\| = \|T^{(n)}(t)f\| \leq c\alpha^n[n!]^\beta, \quad n = 0,1,\ldots. \tag{3.29}$$

Therefore,

$$T(t)f \in \mathscr{E}^{\{\beta\}}(A)\ (\text{resp.,} \mathscr{E}^{(\beta)}(A)), \quad t \in I. \tag{3.30}$$

**"If" part.** Let an orbit $T(\cdot)f$, $f \in X$, of the $C_0$-semigroup $\{T(t) \mid t \geq 0\}$ generated by $A$ be such that (3.30) holds, where $I$ is a subinterval of $[0,\infty)$.

Hence, for arbitrary $t \in I$ and some (any) $\alpha(t) > 0$, there is such a $c(t,\alpha) > 0$ that

$$\|A^n T(t)f\| \leq c(t,\alpha)\alpha(t)^n [n!]^\beta, \quad n = 0,1,2,\dots. \tag{3.31}$$

The inclusions

$$\mathscr{E}^{(\beta)}(A) \subseteq \mathscr{E}^{\{\beta\}}(A) \subseteq C^\infty(A) \tag{3.32}$$

imply, by (i), that (3.28) holds. Recall that

$$T(t)f = e^{tA}f, \quad 0 \leq t < \infty. \tag{3.33}$$

We fix an arbitrary subsegment $[a,b] \subseteq I$. For $n = 0,1,\dots$, we have

$$\begin{aligned}
&\max_{a\leq t\leq b} \|T^{(n)}(t)f\| \\
&\quad = \max_{a\leq t\leq b} \|A^n T(t)f\| \\
&\quad = \max_{a\leq t\leq b} \|A^n e^{tA}f\| \quad \text{(by the properties of the \textit{o.c.} and the \textit{Hahn-Banach theorem})} \\
&\quad = \max_{a\leq t\leq b} \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \left| \left\langle \int_{\sigma(A)} \lambda^n e^{t\lambda} dE_A(\lambda)f, g^* \right\rangle \right| \quad \text{(by the properties of the \textit{o.c.})} \\
&\quad \leq \max_{a\leq t\leq b} \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \left| \int_{\sigma(A)} \lambda^n e^{t\lambda} d\langle E_A(\lambda)f, g^*\rangle \right| \\
&\quad \leq \max_{a\leq t\leq b} \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\quad = \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \max_{a\leq t\leq b} \left[ \int_{\{\lambda\in\sigma(A) \mid \operatorname{Re}\lambda\leq 0\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \right. \\
&\qquad\qquad \left. + \int_{\{\lambda\in\sigma(A) \mid \operatorname{Re}\lambda> 0\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \right] \\
&\quad \leq \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid \operatorname{Re}\lambda\leq 0\}} |\lambda|^n e^{a\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\qquad + \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid \operatorname{Re}\lambda> 0\}} |\lambda|^n e^{b\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\quad \leq \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} |\lambda|^n e^{a\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\qquad + \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} |\lambda|^n e^{b\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad \text{(by (2.9))} \\
&\quad \leq \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M\|A^n e^{aA}f\|\|g^*\| + \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M\|A^n e^{bA}f\|\|g^*\| \\
&\quad \leq 4M\left[\|A^n e^{aA}f\| + \|A^n e^{bA}f\|\right] = 4M\left[\|A^n y(a)\| + \|A^n y(b)\|\right] \quad \text{(by (3.28))} \\
&\quad = 4M\left[\|y^{(n)}(a)\| + \|y^{(n)}(b)\|\right].
\end{aligned} \tag{3.34}$$

Hence, in view of (3.31), we obtain

$$\max_{a\le t\le b}\|T^{(n)}(t)f\| \le 4M\left[c(a,\alpha)+c(b,\alpha)\right]\max\left[\alpha(a),\alpha(b)\right]^n[n!]^\beta, \quad n=0,1,2,\dots, \tag{3.35}$$

which implies that the restriction of $T(\cdot)f$ to the subinterval $I\subseteq[0,T)$ belongs to the Gevrey class $\mathscr{E}^{\{\beta\}}(I,X)$ (resp., $\mathscr{E}^{(\beta)}(I,X)$). □

**4. Infinite differentiable $C_0$-semigroups.** Recall that a $C_0$-semigroup $\{T(t)\mid t\ge 0\}$ in a Banach space $X$ is said to be *infinite differentiable* (a $C^\infty$-semigroup) if, for any $f\in X$, the orbit $T(\cdot)f$ is infinite differentiable on $(0,\infty)$ in the strong sense. Note that, due to the semigroup property $T(t+s)=T(t)T(s)$, $t,s\ge 0$, the first-order strong differentiability of an orbit on $(0,\infty)$ immediately implies its infinite strong differentiability on $(0,\infty)$.

**Theorem 4.1.** *A $C_0$-semigroup generated by a scalar type spectral operator $A$ in a complex Banach space $X$ is infinite differentiable if and only if, for an arbitrary positive $b$, there is a real $a$ such that*

$$\operatorname{Re}\lambda \le a-b\ln|\operatorname{Im}\lambda|, \quad \lambda\in\sigma(A). \tag{4.1}$$

**Proof**

**"Only if" part.** This part immediately follows from the general criterion of the generation of infinite differentiable $C_0$-semigroups [25] (see also [6, 26]).

**"If" part.** Here, unlike in [22], resorting to the general criterion, that is, proving that there is a $C>0$ such that in the region

$$R_b := \{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda > a-b\ln|\operatorname{Im}\lambda|\}\subseteq\rho(A) \tag{4.2}$$

($\rho(\cdot)$ is the *resolvent set* of an operator) the estimate

$$\|R(\lambda,A)\|\le C|\operatorname{Im}\lambda| \tag{4.3}$$

holds, brings about rather formidable difficulties. The reader could try evaluating the distance from a point $\lambda\in R_b$ to the boundary of the region $R_b$ such an approach would inevitably entail.

Utilizing the general criterion not being an option, we are to prove directly that all the orbits of the semigroup generated by $A$ are strongly differentiable on $(0,\infty)$.

Since $A$ generates a $C_0$-semigroup, the latter, as was shown in [22], consists of its exponentials,

$$e^{tA}=\int_{\sigma(A)} e^{t\lambda}\,dE_A(\lambda), \quad t\ge 0, \tag{4.4}$$

and there is a real $\omega$ such that

$$\operatorname{Re}\lambda \le \omega, \quad \lambda \in \sigma(A). \tag{4.5}$$

Without loss of generality, we can regard that

$$\operatorname{Re}\lambda \le 0, \quad \lambda \in \sigma(A), \tag{4.6}$$

that is, we deal with a *contraction* semigroup. Indeed, otherwise, we can consider the $C_0$-semigroup $T(t) := e^{-\omega t}e^{tA}$, $t \ge 0$, which, evidently, satisfies (4.1).

We need to show that, for any $f \in X$,

$$e^{tA}f \in D(A), \quad 0 < t < \infty. \tag{4.7}$$

Let $0 < t < \infty$. Since the constant $b$ can acquire arbitrary positive values, we can set $b := 1/t$. Then, for any Borel set $\sigma \subseteq \sigma(A)$ and arbitrary $f \in X$ and $g^* \in X^*$,

$$\begin{aligned}
&\int_\sigma |\lambda| e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\quad\le \int_\sigma (|\operatorname{Re}\lambda| + |\operatorname{Im}\lambda|)e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\qquad \left(\text{for } \lambda \in \sigma,\ \operatorname{Re}\lambda \le \min(0, a - b\ln|\operatorname{Im}\lambda|) \Longrightarrow \operatorname{Re}\lambda \le 0 \text{ and } |\operatorname{Im}\lambda| \le e^{b^{-1}(a-\operatorname{Re}\lambda)}\right)\\
&\quad\le \int_\sigma (-\operatorname{Re}\lambda + e^{ab^{-1}}e^{b^{-1}(-\operatorname{Re}\lambda)})e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad (\text{since } x \le e^x,\ 0 \le x < \infty)\\
&\quad\le \int_\sigma \left(be^{b^{-1}(-\operatorname{Re}\lambda)} + e^{ab^{-1}}e^{b^{-1}(-\operatorname{Re}\lambda)}\right)e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\quad= \left[b + e^{ab^{-1}}\right]\int_\sigma e^{(t-b^{-1})\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad \left(\text{by the choice } b = \frac{1}{t}\right)\\
&\quad= \left[\frac{1}{t} + e^{at}\right]\int_\sigma 1\, dv(f,g^*,\lambda) = \left[\frac{1}{t} + e^{at}\right]v(f,g^*,\sigma).
\end{aligned} \tag{4.8}$$

This estimate, by [23, Proposition 3.1], implies (4.7).

Indeed,

(i) for any $f \in X$ and $g^* \in X^*$, we have

$$\begin{aligned}
\int_{\sigma(A)} |\lambda| e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) &\le \left[\frac{1}{t} + e^{at}\right]v(f,g^*,\sigma(A)) \quad (\text{by (2.6)})\\
&\le 4M\left[\frac{1}{t} + e^{at}\right]\|f\|\|g^*\|, \quad 0 < t < \infty,
\end{aligned} \tag{4.9}$$

(ii) analogously, for any $0 < t < \infty$ and an arbitrary $f \in X$,

$$\begin{aligned}
&\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |\lambda| e^{t\operatorname{Re}\lambda}>n\}} |\lambda| e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\
&\quad \le \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \left[\frac{1}{t}+e^{at}\right] \int_{\{\lambda\in\sigma(A) \mid |\lambda| e^{t\operatorname{Re}\lambda}>n\}} 1\, dv(f,g^*,\lambda) \quad (\text{by } (2.8)) \\
&\quad \le \left[\frac{1}{t}+e^{at}\right] \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M \left\| E_A(\{\lambda\in\sigma(A) \mid |\lambda| e^{t\operatorname{Re}\lambda}>n\}) f\right\| \|g^*\| \qquad (4.10) \\
&\quad \le 4M\left[\frac{1}{t}+e^{at}\right] \left\| E_A(\{\lambda\in\sigma(A) \mid |\lambda| e^{t\operatorname{Re}\lambda}>n\}) f\right\| \\
&\qquad\qquad\qquad\qquad (\text{by the strong continuity of the } s.m.) \\
&\quad \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Thus, by [23, Proposition 3.1], (4.7) holds. □

**5. Gevrey ultradifferentiable $C_0$-semigroups.** Let $0 < \beta < \infty$. We will call a $C_0$-semigroup $\{T(t) \mid t \ge 0\}$ in a Banach space $X$ an $\mathscr{E}^{\{\beta\}}$-semigroup (resp., an $\mathscr{E}^{(\beta)}$-semigroup) if, for any $f \in X$, the orbit $T(\cdot)f$ belongs to the Gevrey class $\mathscr{E}^{\{\beta\}}((0,\infty),X)$ (resp., $\mathscr{E}^{(\beta)}((0,\infty),X)$). We will call a $C_0$-semigroup a *Gevrey ultradifferentiable* semigroup if, for some $0 < \beta < \infty$, it is an $\mathscr{E}^{\{\beta\}}$-semigroup or, which, due to inclusions (2.16), is the same, an $\mathscr{E}^{(\beta)}$-semigroup.

**Theorem 5.1.** *Let $1 \le \beta < \infty$. A $C_0$-semigroup generated by a scalar type spectral operator $A$ in a complex Banach space $X$ is an $\mathscr{E}^{\{\beta\}}$-semigroup if and only if there are a positive $b$ and a real $a$ such that*

$$\operatorname{Re}\lambda \le a - b|\operatorname{Im}\lambda|^{1/\beta}, \quad \lambda \in \sigma(A). \tag{5.1}$$

**Proof**

**"If" part.** As is easily seen, the sufficiency condition of Theorem 5.1 is stronger than that of Theorem 4.1. Therefore, by Theorem 4.1, we infer that $A$ generates an infinitely differentiable $C_0$-semigroup consisting, according to [22], of its exponentials presented in (4.4). For any $f \in X$ and $n = 1,2,\ldots,$

$$\frac{d^n}{dt^n} e^{tA} f = A^n e^{tA} f, \quad 0 < t < \infty. \tag{5.2}$$

According to Proposition 3.1, we need to show that, for any $f \in X$,

$$e^{tA} f \in \mathscr{E}^{\{\beta\}}(A), \quad 0 < t < \infty. \tag{5.3}$$

In view of inclusions (2.14), it suffices to show that

$$e^{tA} f \in \bigcup_{s>0} D\left(e^{s|A|^{1/\beta}}\right), \quad 0 < t < \infty. \tag{5.4}$$

We fix an arbitrary Borel subset $\sigma$ of $\sigma(A)$ and an arbitrary $t > 0$. We also set $s := t/[1+(2/b)^{\beta}]^{1/\beta} > 0$ (such a peculiar choice of $s$ will make sense later).

For any $f \in X$ and $g^* \in X^*$,

$$\begin{aligned}\int_\sigma & e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ &= \int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,a)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ &\quad + \int_{\{\lambda\in\sigma\,|\,\min(-1,a)<\operatorname{Re}\lambda\le a\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) < \infty.\end{aligned} \tag{5.5}$$

Indeed, the latter integral is finite due to the boundedness of the set $\{\lambda \in \sigma \mid \min(-1,a) < \operatorname{Re}\lambda \le a\}$ (note that, for $a \le -1$, the set is, obviously, empty), the continuity of the integrated function, and the finiteness of the positive measure $v(f,g^*,\cdot)$ (see (2.6)).

For the former of the above two integrals, we have

$$\begin{aligned}\int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,a)\}} & e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ & \left(\lambda\in\sigma,\ \operatorname{Re}\lambda\le\min(-1,a) \Longrightarrow \operatorname{Re}\lambda\le -1 \text{ and } |\operatorname{Im}\lambda| \le b^{-\beta}(a-\operatorname{Re}\lambda)^\beta\right) \\ \le \int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,a)\}} & e^{s[-\operatorname{Re}\lambda + b^{-\beta}(a-\operatorname{Re}\lambda)^\beta]^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda).\end{aligned} \tag{5.6}$$

We consider separately the two possible cases $a \le 0$ and $a > 0$.

If $a \le 0$, then $a - \operatorname{Re}\lambda \le -2\operatorname{Re}\lambda$ for all $\lambda$'s such that $\operatorname{Re}\lambda \le \min(-1,a)$, and we have

$$\begin{aligned}\int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,a)\}} & e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ &\le \int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,a)\}} e^{s[-\operatorname{Re}\lambda + b^{-\beta}(-2\operatorname{Re}\lambda)^\beta]^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ &\qquad \left(\text{since, for } 1\le\beta<\infty,\ x\le x^\beta,\ x\ge 1\right) \\ &\le \int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,a)\}} e^{s[(-\operatorname{Re}\lambda)^\beta + (2/b)^\beta(-\operatorname{Re}\lambda)^\beta]^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ &= \int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,a)\}} e^{(t-s[1+(2/b)^\beta]^{1/\beta})\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ &\qquad \left(\text{by the choice } s = \frac{t}{[1+(2/b)^\beta]^{1/\beta}}\right) \\ &= \int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,a)\}} 1\, dv(f,g^*,\lambda) \\ &= v(f,g^*,\{\lambda\in\sigma \mid \operatorname{Re}\lambda\le\min(-1,a)\}) \quad (\text{by } (2.6)) \\ &\le 4M\|f\|\|g^*\| < \infty.\end{aligned} \tag{5.7}$$

If $a > 0$,

$$\begin{aligned}\int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,a)\}} & e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ &= \int_{\{\lambda\in\sigma\,|\,\operatorname{Re}\lambda\le\min(-1,-a)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ &\quad + \int_{\{\lambda\in\sigma\,|\,\min(-1,-a)<\operatorname{Re}\lambda\le -1\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) < \infty.\end{aligned} \tag{5.8}$$

Indeed, the latter integral is finite due to the boundedness of the set $\{\lambda \in \sigma \mid \min(-a,-1) < \operatorname{Re}\lambda \le -1\}$ (note that, for $a \ge 1$, the set is, obviously, empty), the continuity of the integrated function, and the finiteness of the positive measure $v(f,g^*,\cdot)$ (see (2.6)).

The former of the above two integrals is finite as well:

$$\begin{aligned}
&\int_{\{\lambda\in\sigma\mid\operatorname{Re}\lambda\le\min(-1,-a)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \quad (\text{since, for } x\le -a,\ a-x\le -2x)\\
&\quad\le \int_{\{\lambda\in\sigma\mid\operatorname{Re}\lambda\le\min(-1,-a)\}} e^{s[-\operatorname{Re}\lambda+b^{-\beta}(-2\operatorname{Re}\lambda)^{\beta}]^{1/\beta}} e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\qquad\qquad\qquad\qquad (\text{since, for } 1\le\beta<\infty,\ x\le x^{\beta},\ x\ge 1)\\
&\quad\le \int_{\{\lambda\in\sigma\mid\operatorname{Re}\lambda\le\min(-1,-a)\}} e^{s[(-\operatorname{Re}\lambda)^{\beta}+(2/b)^{\beta}(-\operatorname{Re}\lambda)^{\beta}]^{1/\beta}} e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\quad= \int_{\{\lambda\in\sigma\mid\operatorname{Re}\lambda\le\min(-1,-a)\}} e^{(t-s[1+(2/b)^{\beta}]^{1/\beta})\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \qquad (5.9)\\
&\qquad\qquad\qquad\qquad \left(\text{by the choice } s=\frac{t}{[1+(2/b)^{\beta}]^{1/\beta}}\right)\\
&\quad= \int_{\{\lambda\in\sigma\mid\operatorname{Re}\lambda\le\min(-1,-a)\}} 1\,dv(f,g^*,\lambda)\\
&\quad= v(f,g^*,\{\lambda\in\sigma\mid\operatorname{Re}\lambda\le\min(-1,-a)\}) \quad (\text{by } (2.6))\\
&\quad\le 4M\|f\|\|g^*\|<\infty.
\end{aligned}$$

Thus, we have proved that, for an arbitrary Borel subset $\sigma \subseteq \sigma(A)$, any $f \in X$, and $g^* \in X^*$,

$$\int_{\sigma} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) < \infty, \quad t>0, \tag{5.10}$$

with $s = t/[1+(2/b)^{\beta}]^{1/\beta} > 0$.

This, in particular, implies that, for any $f \in X$ and $g^* \in X^*$,

$$\int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) < \infty, \quad t>0, \tag{5.11}$$

with $s = t/[1+(2/b)^{\beta}]^{1/\beta} > 0$.

Furthermore, for any $f \in X$, $g^* \in X^*$, $t > 0$, and $s = t/[1+(2/b)^{\beta}]^{1/\beta} > 0$,

$$\sup_{\{g^*\in X^*\mid\|g^*\|=1\}} \int_{\{\lambda\in\sigma\mid e^{2s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{5.12}$$

Indeed, as follows from the preceding argument, the specific choice of $s = t/[1+(2/b)^{\beta}]^{1/\beta} > 0$ allows to partition the set $\{\lambda \in \sigma \mid e^{2s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} > n\}$ into two subsets $\sigma_1$ and $\sigma_2$ in such a way that $\sigma_1$ is bounded and

$$e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} = 1, \quad \lambda \in \sigma_2. \tag{5.13}$$

Therefore,

$$
\begin{aligned}
&\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma \mid e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\le \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma_1 \mid e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\quad + \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma_2 \mid e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\qquad \left(\text{since } \sigma_1 \text{ is bounded, there is such a } C>0 \text{ that } e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\le C,\ \lambda\in\sigma_1;\ \text{by (2.8)}\right)\\
&\le \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} C4M\left\|E_A\left(\left\{\lambda\in\sigma_1 \mid e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}\right)f\right\|\left\|g^*\right\|\\
&\quad + \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M\left\|E_A\left(\left\{\lambda\in\sigma_2 \mid e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}\right)f\right\|\left\|g^*\right\|\\
&\le 4CM\left\|E_A\left(\left\{\lambda\in\sigma_1 \mid e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}\right)f\right\|\\
&\quad + 4M\left\|E_A\left(\left\{\lambda\in\sigma_2 \mid e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}\right)f\right\| \quad (\text{by the strong continuity of the } s.m.)\\
&\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned}
\tag{5.14}
$$

According to [23, Proposition 3.1], we have proved that, for any $f\in X$ and $t>0$,

$$
e^{tA}f\in D\left(e^{s|A|^{1/\beta}}\right), \tag{5.15}
$$

where $s=t/[1+(2/b)^{\beta}]^{1/\beta}>0$.

Hence, for any $f\in X$,

$$
e^{tA}f\in\bigcup_{s>0}D\left(e^{s|A|^{1/\beta}}\right)\subseteq\mathscr{E}^{\{\beta\}}(A),\quad 0<t<\infty. \tag{5.16}
$$

**“Only if” part.** We prove this part by *contrapositive*.

Assume the negation of *“for some positive $b$ and real $a$, $\sigma(A)\subseteq\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda\le a-b|\operatorname{Im}\lambda|^{1/\beta}\}$,”* that is, *for any positive $b$ and real $a$, the set $\sigma(A)\setminus\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda\le a-b|\operatorname{Im}\lambda|^{1/\beta}\}\neq\emptyset$*. Whence it is easy to infer that, for any natural $n$, the set

$$
\sigma(A)\setminus\left\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda\le-\frac{1}{n}|\operatorname{Im}\lambda|^{1/\beta}\right\} \tag{5.17}
$$

is *unbounded*.

Hence, we can choose a sequence of points of the complex plane $\{\lambda_n\}_{n=1}^{\infty}$ in the following way:

$$
\begin{gathered}
\lambda_n\in\sigma(A),\quad n=1,2,\ldots,\\
\operatorname{Re}\lambda_n>-\frac{1}{n}|\operatorname{Im}\lambda|^{1/\beta},\quad n=1,2,\ldots,\\
\lambda_0:=0,\quad |\lambda_n|>\max[n,|\lambda_{n-1}|],\quad n=1,2,\ldots.
\end{gathered}
\tag{5.18}
$$

The latter, in particular, implies that the points $\lambda_n$ are *distinct*:

$$\lambda_i \neq \lambda_j, \quad i \neq j. \tag{5.19}$$

Since the set

$$\left\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > -\frac{1}{n}|\operatorname{Im}\lambda|^{1/\beta},\ |\lambda| > \max\left[n, |\lambda_{n-1}|\right]\right\} \tag{5.20}$$

is *open* in $\mathbb{C}$ for any $n = 1, 2, \ldots$, there exists such a $\varepsilon_n > 0$ that this set contains together with the point $\lambda_n$ the *open disk* centered at $\lambda_n$:

$$\Delta_n = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_n| < \varepsilon_n\}, \tag{5.21}$$

that is, for any $\lambda \in \Delta_n$,

$$\begin{gathered} \operatorname{Re}\lambda > -\frac{1}{n}|\operatorname{Im}\lambda|^{1/\beta}, \\ |\lambda| > \max\left[n, |\lambda_{n-1}|\right]. \end{gathered} \tag{5.22}$$

Moreover, since the points $\lambda_n$ are *distinct*, we can regard that the radii of the disks, $\varepsilon_n$, are chosen to be small enough so that

$$\begin{gathered} 0 < \varepsilon_n < \frac{1}{n}, \quad n = 1, 2, \ldots, \\ \Delta_i \cap \Delta_j = \varnothing, \quad i \neq j \quad \text{(the disks are } \textit{pairwise disjoint}\text{)}. \end{gathered} \tag{5.23}$$

Note that, by the properties of the *s.m.*, the latter implies that the subspaces $E_A(\Delta_n)X$, $n = 1, 2, \ldots$, are *nontrivial* since $\Delta_n \cap \sigma(A) \neq \varnothing$ and $\Delta_n$ is *open*, and

$$E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j. \tag{5.24}$$

We can choose a unit vector $e_n$ in each subspace $E_A(\Delta_n)X$ and thereby obtain a vector sequence such that

$$E_A(\Delta_i)e_j = \delta_{ij}e_i \tag{5.25}$$

($\delta_{ij}$ is the *Kronecker delta symbol*).

The latter, in particular, implies that the vectors $\{e_1, e_2, \ldots\}$ are linearly independent and that

$$d_n := \operatorname{dist}\left(e_n, \operatorname{span}\left(\{e_k \mid k \in \mathbb{N},\ k \neq n\}\right)\right) > 0, \quad n = 1, 2, \ldots. \tag{5.26}$$

Furthermore,

$$d_n \not\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.27}$$

Indeed, if we assume the opposite: $d_n \to 0$ as $n \to \infty$, then, for any $n = 1,2,\dots$, we can find an $f_n \in \operatorname{span}(\{e_k \mid k \in \mathbb{N},\ k \neq n\})$ such that $\|e_n - f_n\| < d_n + 1/n$, which immediately implies that $e_n = E_A(\Delta_n)(e_n - f_n) \to 0$. Thus, such an assumption leads to a contradiction.

Therefore, there is a positive $\varepsilon$ such that

$$d_n \geq \varepsilon, \quad n = 1,2,\dots. \tag{5.28}$$

As follows from the *Hahn-Banach theorem*, for each $n = 1,2,\dots$, there is an $e_n^* \in X^*$ such that

$$\begin{gathered} \|e_n^*\| = 1, \\ \langle e_i, e_j^* \rangle = \delta_{ij} d_i. \end{gathered} \tag{5.29}$$

Let

$$g^* := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n^*. \tag{5.30}$$

Hence,

$$\begin{aligned} \langle e_n, g^* \rangle &= \frac{d_n}{n^2} \quad (\text{by } (5.28)) \\ &\geq \frac{\varepsilon}{n^2}. \end{aligned} \tag{5.31}$$

Concerning the sequence of the real parts, $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$, there are two possibilities: it is either *bounded* or not. We consider separately each of them.

First, assume that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is *bounded*, that is, there is such an $\omega > 0$ that

$$|\operatorname{Re}\lambda_n| \leq \omega, \quad n = 1,2,\dots. \tag{5.32}$$

Let

$$f := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n. \tag{5.33}$$

As can be easily deduced from (5.24),

$$\begin{gathered} E_A(\Delta_n) f = \frac{1}{n^2} e_n, \quad n = 1,2,\dots, \\ E_A\left(\bigcup_{n=1}^{\infty} \Delta_n\right) f = f. \end{gathered} \tag{5.34}$$

Also, for $n = 1,2,\dots,$

$$\begin{aligned} v(f, g^*, \Delta_n) &\geq |\langle E_A(\Delta_n) f, g^* \rangle| \quad (\text{by } (5.34)) \\ &= \left| \left\langle \frac{1}{n^2} e_n, g^* \right\rangle \right| \quad (\text{by } (5.31)) \\ &= \frac{d_n}{n^4} \geq \frac{\varepsilon}{n^4}. \end{aligned} \tag{5.35}$$

For an arbitrary $s > 0$, we have

$$\begin{aligned}
&\int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad (\text{by (5.34)})\\
&\quad = \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re}\lambda} dv\left(E_A\left(\bigcup_{n=1}^{\infty}\Delta_n\right)f,g^*,\lambda\right) \quad (\text{by the properties of the } o.c.)\\
&\quad = \int_{\bigcup_{n=1}^{\infty}\Delta_n} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\quad = \sum_{n=1}^{\infty}\int_{\Delta_n} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\qquad\qquad (\text{for } \lambda\in\Delta_n, \text{ by (5.22), (5.32), and (5.23)}, |\lambda|\geq n, \text{ and } \operatorname{Re}\lambda =\\
&\qquad\qquad \operatorname{Re}\lambda_n - (\operatorname{Re}\lambda_n - \operatorname{Re}\lambda) \geq \operatorname{Re}\lambda_n - |\lambda_n-\lambda| \geq -\omega-\varepsilon_n \geq -\omega - 1)\\
&\quad \geq \sum_{n=1}^{\infty} e^{sn^{1/\beta}} e^{-(\omega+1)} v(f,g^*,\Delta_n) \quad (\text{by (5.35)})\\
&\quad \geq e^{-(\omega+1)}\sum_{n=1}^{\infty}\frac{\varepsilon e^{sn^{1/\beta}}}{n^4} = \infty.
\end{aligned} \tag{5.36}$$

This, by [23, Proposition 3.1], implies that

$$e^A f \notin \bigcup_{t>0} D\left(e^{t|A|^{1/\beta}}\right). \tag{5.37}$$

Then, by (2.14), moreover,

$$e^A f \notin \mathscr{E}^{\beta}(A). \tag{5.38}$$

Hence, by Proposition 3.1, we infer that the orbit $e^{tA}f$, $t \geq 0$, does not belong to $\mathscr{E}^{\{\beta\}}((0,\infty),X)$.

Now, suppose that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is *unbounded*. The sequence being *bounded above*, since $A$ generates a $C_0$-semigroup [11] (see also [22]), this means that there is a subsequence $\{\operatorname{Re}\lambda_{n(k)}\}_{k=1}^{\infty}$ such that

$$\operatorname{Re}\lambda_{n(k)} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty. \tag{5.39}$$

Thus, without loss of generality, we can regard that

$$\operatorname{Re}\lambda_{n(k)} \leq -k, \quad k = 1,2,\ldots. \tag{5.40}$$

Consider the vector

$$f := \sum_{k=1}^{\infty}\frac{1}{k^2}e_{n(k)}. \tag{5.41}$$

By (5.24),

$$E_A(\Delta_n(k))f = \frac{1}{k}e_{n(k)}, \quad k = 1,2,\ldots,$$
$$E_A\left(\bigcup_{k=1}^{\infty}\Delta_{n(k)}\right)f = f. \tag{5.42}$$

For an arbitrary $s > 0$, we similarly have

$$\int_{\sigma(A)} e^{s|\lambda|^{1/\beta}}e^{\operatorname{Re}\lambda}dv(f,g^*,\lambda) = \sum_{k=1}^{\infty}\int_{\Delta_{n(k)}} e^{s|\lambda|^{1/\beta}}e^{\operatorname{Re}\lambda}dv(f,g^*,\lambda) = \infty. \tag{5.43}$$

Indeed, for all $\lambda \in \Delta_{n(k)}$, based on (5.23), (5.40), and (5.22), we have

$$\begin{aligned}\operatorname{Re}\lambda &= \operatorname{Re}\lambda_{n(k)} - (\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda) \le \operatorname{Re}\lambda_{n(k)} + |\lambda_{n(k)} - \lambda| \\ &\le \operatorname{Re}\lambda_{n(k)} + \varepsilon_{n(k)} \le -k+1 \le 0, \\ &\qquad -\frac{1}{n(k)}|\operatorname{Im}\lambda|^{1/\beta} < \operatorname{Re}\lambda.\end{aligned} \tag{5.44}$$

Therefore, for $\lambda \in \Delta_{n(k)}$,

$$-\frac{1}{n(k)}|\operatorname{Im}\lambda|^{1/\beta} < \operatorname{Re}\lambda \le -k+1 \le 0. \tag{5.45}$$

Whence, for $\lambda \in \Delta_{n(k)}$,

$$\operatorname{Re}\lambda \le -k+1 \le 0, \quad |\lambda| \ge |\operatorname{Im}\lambda| \ge [n(k)(-\operatorname{Re}\lambda)]^{\beta}. \tag{5.46}$$

Using these estimates, we have

$$\begin{aligned}&\int_{\Delta_{n(k)}} e^{s|\lambda|^{1/\beta}}e^{\operatorname{Re}\lambda}dv(f,g^*,\lambda) \\ &\quad \ge \int_{\Delta_{n(k)}} e^{[sn(k)-1](-\operatorname{Re}\lambda)}dv(f,g^*,\lambda) \\ &\quad \text{(for all } k\text{'s sufficiently large so that } sn(k)-1>0 \text{ and } k-1\ge 1) \\ &\quad \ge e^{[sn(k)-1](k-1)}v(f,g^*,\Delta_{n(k)}) \quad \text{(by (5.35))} \\ &\quad \ge \frac{\varepsilon e^{[sn(k)-1]}}{n(k)^4} \longrightarrow \infty \quad \text{as } k \longrightarrow \infty.\end{aligned} \tag{5.47}$$

Similarly, we infer that the orbit $e^{tA}f$, $t \ge 0$, does not belong to the class $\mathscr{E}^{\{\beta\}}((0,\infty),X)$.

Having analyzed all the possibilities concerning $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$, we infer that the negation of *"for some positive $b$ and real $a$, $\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le a - b|\operatorname{Im}\lambda|^{1/\beta}\}$"* implies that *not every orbit of the $C_0$-semigroup $\{e^{tA} \mid t \ge 0\}$ belongs to the Gevrey class* $\mathscr{E}^{\{\beta\}}((0,\infty),H)$, that is, $\{e^{tA} \mid t \ge 0\}$ is not an $\mathscr{E}^{\{\beta\}}$-semigroup.

Thus, the "only if" part has been proved by *contrapositive*. □

In particular, for $\beta = 1$, we obtain a criterion of the generation of an analytic $C_0$-semigroup by a *scalar type spectral operator* [22] (see also [30]).

Observe that, for $0 < \beta < 1$, all the orbits of the $C_0$-semigroup $\{e^{tA} \mid t \geq 0\}$ are *entire* functions, which immediately implies that $A$ is bounded (see [20]).

**6. A concluding remark.** Similar results for a *normal operator* in a complex Hilbert space are discussed in a more general context in [18, 19] (see also [20, 21]).

## References


[1] J. M. Ball, *Strongly continuous semigroups, weak solutions, and the variation of constants formula*, Proc. Amer. Math. Soc. **63** (1977), no. 2, 370–373.

[2] N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. **64** (1958), 217–274.

[3] N. Dunford and J. T. Schwartz, *Linear Operators. Part I. General Theory*, Pure and Applied Mathematics, vol. 7, Interscience Publishers, New York, 1958, with the assistance of W. G. Bade and R. G. Bartle.

[4] ______, *Linear Operators. Part II: Spectral Theory. Self Adjoint Operators In Hilbert Space*, Interscience Publishers, New York, 1963, with the assistance of William G. Bade and Robert G. Bartle.

[5] ______, *Linear Operators. Part III: Spectral Operators*, Interscience Publishers, New York, 1971.

[6] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.

[7] M. Gevrey, *Sur la nature analytique des solutions des équations aux dérivées partielles*, Ann. Ec. Norm. Sup. Paris **35** (1918), 129–190 (French).

[8] S. Goldberg, *Unbounded Linear Operators: Theory and Applications*, McGraw-Hill, New York, 1966.

[9] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, Clarendon Press, Oxford University Press, New York, 1985.

[10] R. Goodman, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc. **143** (1969), 55–76.

[11] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, vol. 31, American Mathematical Society, Rhode Island, 1957.

[12] H. Komatsu, *Ultradistributions and hyperfunctions*, Hyperfunctions and Pseudo-Differential Equations (Proc. Conf. on the Theory of Hyperfunctions and Analytic Functionals and Applications, R. I. M. S., Kyoto Univ., Kyoto, 1971; Dedicated to the Memory of André Martineau), Lecture Notes in Math., vol. 287, Springer, Berlin, 1973, pp. 164–179.

[13] ______, *Ultradistributions. I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20** (1973), 25–105.

[14] ______, *Microlocal analysis in Gevrey classes and in complex domains*, Microlocal Analysis and Applications (Montecatini Terme, 1989), Lecture Notes in Math., vol. 1495, Springer-Verlag, Berlin, 1991, pp. 161–236.

[15] S. G. Kreĭn and M. I. Khazan, *Differential equations in a Banach space*, Mathematical Analysis, Vol. 21, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983, pp. 130–264.

[16] S. Mandelbrojt, *Series de Fourier et Classes Quasi-Analytiques de Fonctions*, Gauthier-Villars, Paris, 1935.

[17] M. V. Markin, *On the Carleman classes of vectors of a scalar type spectral operator*, to appear in Int. J. Math. Math. Sci.

[18] ______, *On the strong smoothness of weak solutions of an abstract evolution equation. I. Differentiability*, Appl. Anal. **73** (1999), no. 3-4, 573–606.

[19] ______, *On the strong smoothness of weak solutions of an abstract evolution equation. II. Gevrey ultradifferentiability*, Appl. Anal. **78** (2001), no. 1-2, 97–137.

[20] ______, *On the strong smoothness of weak solutions of an abstract evolution equation. III. Gevrey ultradifferentiability*, Appl. Anal. **78** (2001), no. 1-2, 139–152.

[21] ______, *On the strong smoothness of weak solutions of an abstract evolution equation. IV. Carleman ultradifferentiability*, to appear in Appl. Anal.

[22] ______, *A note on the spectral operators of scalar type and semigroups of bounded linear operators*, Int. J. Math. Math. Sci. **32** (2002), no. 10, 635–640.

[23] ______, *On an abstract evolution equation with a spectral operator of scalar type*, Int. J. Math. Math. Sci. **32** (2002), no. 9, 555–563.

[24] E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615.

[25] A. Pazy, *On the differentiability and compactness of semigroups of linear operators*, J. Math. Mech. **17** (1968), 1131–1141.

[26] ______, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[27] A. Plesner, *Spectral Theory of Linear Operators*, Izdat. "Nauka", Moscow, 1965.

[28] V. V. Vasil'ev, S. G. Kreĭn, and S. I. Piskarëv, *Operator semigroups, cosine operator functions, and linear differential equations*, Mathematical Analysis, Vol. 28, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990, pp. 87–202.

[29] J. Wermer, *Commuting spectral measures on Hilbert space*, Pacific J. Math. **4** (1954), 355–361.

[30] K. Yosida, *On the differentiability of semigroups of linear operators*, Proc. Japan Acad. **34** (1958), 337–340.

[31] ______, *Functional Analysis*, Die Grundlehren der mathematischen Wissenschaften, vol. 123, Academic Press, New York, 1965.

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