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THE RADIUS OF STARLIKENESS FOR CONVEX FUNCTIONS OF COMPLEX ORDER

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We will give the relation between the class of Janowski starlike functions of complex order and the class of Janowski convex functions of complex order. As a corollary of this relation, we obtain the radius of starlikeness for the class of Janowski convex functions of complex order.

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1. Introduction. Let F be the class of analytic functions in $D = \{z \mid |z| < 1\}$, and let S denote those functions in F that are univalent and normalized by $f(0) = 0$, $f'(0) = 1$. Furthermore, let Ω be the family of functions $\omega(z)$ regular in D and satisfying $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$.

For arbitrary fixed numbers $-1 \leq B < A \leq 1$, denoted by $P(A,B)$, the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \tag{1.1}$$

which is regular in D on the condition such that

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)} \tag{1.2}$$

for some functions $\omega(z) \in \Omega$ and every $z \in D$. This class was introduced by Janowski [7].

Moreover, let $S^*(A,B,b)$ be denoted by the family of functions $f(z) \in S$ such that $f(z)$ is in $S^*(A,B,b)$ if and only if $f(z)/z \neq 0$,

$$1 + \frac{1}{b}\left(z \cdot \frac{f'(z)}{f(z)} - 1\right) = p(z), \quad (b \neq 0, \text{ complex}) \tag{1.3}$$

for some functions $p(z) \in P(A,B)$ and all z in D.

Finally, let $C(A,B,b)$ denote the family of functions which are regular:

$$1 + \frac{1}{b} \cdot z \cdot \frac{f''(z)}{f'(z)} = p(z), \quad (b \neq 0, \text{ complex}) \tag{1.4}$$

for some functions $p(z) \in P(A,B)$ and every z in D.

We note that $P(-1,1)$ is the class of Caratheodory functions, and therefore the class $C(A,B,b)$ contains the following classes. $b=1$, $C(1,-1,1)=C$ is the well-known class of convex functions [2], and $C(1,-1,b)=C(b)$ is the class of convex functions of complex order [7, 8]. $C(1,-1,1-\beta)$, $(0\le\beta<1)$ is the class of convex functions of order β [9]. For $A=1$, $B=-1$, $b=e^{-i\lambda}\cdot\cos\lambda$, $|\lambda|<\pi/2$ is the class of functions for which $zf'(z)$ is λ-spirallike [3, 6, 11, 12, 13, 14]. For $A=1$, $B=-1$, $b=(1-\beta)e^{-i\lambda}\cdot\cos\lambda$, $0\le\beta<1$, $|\lambda|<\pi/2$ is the class of functions for which $zf'(z)$ is λ-spirallike of order β [3, 6, 11, 12, 13, 14].

2. Representation theorem for the class $S^*(A,B,b)$. The following lemma, well known as Jack's lemma, is required in our investigation.

Lemma 2.1 [4, 5]. *Let $w(z)$ be a nonconstant and analytic function in the unit disc D with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r$ at the point z_0, then $z_0w'(z_0)=kw(z_0)$ and $k\ge 1$.*

Lemma 2.2. *$e^{-i\alpha}f(e^{i\alpha}z)$, $\alpha\in[0,2\pi)$ is in $C(A,B,b)$ whenever $f(z)$ is in $C(A,B,b)$.*

Proof. If $f(z)\in C(A,B,b)$, then

$$g(z)=e^{-i\alpha}f(e^{i\alpha}z)\Longrightarrow 1+\frac{1}{b}z\frac{g'(z)}{g(z)}=1+\frac{1}{b}(e^{i\alpha}z)\frac{f'(e^{i\alpha}z)}{f(e^{i\alpha}z)}. \tag{2.1}$$

□

We note that similarly the class $S^*(A,B,b)$ is invariant under the rotation so that $e^{-i\alpha}f(e^{i\alpha}z)$, $\alpha\in[0,2\pi)$ is in $S^*(A,B,b)$ whenever $f(z)$ is in $S^*(A,B,b)$.

Lemma 2.3. *If $g(z)\in S^*(A,B,b)$, then*

$$g(z)=\begin{cases} z(1+Bw(z))^{b(A-B)/B}, & B\ne 0,\ k=1,\\ ze^{bAw(z)}, & B=0,\ k=1,\end{cases} \tag{2.2}$$

for some $w(z)\in\Omega$ and for all z in D, and conversely.

Proof. The proof of this lemma is completed in four steps, and we have used Nicola Tuneski's technique for the special case of $k=1$ [15].

First step. If $B\ne 0$ and

$$g(z)=z(1+Bw(z))^{b(A-B)/B}, \tag{2.3}$$

then by taking logarithmic derivative of (2.3) followed by a brief computation using Jack's lemma and the definition of subordination, we obtain

$$1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right)=\frac{1+Aw(z)}{1+Bw(z)},\quad \text{for } k=1, \tag{2.4}$$

and so from the definition of $S^*(A,B,b)$ it follows that $g(z)\in S^*(A,B,b)$. (See [10].)

Second step. If $B = 0$, then we have $g(z) = ze^{bAw(z)}$. Similarly, we get

$$1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right) = 1+Aw(z), \quad \text{for } k=1. \tag{2.5}$$

The equality shows that $g(z) \in S^*(A,B,b)$.

Third step. Conversely, if $g(z) \in S^*(A,B,b)$ and $B \neq 0$, then we have

$$1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right) = \frac{1+Aw(z)}{1+Bw(z)}. \tag{2.6}$$

Equation (2.6) can be written in the form

$$\frac{g'(z)}{g(z)} = \frac{b(A-B)(w(z)/z)}{1+Bw(z)}+\frac{1}{z}. \tag{2.7}$$

If we use Jack's lemma in (2.7) for $k = 1$, we obtain

$$\frac{g'(z)}{g(z)} = \frac{b(A-B)w'(z)}{1+Bw(z)}+\frac{1}{z}. \tag{2.8}$$

Integrating both sides of equality (2.8), we get (2.3).

Fourth step. Again, conversely, if $g(z) \in S^*(A,B,b)$ and $B = 0$, then in the same way we obtain $g(z) = ze^{bAw(z)}$ which completes the proof. □

Lemma 2.4. *Let $f(z)$ be regular and analytic in D, and normalized so that $f(0) = 0$, $f'(0) = 1$. A necessary and sufficient condition for $f(z) \in C(A,B,b)$ is that for each member $g(z) = z+b_1z+b_2z^2+\cdots$ of $S^*(A,B,b)$ the following equation holds:*

$$g(z,\zeta) = z\left(\frac{f(z)-f(\zeta)}{z-\zeta}\right)^2, \quad \zeta,z \in D,\ \zeta \neq z,\ \zeta = nz,\ |n| \leq 1. \tag{2.9}$$

Proof. If $f(z) \in C(A,B,b)$, then this function is analytic, regular, and continuous in the unit disc. Therefore, equality (2.9) can be written in the form

$$g(z) = z\left(f'(z)\right)^2. \tag{2.10}$$

If we take the logarithmic derivative of equality (2.10) followed by simple calculations, we get

$$1+\frac{1}{2b}\left(z\frac{g'(z)}{g(z)}-1\right) = 1+\frac{1}{b}z\frac{f''(z)}{f'(z)} = \frac{1+Aw(z)}{1+Bw(z)}. \tag{2.11}$$

On the other hand, b is a complex number and $b \neq 0$. Therefore, $b_1 = 2b$ is a complex number and $2b \neq 0$, thus (2.11) can be written in the form

$$1+\frac{1}{b_1}\left(z\frac{g'(z)}{g(z)}-1\right) = 1+\frac{1}{b}z\frac{f''(z)}{f'(z)}. \tag{2.12}$$

Considering equality (2.12), the definition of $C(A,B,b)$, and the definition of $S^*(A,B,b)$, we obtain $g(z) \in S^*(A,B,2b)$.

Conversely, if $g(z) \in S^*(A,B,b)$, and $g(z) = z((f(z)-f(\zeta))/(z-\zeta))$ holds, then from Lemma 2.3 we get

$$g(z) = z\left(\frac{f(z)-f(\zeta)}{z-\zeta}\right)^2 = \begin{cases} z(1+Bw(z))^{b(A-B)/B}, & B \neq 0, \\ ze^{bAw(z)}, & B = 0. \end{cases} \tag{2.13}$$

If we take the logarithmic derivative with respect to z of (2.13) followed by simple calculations, we get

$$\begin{aligned} 1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right) &= \frac{1}{b}\left[\frac{2zf'(z)}{f(z)-f(\zeta)} - \frac{z+\zeta}{z-\zeta}\right]+1-\frac{1}{b} \\ &= \frac{1+Aw(z)}{1+Bw(z)}, \quad B \neq 0, \\ 1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right) &= \frac{1}{b}\left[\frac{2zf'(z)}{f(z)-f(\zeta)} - \frac{z+\zeta}{z-\zeta}\right]+1-\frac{1}{b} \\ &= 1+Aw(z), \quad B = 0. \end{aligned} \tag{2.14}$$

Furthermore, if we write $F(z,\zeta) = (1/b)[2zf'(z)/(f(z)-f(\zeta)) - (z+\zeta)/(z-\zeta)] + 1-1/b$, then we have

$$\lim_{\zeta\to z} F(z,\zeta) = 1+\frac{1}{b}z\frac{f''(z)}{f'(z)}. \tag{2.15}$$

Considering relations (2.14) and (2.15) together, we obtain $f(z) \in C(A,B,b)$. □

Corollary 2.5. *If $f(z) \in C(A,B,b)$, then*

$$2\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right]-1 = p(z) = \frac{1+Aw(z)}{1+Bw(z)}. \tag{2.16}$$

Proof. If we take $\zeta = 0$ in $F(z,\zeta)$, we obtain the desired result of this corollary. □

3. The radius of starlikeness for the class $C(A,B,b)$

Lemma 3.1. *If $f(z) \in C(A,B,b)$, then*

$$\left| z\frac{f'(z)}{f(z)} - \frac{2-[B^2-b(2B^2-AB)r^2]}{2(1-B^2r^2)} \right| \leq \frac{|b|(A-B)r}{2(1-B^2r^2)}. \tag{3.1}$$

Proof. If $p(z) \in P(A,B)$, then

$$\left| p(z) - \frac{1-ABr^2}{1-B^2r^2} \right| \leq \frac{(A-B)r}{1-B^2r^2}. \tag{3.2}$$

The inequality (3.2) was proved by Janowski [7]. Considering Corollary 2.5 and inequality (3.1), then we get

$$\left|2\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)-1\right]-\frac{1-ABr^2}{1-B^2r^2}\right| \le \frac{(A-B)r}{1-B^2r^2}. \tag{3.3}$$

After brief calculations from (3.3), we obtain (3.1). □

Theorem 3.2. *The radius of starlikeness for the class $C(A,B,b)$ is*

$$r_s = \frac{4}{|b|(A-B)+\sqrt{|b|^2(A-B)^2+8\left[2B^2+(AB-B^2)\operatorname{Re} b\right]}}. \tag{3.4}$$

This radius is sharp, because the extremal function is

$$f_*(z) = \begin{cases} \displaystyle\int_0^z (1+B\zeta)^{b(A-B)/B}\,d\zeta, & B \neq 0, \\ \displaystyle\int_0^z e^{Ab\zeta}\,d\zeta, & B = 0. \end{cases} \tag{3.5}$$

Proof. After the brief calculations from inequality (3.1), we get

$$\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) \ge \frac{2-|b|(A-B)r-\left[2B^2+(AB-B^2)\operatorname{Re} b\right]r^2}{1-B^2r^2}. \tag{3.6}$$

Hence for $r < r_s$ the right-hand side of inequality (3.6) is positive. This implies that (3.4) holds.

Also note that inequality (3.6) becomes an equality for the function $f_*(z)$. It follows that (3.4) holds. □

Corollary 3.3. *If $A = 1$, $B = -1$, $b = 1$, then $r_s = 1$. This is the radius of starlikeness of convex functions which is well known (see* [1, Volume II, page 88]*).*

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