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SOME CRITERIA FOR UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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We derive some criteria for univalence of certain integral operators for analytic functions in the open unit disk.

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1. Introduction. Let $\mathcal{A}$ be the class of the functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in $\mathbb{U}$. Miller and Mocanu [1] have considered many integral operators for functions $f(z)$ belonging to the class $\mathcal{A}$. In this paper, we consider the integral operators

$$F_\alpha(z) = \left\{\frac{1}{\alpha}\int_0^z (f(u))^{1/\alpha} u^{-1} du\right\}^\alpha \quad (z \in \mathbb{U}) \tag{1.1}$$

for $f(z) \in \mathcal{A}$ and for some $\alpha \in \mathbb{C}$. It is well known that $F_\alpha(z) \in \mathcal{S}$ for $f(z) \in \mathcal{S}^*$ and $\alpha > 0$, where $\mathcal{S}^*$ denotes the subclass of $\mathcal{S}$ consisting of all starlike functions $f(z)$ in $\mathbb{U}$.

2. Preliminary results. To discuss our integral operators, we need the following theorems.

Theorem 2.1 [3]. *Let α be a complex number with $\operatorname{Re}(\alpha) > 0$ and $f(z) \in \mathcal{A}$. If $f(z)$ satisfies*

$$\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left|\frac{zf''(z)}{f'(z)}\right| \leqq 1, \tag{2.1}$$

for all $z \in \mathbb{U}$, then the integral operator

$$G_\alpha(z) = \left\{\alpha \int_0^z u^{\alpha-1} f'(u) du\right\}^{1/\alpha} \tag{2.2}$$

is in the class $\mathcal{S}$.

Theorem 2.2 [4]. *Let α be a complex number with $\mathrm{Re}(\alpha) > 0$ and $f(z) \in \mathcal{A}$. If $f(z)$ satisfies (2.1) for all $z \in \mathbb{U}$, then, for any complex number β with $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha)$, the integral operator*

$$G_\beta(z) = \left\{\beta \int_0^z u^{\beta-1} f'(u)\, du\right\}^{1/\beta} \tag{2.3}$$

is in the class $\mathcal{S}$.

Example 2.3. Defining the function $f(z)$ by

$$f(z) = \int_0^z \left(\frac{1+u^{\mathrm{Re}(\alpha)}}{1-u^{\mathrm{Re}(\alpha)}}\right)^{1/2} du \tag{2.4}$$

with $\mathrm{Re}(\alpha) > 0$, we have that

$$\frac{1-z^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \left(\frac{zf''(z)}{f'(z)}\right) = z^{\mathrm{Re}(\alpha)}. \tag{2.5}$$

Thus the function $f(z)$ satisfies the condition of Theorem 2.2. Therefore, for $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha)$,

$$G_\beta(z) = \left\{\beta \int_0^z u^{\beta-1} \left(\frac{1+u^{\mathrm{Re}(\alpha)}}{1-u^{\mathrm{Re}(\alpha)}}\right)^{1/2} du\right\}^{1/\beta} \tag{2.6}$$

is in the class $\mathcal{S}$.

Theorem 2.4 [2]. *If the function $g(z)$ is regular in $\mathbb{U}$, then, for all $\xi \in \mathbb{U}$ and $z \in \mathbb{U}$, $g(z)$ satisfies*

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)} \right| \leqq \left| \frac{\xi - z}{1 - \overline{z}\xi} \right|, \tag{2.7}$$

$$|g'(z)| \leqq \frac{1 - |g(z)|^2}{1 - |z|^2}. \tag{2.8}$$

The equalities hold only in the case $g(z) = \varepsilon((z+u)/(1+\overline{u}z))$, where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 2.5 [2]. For $z = 0$, from inequality (2.7),

$$\left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)} \right| \leqq |\xi| \tag{2.9}$$

and, hence

$$|g(\xi)| \leqq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}. \tag{2.10}$$

Considering $g(0) = a$ and $\xi = z$, we see that

$$|g(z)| \leqq \frac{|z| + |a|}{1 + |a||z|} \tag{2.11}$$

for all $z \in \mathbb{U}$.

Schwarz lemma [2]. *If the function $g(z)$ is regular in $\mathbb{U}$, $g(0)=0$, and $|g(z)| \leqq 1$, for all $z \in \mathbb{U}$, then*

$$|g(z)| \leqq |z| \tag{2.12}$$

for all $z \in \mathbb{U}$, and $|g'(0)| \leqq 1$. The equality in (2.12) for $z \neq 0$ holds only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3. Main results

Theorem 3.1. *Let α be a complex number with $\operatorname{Re}(1/\alpha) = a > 0$ and the function $g(z) \in \mathcal{A}$ satisfying*

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| \leqq 1 \quad (z \in \mathbb{U}). \tag{3.1}$$

Then, for

$$|\alpha| \geqq \frac{2}{(2a+1)^{(2a+1)/2a}}, \tag{3.2}$$

the integral operator

$$F_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z (g(u))^{1/\alpha} u^{-1} du \right\}^\alpha \tag{3.3}$$

is in the class $\mathcal{S}$.

Proof. Let $1/\alpha = \beta$. Then we have

$$F_{1/\beta}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{g(u)}{u} \right)^\beta du \right\}^{1/\beta}. \tag{3.4}$$

We consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u} \right)^\beta du. \tag{3.5}$$

Then the function

$$h(z) = \left(\frac{1}{|\beta|} \right) \frac{zf''(z)}{f'(z)} \tag{3.6}$$

is regular in $\mathbb{U}$ and the constant $|\beta|$ satisfies the inequality

$$|\beta| \leqq \frac{(2a+1)}{2}^{(2a+1)/2a}. \tag{3.7}$$

From (3.5) and (3.6), we have that

$$h(z) = \frac{\beta}{|\beta|} \left(\frac{zg'(z)}{g(z)} - 1 \right). \tag{3.8}$$

Using (3.8) and (3.1), we obtain

$$\left|h(z)\right| \leqq 1 \quad (z \in \mathbb{U}). \tag{3.9}$$

Noting that $h(0) = 0$ and applying the Schwarz lemma for $h(z)$, we get

$$\frac{1}{|\beta|}\left|\frac{zf''(z)}{f'(z)}\right| \leqq |z| \quad (z \in \mathbb{U}) \tag{3.10}$$

and hence we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leqq |\beta|\left(\frac{1-|z|^{2a}}{a}\right)|z| \quad (z \in \mathbb{U}). \tag{3.11}$$

Because

$$\max_{|z|\leq 1}\left(\frac{1-|z|^{2a}}{a}|z|\right) = \frac{2}{(2a+1)^{(2a+1)/2a}}, \tag{3.12}$$

from (3.11) and (3.7), we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 \tag{3.13}$$

for $z \in \mathbb{U}$. From (3.13) and Theorem 2.1, it follows that (2.4) belongs to the class $\mathcal{S}$.

By means of (2.4) and (3.5), we have that the integral operator $F_{1/\beta}(z)$ is in the class $\mathcal{S}$, and hence we conclude that the integral operator $F_\alpha(z)$ is in the class $\mathcal{S}$. □

Example 3.2. If we take the function $g(z) = ze^z$ and $\alpha = 1/a > 0$, then

$$g(z) = z + a_2z^2 + a_3z^3 + \cdots \tag{3.14}$$

is analytic in $\mathbb{U}$ and

$$\left|\frac{zg'(z)}{g(z)} - 1\right| = |z| < 1 \quad (z \in \mathbb{U}). \tag{3.15}$$

Since the function $g(z)$ satisfies the condition of Theorem 3.1, we have

$$T_\alpha(z) = \left\{\frac{1}{\alpha}\int_0^z e^{u/\alpha}u^{1/\alpha-1}du\right\}^\alpha \in \mathcal{S}. \tag{3.16}$$

Theorem 3.3. *Let α, β be complex numbers with* $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha) > 0$ *and the function* $g(z) \in \mathcal{A}$ *satisfying*

$$\left|\frac{zg'(z) - g(z)}{zg(z)}\right| \leqq 1 \quad (z \in \mathbb{U}). \tag{3.17}$$

Then, for

$$|\alpha| \geqq \max_{|z|\leqq 1}\left\{\left(\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\right)|z|\left(\frac{|z|+|a_2|}{1+|a_2||z|}\right)\right\}, \tag{3.18}$$

the integral operator

$$F_{\alpha,\beta}(z) = \left\{\beta \int_0^z (g(u))^{1/\alpha} u^{\beta-1/\alpha-1} du\right\}^{1/\beta} \tag{3.19}$$

is in the class $\mathcal{S}$.

Proof. We have

$$F_{\alpha,\beta}(z) = \left\{\beta \int_0^z u^{\beta-1} \left(\frac{g(u)}{u}\right)^{1/\alpha} du\right\}^{1/\beta}. \tag{3.20}$$

We consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{1/\alpha} du, \tag{3.21}$$

which is regular in $\mathbb{U}$. The function

$$p(z) = |\alpha| \frac{f''(z)}{f'(z)}, \tag{3.22}$$

where the constant $|\alpha|$ satisfies inequality (3.18), is regular in $\mathbb{U}$. From (3.22) and (3.21), we obtain

$$p(z) = \frac{|\alpha|}{\alpha}\left\{\frac{zg'(z)-g(z)}{zg(z)}\right\} \tag{3.23}$$

and using (3.17), we have

$$|p(z)| < 1 \quad (z \in \mathbb{U}) \tag{3.24}$$

and $|p(0)| = |a_2|$. Applying Remark 2.5, we obtain

$$\left|\alpha \frac{f''(z)}{f'(z)}\right| \le \frac{|z|+|a_2|}{1+|a_2||z|} \quad (z \in \mathbb{U}). \tag{3.25}$$

It follows that

$$\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left|\frac{zf''(z)}{f'(z)}\right| \leqq \left(\frac{1}{|\alpha|}\right)\left(\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}\right)|z|\left(\frac{|z|+|a_2|}{1+|a_2||z|}\right) \tag{3.26}$$

for all $z \in \mathbb{U}$. We consider the function

$$Q(x) = \left(\frac{1-x^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}\right)x\left(\frac{x+|a_2|}{1+|a_2|x}\right) \quad (x = |z|;\ x \in [0,1]). \tag{3.27}$$

Because $Q(1/2) > 0$, $Q(x)$ satisfies

$$\max_{x\in[0,1]} Q(x) > 0. \tag{3.28}$$

Using this fact, (3.26) gives us that

$$\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}\left|z\frac{f''(z)}{f'(z)}\right| \leqq \frac{1}{|\alpha|}\max_{|z|\leqq 1}\left\{\left(\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}\right)|z|\left(\frac{|z|+|a_2|}{1+|a_2||z|}\right)\right\}. \tag{3.29}$$

From (3.29) and (3.18), we obtain (2.1). Using (2.1) and Theorem 2.2, we obtain that the integral operator (2.4) belongs to the class $\mathcal{S}$. Therefore, it follows from (2.4) and (3.21) that $F_{\alpha,\beta}(z)$ is in the class $\mathcal{S}$. □

Corollary 3.4. *Let α be a complex number with $\operatorname{Re}(\alpha) > 0$ and the function $g(z) \in \mathcal{A}$ satisfying (3.18). Then, for*

$$\max_{|z|\leqq 1}\left\{\left(\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}\right)|z|\left(\frac{|z|+|a_2|}{1+|a_2||z|}\right)\right\} \leqq |\alpha| \leqq 1, \tag{3.30}$$

the integral operator (3.3) is in the class $\mathcal{S}$.

Proof. From Theorem 3.3 for $\beta = 1/\alpha$, the condition $\operatorname{Re}(\beta) \geqq \operatorname{Re}(\alpha) > 0$ is identical with $|\alpha| < 1$ and we have $F_{\alpha,\beta}(z) = F_\alpha(z)$. □

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