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# RESOLVENT ESTIMATES OF ELLIPTIC DIFFERENTIAL AND FINITE-ELEMENT OPERATORS IN PAIRS OF FUNCTION SPACES

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We present some resolvent estimates of elliptic differential and finite-element operators in pairs of function spaces, for which the first space in a pair is endowed with stronger norm. In this work we deal with estimates in (Lebesgue, Lebesgue), (Hölder, Lebesgue), and (Hölder, Hölder) pairs of norms. In particular, our results are useful for the stability and error analysis of semidiscrete and fully discrete approximations to parabolic partial differential problems with rough and distribution-valued data.

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**1. Introduction.** The aim of this work is twofold. The main objective is to get some new resolvent estimates of elliptic finite-element operators which are intended for use in applications of the finite-element method to parabolic initial boundary value problems. It however turns out that this might be successfully achieved when using related estimates for elliptic partial differential operators; some of them seem, apart from everything else, to be of independent interest. We therefore start with showing resolvent estimates for differential operators. In both cases, continuous and discrete, we make an emphasis on deriving such estimates in pairs of function spaces, for which the first space in a pair is endowed with stronger norm. More precisely, our consideration deals with estimates in (Lebesgue, Lebesgue), (Hölder, Lebesgue), and (Hölder, Hölder) pairs of norms.

It is well known that resolvent estimates of elliptic partial differential operators are most important for applications of semigroup theory to the analysis of parabolic initial boundary value problems. In fact, such estimates allow one to study the problem of generation of analytic semigroups, associated to the elliptic operators, in different function spaces. The classical papers of Agmon [1] and Stewart [31] are concerned with generation in Lebesgue spaces and in the space of continuous functions, respectively. For a recent work concerning generation in Besov and Hölder spaces we refer to Grisvard [19], Campanato [8], and Lunardi [21]. The problem of generation in Hölder and Sobolev spaces of integral positive orders was examined by Colombo and Vespri [12] and Mora [23]. A detailed discussion of generation results in different function spaces can be found, for example, in Lunardi [22].

We further remark that resolvent estimates for elliptic finite-element operators are of great significance for applications of operator theory to the analysis of both spatially semidiscrete and fully discrete approximations to parabolic PDE problems (cf., e.g., [7]).

For an earlier work related to this subject we refer (in the chronological order) to Fujii [18], Schatz et al. [28], Nitsche and Wheeler [24], Wahlbin [36], Thomée and Wahlbin [33], Rannacher [26], Chen [10], Crouzeix et al. [13], Palencia [25], Bakaev et al. [6], Schatz et al. [29], Thomée and Wahlbin [34], Bakaev [4], Crouzeix and Thomée [14], and Bakaev et al. [7] (for brief discussions of the above work, see, e.g., [4, 29]).

Let $\Omega = \operatorname{Int}\Omega$ be a convex bounded domain in $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\Omega$. For simplicity we assume $\Omega$ to be of class $\mathscr{C}^\infty$, meaning that in the case $d = 1$, $\Omega$ is an open interval of the real axis. We introduce a linear operator $A$ by

$$A = -\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}, \tag{1.1}$$

with homogeneous Dirichlet boundary conditions on $\partial\Omega$. The operator $A$ will be considered in complex Lebesgue, Sobolev, and Hölder spaces on $\Omega$ (or on $\overline{\Omega}$), for which we use the standard notation $L_p$, $W_p^l$, and $\mathscr{C}^\xi$, respectively. The respective norms will be denoted by $\|\cdot\|_p$, $\|\cdot\|_{p;l}$, and $|\cdot|_\xi$. In this work it will be convenient to identify denoted differently but, at the same time, equivalent norms, although they are usually intended for use in different underlying spaces. In such a way, we will often write $|v|_0$ instead of $\|v\|_\infty$ and $|v|_l$ instead of $\|v\|_{\infty;l}$, even for measuring functions $v$ in $L_\infty$ and $W_\infty^l$, respectively. Let further $\dot{\mathscr{C}}^0 = \{v \in \mathscr{C}^0 : v = 0 \text{ on } \partial\Omega\}$, $\dot{\mathscr{C}}^\xi = \mathscr{C}^\xi \cap \dot{\mathscr{C}}^0$, and $\dot{W}_\infty^l = W_\infty^l \cap \dot{\mathscr{C}}^0$. Given two function spaces $E_1$ and $E_2$ (among those introduced above), if $B : E_1 \to E_2$ is a linear bounded operator, the symbol $|\|B\||_{E_1 \to E_2}$ will stand for the corresponding operator norm. Below, in connection with the finite-element space, we will also need to use a special operator norm which will be specified additionally. In what follows we will sometimes work with functions defined on domains different from $\Omega$ (or $\overline{\Omega}$). We therefore denote by $\|\cdot\|_{p;\mathfrak{D}}$, $\|\cdot\|_{p;l;\mathfrak{D}}$, and $|\cdot|_{\xi;\mathfrak{D}}$ the norms in the spaces $L_p(\mathfrak{D})$, $W_p^l(\mathfrak{D})$, and $\mathscr{C}^\xi(\overline{\mathfrak{D}})$, respectively (identifying, as above, equivalent norms). For our subsequent needs we also denote $(v, w)_{\mathfrak{D}} = \int_{\mathfrak{D}} v\overline{w}\,dx$, and we will write $(\cdot,\cdot)$ instead of $(\cdot,\cdot)_\Omega$.

As already mentioned above, for elliptic partial differential operators, there are known resolvent estimates in different function spaces which allow one to show that the above operator $A$ generates a holomorphic semigroup $e^{-tA}$ in these spaces. Such estimates are usually stated under quite general restrictions on the operator in question and, as a rule, the sector of analyticity of the semigroup $e^{-tA}$ is not specified. For our more concrete situation, it is possible to derive refined estimates, which are given below. Moreover, we generalize these results and present as well related estimates in pairs of spaces.

For finite-element versions of elliptic operators, resolvent estimates can be easily derived in $L_2$-norm, but it is not a simple problem to show them in the case of $L_p$-norm, $p \neq 2$. Note that if one has a suitable estimate in $L_\infty$-norm, by duality and interpolation it immediately extends to the whole scale of Lebesgue norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. It is worth mentioning that the problem in the case of $L_\infty$-norm is brought to an end at least for second-order operators with real-valued sufficiently smooth coefficients due to [4] (the case with at least quadratic elements) and [7] (the case with linear elements) in the sense that these results yield a uniform resolvent estimate in $L_\infty$-norm which is valid in any closed sector outside the real positive semiaxis. We are not however informed of any estimates in Hölder norms or estimates involving two different norms. In particular, for the time being, no uniform estimate of the gradient is known in $L_\infty$-norm. In what follows we show such estimates in the case of finite-element discretization with piecewise linear test functions. As concerns the case with higher-degree elements, it seems to be simpler for analysis for the reason that the $L_\infty$-norm of the Ritz projection is uniformly bounded in this situation (unlike in the case with linear elements; cf. [30]).

Throughout this work we will denote by $C$ and $c$ generic constants, subject to $C \ge 0$ and $c > 0$, whose sizes will be unessential for our subsequent analysis.

Let now $h \in (0, h_0]$, $h_0 > 0$, be a small parameter and let $\mathcal{T}_h = \{\tau_j\}_{j=1}^{J_h}$ be a triangulation of $\Omega_h = \mathrm{Int}(\cup_j \overline{\tau_j}) \subset \Omega$ into mutually disjoint open face-to-face simplices $\tau_j$. It will be convenient to take $h = \max_j \operatorname{diam} \tau_j$. We assume that the vertices of simplices $\tau_j$ which belong to $\partial\Omega_h$ lie also on $\partial\Omega$ and that the family $\{\mathcal{T}_h\}$ of triangulations is globally quasiuniform in the sense that $\min_j \operatorname{vol}(\tau_j) \ge ch^d$. It follows from our assumptions that $\operatorname{dist}(x, \partial\Omega) \le Ch^2$ for $x \in \partial\Omega_h$. Let further $S_h$ be the finite-dimensional space of all continuous complex-valued piecewise linear functions, associated with $\mathcal{T}_h$, that vanish outside $\Omega_h$. To the operator $A$ we associate its finite-element version $A_h : S_h \to S_h$ by

$$(A_h \psi, \chi) = (\nabla \psi, \nabla \chi) \quad \forall \psi, \chi \in S_h. \tag{1.2}$$

If we consider the parabolic initial boundary value problem

$$u_t + Au = 0, \quad t > 0, \qquad u(0) = v, \tag{1.3}$$

associated to the operator $A$, its solution is given, with the aid of the semigroup $e^{-tA}$, by $u(t) = e^{-tA}v$. Then a spatial semidiscrete approximation to $u(t)$ may be taken as $u_h(t) = e^{-tA_h}v_h$, with $v_h$ a suitable approximation to $v$ in (1.3). For any $h \in (0, h_0]$, the operator $A_h$ is bounded, which yields that the exponential function $e^{-tA_h}$ is a bounded operator as well (for any fixed $h \in (0, h_0]$ and $t > 0$). The known resolvent estimates allow one to show that $e^{-tA_h}$ is in fact uniformly bounded with respect to $h \in (0, h_0]$ and $t \ge 0$, in the operator norms of the corresponding function spaces. This fact is most essential for the analysis of stability and convergence of spatial semidiscrete approximations to (1.3). Moreover, for one-step methods approximating (1.3) both in space and in time

$$U^n = r(-kA_h)^n v_h, \tag{1.4}$$

where $r(z)$ is a rational function (it just specifies the method), the stability of $U^n$ and its convergence to the solution of (1.3) can be shown by making use of the same resolvent estimates for the operator $A_h$ (cf., e.g., Bakaev [3] or Thomée [32]). It is worth mentioning that, for the analysis of fully discrete approximations based on the application of A($\varphi$)-stable methods, it is essential to use resolvent estimates of the operator $A_h$ that would be valid outside any closed sector around the real positive semiaxis; just such estimates were found in $L_\infty$-norm in [4, 7]. Using duality and interpolation arguments, it is possible to extend the results of [4, 7] to the whole scale $L_p$, $1 \le p \le \infty$, of Lebesgue norms. In this work we show resolvent estimates of $A_h$ in (Lebesgue, Lebesgue), (Hölder, Lebesgue), and (Hölder, Hölder) pairs of norms; all of them are valid outside any sector containing the real positive semiaxis. In particular, a uniform resolvent estimate in $L_\infty$-norm for the gradient can be obtained as a direct consequence of Theorem 3.6 below. Our results can be used, for example, for showing the stability and convergence of $U^n$ in (1.4) in the corresponding pairs of norms. For further possible applications of the below estimates we can refer to [5].

Our main results are collected in the following two sections. In Section 2 we show resolvent estimates in pairs of spaces for the operator $A$. These assertions are further applied in Section 3 in order to obtain analogous estimates for the finite-element operator $A_h$. All the restrictions stated above are assumed to be in force throughout this work. The below technical tools are similar to those used in [4, 7]; they are based on comparing the discrete resolvent to the continuous one and on making use of estimates for the continuous problem.

**2. Resolvent estimates of the operator $A$.** In this section we present, as mentioned above, resolvent estimates for the elliptic differential operator $A$ in pairs of function spaces. It will be convenient, given $\varphi \in (0,\pi/2)$, to denote

$$\Sigma_\varphi = \{\lambda : \lambda \neq 0,\ |\arg\lambda| \leq \varphi\} \cup \{0\}. \tag{2.1}$$

We start by showing some auxiliary results, the first of which will be related to real interpolation theory.

Let $(\dot{\mathscr{C}}^0, \dot{W}^1_\infty)_{\theta,\infty}$, $0 < \theta < 1$, be the interpolation space between $\dot{\mathscr{C}}^0$ and $\dot{W}^1_\infty$ constructed by the $K$-method (cf., e.g., Triebel [35, pages 23-24]). The norm in $(\dot{\mathscr{C}}^0, \dot{W}^1_\infty)_{\theta,\infty}$ is given by

$$\|v\|_{(\dot{\mathscr{C}}^0,\dot{W}^1_\infty)_{\theta,\infty}} = \sup_{s>0} s^{-\theta} \inf_{\substack{v=w_0(s)+w_1(s)\\ w_0(s)\in\dot{\mathscr{C}}^0,\ w_1(s)\in\dot{W}^1_\infty}} \Big( |w_0(s)|_0 + s\|w_1(s)\|_{\infty;1} \Big). \tag{2.2}$$

**Lemma 2.1.** *For any fixed $0 < \alpha < 1$,*

$$\|v\|_{(\dot{\mathscr{C}}^0,\dot{W}^1_\infty)_{\alpha,\infty}} \leq C|v|_\alpha \quad \forall v \in \dot{\mathscr{C}}^\alpha. \tag{2.3}$$

This result can be shown by direct use of the definition of $\|v\|_{(\dot{\mathscr{C}}^0,\dot{W}^1_\infty)_{\alpha,\infty}}$ and by application of standard methods in real interpolation theory. We also note that the same estimate with $\dot{\mathscr{C}}^1$ in place of $\dot{W}^1_\infty$ follows from Lunardi [21, Proposition 2.6 and Remark 2.9], but the norm in $(\dot{\mathscr{C}}^0, \dot{W}^1_\infty)_{\alpha,\infty}$ is dominated by that in $(\dot{\mathscr{C}}^0, \dot{\mathscr{C}}^1)_{\alpha,\infty}$.

Let further $G(s;x,y)$, $s > 0$, be the Green's function for the operator $(sI + A)^{-1}$. We will now obtain some pointwise estimates which will be quite useful in the sequel.

**Lemma 2.2.** *For any fixed $0 \leq \alpha < 2/d$ ($0 \leq \alpha \leq 1$ if $d = 1$),*

$$|G(s;x,y)| \leq C\frac{s^{d\alpha/2-1}}{2d^{-1}-\alpha}\frac{\exp\left(-c\sqrt{s}|x-y|\right)}{|x-y|^{d(1-\alpha)}} + C(s+1)^{-1} \quad \forall x,y \in \Omega,\ s > 0. \tag{2.4}$$

*Furthermore, if $\alpha$ is restricted by $0 \leq \alpha < 2/(d+1)$, then for all $x,y \in \Omega$ and $s > 0$,*

$$|\nabla_x G(s;x,y)| \leq C\frac{s^{(d+1)\alpha/2-1}}{2(d+1)^{-1}-\alpha}\frac{\exp\left(-c\sqrt{s}|x-y|\right)}{|x-y|^{(d+1)(1-\alpha)}} + C(s+1)^{-1}. \tag{2.5}$$

**Proof.** We denote for short $\rho = |x-y|$. For $\mathfrak{G}(t;x,y)$, the Green's function of the related parabolic problem (1.3), we have (cf. [17])

$$|\mathfrak{G}(t;x,y)| \le C(t+\rho^2)^{-d/2}\exp\left(-\frac{c\rho^2}{t}\right)+C\exp(-ct), \quad t>0. \tag{2.6}$$

Now, in order to show (2.4), we use the representation

$$G(s;x,y) = \int_0^\infty e^{-st}\mathfrak{G}(t;x,y)\,dt, \tag{2.7}$$

which yields with the aid of (2.6), since $t+\rho^2 \ge t^\alpha \rho^{2(1-\alpha)}$,

$$\begin{aligned} |G(s;x,y)| &\le C\int_0^\infty (t+\rho^2)^{-d/2}\exp\left(-st-\frac{c\rho^2}{t}\right)dt+\frac{C}{s+1} \\ &\le Cs^{d\alpha/2-1}\rho^{-(1-\alpha)d}\int_0^\infty t^{-d\alpha/2}\exp\left(-t-\frac{cs\rho^2}{t}\right)dt+\frac{C}{s+1}. \end{aligned} \tag{2.8}$$

The integral on the right-hand side of the last estimate converges for $0 \le \alpha < 2/d$, and we see that (2.4) follows by using the evident estimate

$$\exp\left(-\frac{t}{2}-\frac{cs\rho^2}{t}\right) \le \exp\left(-c\sqrt{s}\rho\right). \tag{2.9}$$

The second stated inequality (2.5) can be shown similarly by applying the estimate (see, e.g., [17])

$$|\nabla_x\mathfrak{G}(t;x,y)| \le C(t+\rho^2)^{-(d+1)/2}\exp\left(-\frac{c\rho^2}{t}\right)+C\exp(-ct), \quad t>0. \tag{2.10}$$

(Note that (2.5) is formally obtained by comparing (2.10) to (2.6) and substituting $(d+1)$ for $d$ into (2.4).) □

Now we turn to showing resolvent estimates themselves. The first result is not original but it is needed for our subsequent purposes.

**Theorem 2.3.** *For any fixed* $\varphi \in (0,\pi/2)$,

$$|(\lambda I-A)^{-1}v|_j \le C(1+|\lambda|)^{-1+j/2}|v|_0 \quad \forall\lambda \notin \operatorname{Int}\Sigma_\varphi,\ j=0,1,\ v\in L_\infty, \tag{2.11}$$

$$|A(\lambda I-A)^{-1}v|_0 \le C(1+|\lambda|)^{-1/2}|v|_1 \quad \forall\lambda \notin \operatorname{Int}\Sigma_\varphi,\ v\in \dot{W}^1_\infty. \tag{2.12}$$

In principle, (2.11) is an almost direct consequence of a quite general result of Stewart [31] which has been sharpened in [7]. In this work we accept in fact just the same starting assumptions as in [7] (which are more restrictive than the assumptions in [31]) and we take the resolvent estimate (2.11) itself actually in the same form as it is stated in [7]. The minor difference is however that (2.11) is stated in [7] only for $v \in \mathscr{C}$, but, since the resolvent is an integral operator, the result is clearly still valid for $v \in L_\infty$. For a proof of (2.12), see [7].

In what follows we will also give analogues of (2.11) and (2.12) in $L_p$-norm. Moreover, an analogue of (2.11) with $j=0$ can be stated right now.

**Theorem 2.4.** *For any fixed $\varphi \in (0,\pi/2)$,*

$$\|(\lambda I - A)^{-1} v\|_p \le C(1+|\lambda|)^{-1}\|v\|_p \quad \forall 1 \le p \le \infty,\ \lambda \notin \operatorname{Int}\Sigma_\varphi,\ v \in L_p. \tag{2.13}$$

In fact, (2.13) with $p = 1$ follows from (2.11) with $j = 0$ by duality and further the result extends, by interpolation, to the general case $1 \le p \le \infty$.

We will also need to use below some estimates in Hölder norms.

**Theorem 2.5.** *For any fixed $\varphi \in (0,\pi/2)$ and for $\alpha = 0,1$,*

$$|(\lambda I - A)^{-1} v|_{1+\alpha} \le C(1+|\lambda|)^{-1+\alpha/2}|v|_1 \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ v \in \dot{W}^1_\infty. \tag{2.14}$$

**Proof.** We will first show that the assertion is true if $|\lambda| \ge R$, with $R > 0$ sufficiently large. We start by taking $\alpha = 0$. Given $\lambda \notin \operatorname{Int}\Sigma_\varphi$, we let $\mu = -|\lambda|$. Applying now [12, Theorem 3.2] with $p \to \infty$ and fixing $R > 0$ sufficiently large, we find that (2.14) with $\alpha = 0$ holds with $\mu$ written for $\lambda$, for $|\mu| \ge R$. We therefore have, since $|\mu| = |\lambda|$,

$$|(\mu I - A)^{-1} v|_1 \le C(1+|\mu|)^{-1}|v|_1 = C(1+|\lambda|)^{-1}|v|_1 \quad \forall |\lambda| \ge R. \tag{2.15}$$

At the same time, it follows from (2.11) with $j = 1$ and (2.12) that, for $|\lambda| \ge R$,

$$|(\mu I - A)^{-1} A(\lambda I - A)^{-1} v|_1 \le C(1+|\mu|)^{-1/2}|A(\lambda I - A)^{-1} v|_0 \le C(1+|\lambda|)^{-1}|v|_1. \tag{2.16}$$

Next, in view of the identity

$$\lambda(\lambda I - A)^{-1} = I + A(\lambda I - A)^{-1}, \tag{2.17}$$

a simple calculation shows, since $|\mu| = |\lambda|$, that

$$\begin{aligned} |(\mu I - A)^{-1}\mu(\lambda I - A)^{-1} v|_1 &= |(\mu I - A)^{-1}\lambda(\lambda I - A)^{-1} v|_1 \\ &\le |(\mu I - A)^{-1} v|_1 + |(\mu I - A)^{-1} A(\lambda I - A)^{-1} v|_1. \end{aligned} \tag{2.18}$$

Using (2.15), (2.16), (2.18), the identity

$$(\lambda I - A)^{-1} = (\mu I - A)^{-1}(\mu I - A)(\lambda I - A)^{-1}, \tag{2.19}$$

and the fact that $|\mu| = |\lambda|$, we thus obtain for $|\lambda| \ge R$,

$$\begin{aligned} |(\lambda I - A)^{-1} v|_1 &\le |(\mu I - A)^{-1}\mu(\lambda I - A)^{-1} v|_1 + |(\mu I - A)^{-1} A(\lambda I - A)^{-1} v|_1 \\ &\le C(1+|\lambda|)^{-1}|v|_1. \end{aligned} \tag{2.20}$$

This yields that (2.14) with $\alpha = 0$ holds for $|\lambda| \ge R$.

Applying further [12, Theorem 3.2] with $s = 1$ and using again (2.19), we get

$$\begin{aligned} |(\lambda I - A)^{-1} v|_2 &\le |(\mu I - A)^{-1}(\mu - \lambda)(\lambda I - A)^{-1} v|_2 + |(\mu I - A)^{-1} v|_2 \\ &\le C(1+|\mu|)^{-1/2}\Big(|(\mu - \lambda)(\lambda I - A)^{-1} v|_1 + |v|_1\Big), \end{aligned} \tag{2.21}$$

and with the aid of estimate (2.14) with $\alpha = 0$ (already proved), we see that in fact (2.14) holds also with $\alpha = 1$ if $\lambda \geq R$ and $R > 0$ is sufficiently large.

It remains to show that (2.14) is also in force for $\lambda \leq R$, with any fixed $R \geq 0$. Note that it would suffice in fact to prove that for $\alpha = 0, 1$,

$$|(\lambda I - A)^{-1} v|_{1+\alpha} \leq C|v|_1 \quad \forall \lambda \notin \operatorname{Int} \Sigma_\varphi,\ |\lambda| \leq R,\ v \in \dot{W}^1_\infty. \tag{2.22}$$

For $\alpha = 0$, the last estimate immediately follows by (2.11) with $j = 1$ since $|v|_0 \leq C|v|_1$. In order to show that (2.22) holds as well for $\alpha = 1$, we write, using well-known inequalities between Hölder norms, the estimate $|w|_{5/2} \leq C|Aw|_{1/2}$ (cf. [2]), and the above identity (2.17),

$$\begin{aligned} |(\lambda I - A)^{-1} v|_2 &\leq C|(\lambda I - A)^{-1} v|_{5/2} \leq C|A(\lambda I - A)^{-1} v|_{1/2} \\ &\leq C\Big(|v|_1 + |\lambda|\,|(\lambda I - A)^{-1} v|_1\Big). \end{aligned} \tag{2.23}$$

Now combining this and (2.22) with $\alpha = 0$ (already shown) leads easily to (2.22) with $\alpha = 1$.

This completes the proof. □

We are now ready to show resolvent estimates in pairs of Lebesgue spaces. We will state first a result for the particular case when $\lambda = -s$, $s > 0$, and when one of the spaces is $L_\infty$.

**Theorem 2.6.** *For any fixed $d/2 < q \leq \infty$ (for $1 \leq q \leq \infty$ if $d = 1$),*

$$\|(sI + A)^{-1} v\|_\infty \leq C(s+1)^{d/(2q)-1} \|v\|_q \quad \forall s \geq 0,\ v \in L_q. \tag{2.24}$$

**Proof.** A suitable argument somewhat changes in details for $d = 1$ and for $d \geq 2$, and one has to distinguish between these cases. The case $d = 1$ is however simpler and we will further concentrate on the situation when $d \geq 2$.

Assuming therefore that $d \geq 2$, we select a fixed $\alpha$ such that $q > 1/\alpha > d/2$. Defining further $q'$ by $1/q' := 1 - 1/q$ and noting that $0 < \alpha < 2/d$ and $d - dq'(1-\alpha) > 0$, we apply (2.4) to obtain for all $v \in L_q$ with $\|v\|_q = 1$ (as above we denote $\rho = |x - y|$), if $s > 1$,

$$\begin{aligned} \|(sI + A)^{-1} v\|_\infty &\leq \sup_{x \in \Omega} \left( \int_\Omega |G(s; x, y)|^{q'} dy \right)^{1/q'} \\ &\leq C s^{d\alpha/2 - 1} \sup_{x \in \Omega} \left( \int_\Omega \frac{\exp(-cq'\sqrt{s}\rho)}{\rho^{dq'(1-\alpha)}} dy \right)^{1/q'} \\ &\leq C s^{d\alpha/2 - 1} \left( \int_0^\infty \exp(-cq'\sqrt{s}z) z^{d-1-dq'(1-\alpha)} dz \right)^{1/q'} \\ &\leq C s^{d/(2q)-1}, \end{aligned} \tag{2.25}$$

which shows the claim at least for $s > 1$.

Thus it remains to prove (2.24) for $0 \le s \le 1$. Clearly, it would suffice to show instead that

$$\big|\big|(sI+A)^{-1}v\big|\big|_\infty \le C\|v\|_q \quad \text{for } 0 \le s \le 1. \tag{2.26}$$

The last estimate follows however easily from (2.17), (2.13) with $p = \infty$ and $\lambda = -s$, and the inequality $\|A^{-1}v\|_\infty \le C\|v\|_q$ (valid for $q > d/2$), with the aid of

$$\big|\big|(sI+A)^{-1}v\big|\big|_\infty = \big|\big|A(sI+A)^{-1}A^{-1}v\big|\big|_\infty \le C\big|\big|A^{-1}v\big|\big|_\infty. \tag{2.27}$$

This completes the proof. □

Now we can obtain sectorial estimates in $(L_p, L_q)$ pairs.

**Theorem 2.7.** *For any fixed* $\varphi \in (0, \pi/2)$ *and* $1 \le q \le p \le \infty$ *such that* $1/q - 1/p < 2/d$ *(for any fixed* $1 \le q \le p \le \infty$ *if* $d = 1$*),*

$$\big|\big|(\lambda I - A)^{-1}v\big|\big|_p \le C(1+|\lambda|)^{(d/2)(1/q-1/p)-1}\|v\|_q \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ v \in L_q. \tag{2.28}$$

**Proof.** As above, we will restrict our consideration to the case $d \ge 2$.

Assuming first that $1/q < 2/d$, by (2.17), (2.13) with $p = \infty$, and (2.24), we find, for all $\lambda \notin \operatorname{Int}\Sigma_\varphi$,

$$\begin{aligned}
\big|\big|(\lambda I - A)^{-1}v\big|\big|_\infty &= \big|\big|(|\lambda| I + A)(\lambda I - A)^{-1}(|\lambda| I + A)^{-1}v\big|\big|_\infty \\
&\le \Big(1 + 2|\lambda|\,\big|\big|\big|(\lambda I - A)^{-1}\big|\big|\big|_{L_\infty \to L_\infty}\Big)\big|\big|(|\lambda| I + A)^{-1}v\big|\big|_\infty \\
&\le C\Big|\Big|(|\lambda| I + A)^{-1}v\Big|\Big|_\infty \le C(1+|\lambda|)^{d/(2q)-1}\|v\|_q,
\end{aligned} \tag{2.29}$$

which shows the claim for $p = \infty$ and $1/q < 2/d$. Observe also that, by duality, this yields that (2.28) holds as well for $q = 1$ and $p \ge 1$ such that $1 - 1/p < 2/d$.

We now consider the general case with arbitrary $p$ and $q$ such that $1/q - 1/p < 2/d$. Define $q_0$ and $p_1$ by

$$\frac{1}{q_0} := \frac{1}{q} - \frac{1}{p} =: 1 - \frac{1}{p_1}. \tag{2.30}$$

Noting that $1/q_0 = 1 - 1/p_1 < 2/d$, we see that in view of the above reasonings, (2.28) holds for $p = \infty$, $q = q_0$ and for $p = p_1$, $q = 1$. For arbitrary $p$ and $q$, with $1/q - 1/p < 2/d$, the desired result is thus obtained by interpolation. □

Further we will obtain resolvent estimates in (Hölder, Lebesgue) pairs of spaces. As above, we will consider first the particular case when $\lambda = -s$, $s \ge 0$.

**Theorem 2.8.** *For any fixed* $0 \le \xi \le 1$ *and* $q \in (d/(2-\xi), \infty]$,

$$\big|(sI+A)^{-1}v\big|_\xi \le C(s+1)^{\xi/2+d/(2q)-1}\|v\|_q \quad \forall s \ge 0,\ v \in L_q. \tag{2.31}$$

**Proof.** Note that, in the case $\xi = 1$ and $q > d$, the result is obtained by using (2.5) and the estimate $|A^{-1}v|_1 \le C\|v\|_q$ (cf. the proof of Theorem 2.6). In particular, (2.31) holds for $\xi = 1$ and $q = \infty$. Since, by (2.13) with $p = \infty$, it holds as well for $\xi = 0$ and $q = \infty$, by interpolation, we have shown the result in fact for $0 \le \xi \le 1$ and $q = \infty$.

We turn to the general case. Let $\xi \in [0,1]$ and $q \in (d/(2-\xi),\infty]$ be fixed. We can write, with some $\varepsilon > 0$,

$$\frac{1}{q} = \frac{2-\xi}{d} - \varepsilon. \tag{2.32}$$

Assuming that $\varepsilon < 1/d$ and defining then $q_0$ and $q_1$ by

$$\frac{1}{q_0} := \frac{2}{d} - \varepsilon, \qquad \frac{1}{q_1} := \frac{1}{d} - \varepsilon, \tag{2.33}$$

we conclude that (2.31) holds for $\xi = 0$ and $q = q_0$ by (2.24) with $q = q_0$, and it will hold, as already shown, for $\xi = 1$ and $q = q_1$ since $q_1 > d$. The desired result in the general case will thus follow by interpolation.

If it happens that, in (2.32), $\varepsilon \ge 1/d$, which means that $q$ is sufficiently large, the result nevertheless will follow by interpolation since we have already shown that it is in force for $q = \infty$ and for $q$ sufficiently close to $d/(2-\xi)$. □

Sectorial estimates in $(\mathscr{C}^\xi, L_q)$ pairs will then be obtained as follows.

**Theorem 2.9.** *For any fixed $\varphi \in (0,\pi/2)$, $0 \le \xi \le 1$, and $q \in (d/(2-\xi),\infty]$,*

$$|(\lambda I - A)^{-1} v|_\xi \le C(1+|\lambda|)^{\xi/2+d/(2q)-1}\|v\|_q \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ v \in L_q. \tag{2.34}$$

**Proof.** It follows from (2.17) and (2.13) with $p = q$ that for all $\lambda \notin \operatorname{Int}\Sigma_\varphi$,

$$\|(|\lambda| I + A)(\lambda I - A)^{-1} v\|_q \le \|v\|_q + 2|\lambda|\,\|(\lambda I - A)^{-1} v\|_q \le C\|v\|_q. \tag{2.35}$$

Using this thus yields

$$\begin{aligned} |(\lambda I - A)^{-1} v|_\xi &\le \left|\left|\left|(|\lambda| I + A)^{-1}\right|\right|\right|_{L_q \to \mathscr{C}^\xi} \|(|\lambda| I + A)(\lambda I - A)^{-1} v\|_q \\ &\le C\left|\left|\left|(|\lambda| I + A)^{-1}\right|\right|\right|_{L_q \to \mathscr{C}^\xi} \|v\|_q. \end{aligned} \tag{2.36}$$

It remains to apply (2.31) with $s = |\lambda|$. □

We next show analogues of (2.11) with $j = 1$ and of (2.12), in $L_p$-norm.

**Theorem 2.10.** *For any fixed $\varphi \in (0,\pi/2)$ and $1 \le q \le p \le \infty$ such that $1/q - 1/p < 1/d$, for all $\lambda \notin \operatorname{Int}\Sigma_\varphi$ and $v \in L_q$,*

$$\|(\lambda I - A)^{-1} v\|_{p;1} \le C(1+|\lambda|)^{(d/2)(1/q-1/p)-1/2}\|v\|_q. \tag{2.37}$$

**Proof.** We can write, with some $\varepsilon \in (0,1/d]$,

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{d} - \varepsilon. \tag{2.38}$$

Let further $q_1$ and $p_0$ be defined by

$$\frac{1}{q_1} := \frac{1}{d} - \varepsilon, \qquad 1 - \frac{1}{p_0} := \frac{1}{d} - \varepsilon. \tag{2.39}$$

By (2.31) with $\xi = 1$, we have, since $1/q_1 < 1/d$,

$$\|(sI + A)^{-1} v\|_{\infty;1} \le C(s+1)^{d/(2q_1)-1/2} \|v\|_{q_1} \quad \forall s \ge 0,\ v \in L_{q_1}. \tag{2.40}$$

We now define $q_0$ by $1/q_0 := 1 - 1/p_0$ (note that $1/q_0 < 1/d$) and select some $\alpha \in ((1/(d+1))(1 + d/q_0), 2/(d+1))$. Then

$$(\nabla(sI + A)^{-1} v, \psi) = \int_\Omega \int_\Omega \overline{\psi(x)} \nabla_x G(s; x, y) v(y)\, dy\, dx, \tag{2.41}$$

which yields, using (2.5) and Hölder's inequality, for all $s > 0$, $v \in L_1$, and $\psi \in L_{q_0}$,

$$\begin{aligned} |(\nabla(sI + A)^{-1} v, \psi)| &\le \|v\|_1 \sup_{y\in\Omega} \int_\Omega |\nabla_x G(s; x, y)| |\psi(x)| \, dx \\ &\le C s^{d/(2q_0)-1/2} \|v\|_1 \|\psi\|_{q_0}. \end{aligned} \tag{2.42}$$

This implies in its turn

$$\|(sI + A)^{-1} v\|_{p_0;1} \le C(s+1)^{(d/2)(1-1/p_0)-1/2} \|v\|_1 \quad \forall s \ge 1,\ v \in L_1. \tag{2.43}$$

As above (cf. the argument used in the proof of Theorem 2.6), the restriction $s \ge 1$ can be replaced by $s \ge 0$.

By interpolation, we obtain from (2.40) and (2.43),

$$\|(sI + A)^{-1} v\|_{p;1} \le C(s+1)^{(d/2)(1/q-1/p)-1/2} \|v\|_q \quad \forall s \ge 0,\ v \in L_q. \tag{2.44}$$

Using this finally yields for all $\lambda \notin \operatorname{Int}\Sigma_\varphi$ and $v \in L_q$, with the aid of (2.17),

$$\begin{aligned} \|(\lambda I - A)^{-1} v\|_{p;1} &= \|(|\lambda| I + A)^{-1} (|\lambda| I + A)(\lambda I - A)^{-1} v\|_{p;1} \\ &\le C(1 + |\lambda|)^{(d/2)(1/q-1/p)-1/2} \|(|\lambda| I + A)(\lambda I - A)^{-1} v\|_q \\ &\le C(1 + |\lambda|)^{(d/2)(1/q-1/p)-1/2} \left( |\lambda| \|(\lambda I - A)^{-1} v\|_q + \|v\|_q \right), \end{aligned} \tag{2.45}$$

whence the claim follows by (2.13) with $q$ substituted for $p$. □

**Theorem 2.11.** *For any fixed* $\varphi \in (0, \pi/2)$,

$$\|A(\lambda I - A)^{-1} v\|_p \le C(1 + |\lambda|)^{-1/2} \|v\|_{p;1} \quad \forall 1 \le p \le \infty,\ \lambda \notin \operatorname{Int}\Sigma_\varphi,\ v \in W_p^1. \tag{2.46}$$

**Proof.** For all $\psi \in L_{p'}$ with $1/p + 1/p' = 1$, we can write

$$(A(\lambda I - A)^{-1} v, \psi) = (\nabla v, \nabla(\bar{\lambda} I - A)^{-1} \psi), \tag{2.47}$$

whence, in view of (2.37) with $p = q$, in which we substitute $p'$ for $p$,

$$||(A(\lambda I - A)^{-1}v,\psi)|| \le C\|v\|_{p;1}(1+|\lambda|)^{-1/2}\|\psi\|_{p'}, \tag{2.48}$$

which shows the claim. □

Now we give some generalizations of our above results which will be expressed in terms of estimates involving the iterated resolvent operator.

**Theorem 2.12.** *Let $\xi \in [0,1]$, $q \in [1,\infty]$, and $m \in \mathbb{N}\cup\{0\}$ be such that*

$$\frac{1}{q} < \frac{2m+2-\xi}{d}. \tag{2.49}$$

*Then for any fixed $\varphi \in (0,\pi/2)$,*

$$|(\lambda I - A)^{-(m+1)}v|_{\xi} \le C(1+|\lambda|)^{\xi/2+d/(2q)-(m+1)}\|v\|_q \quad \forall\lambda \notin \operatorname{Int}\Sigma_{\varphi},\ v \in L_q. \tag{2.50}$$

**Proof.** Let $q_0, q_1, \dots, q_{m-1}$ be chosen such that

$$q \le q_{m-1} \le \cdots \le q_0 \le \infty, \tag{2.51}$$

$$\begin{gathered}\frac{1}{q_0} < \frac{2-\xi}{d},\\ \frac{1}{q_1} - \frac{1}{q_0} < \frac{2}{d},\\ \vdots\\ \frac{1}{q} - \frac{1}{q_{m-1}} < \frac{2}{d}.\end{gathered} \tag{2.52}$$

Then applying (2.34), with $q_0$ substituted for $q$, and using further repeatedly (2.28), we get

$$\begin{aligned}|(\lambda I - A)^{-(m+1)}v|_{\xi} &\le C(1+|\lambda|)^{\xi/2+d/(2q_0)-1}||(\lambda I - A)^{-m}v||_{q_0}\\ &\le C(1+|\lambda|)^{\xi/2+d/(2q_1)-2}||(\lambda I - A)^{-(m-1)}v||_{q_1}\\ &\le \cdots \le C(1+|\lambda|)^{\xi/2+d/(2q)-(m+1)}\|v\|_q \quad \forall\lambda \notin \operatorname{Int}\Sigma_{\varphi},\ v \in L_q,\end{aligned} \tag{2.53}$$

which is the desired result. □

**Theorem 2.13.** *Let $m \in \mathbb{N}\cup\{0\}$ and $1 \le q \le p \le \infty$ be such that*

$$\frac{1}{q} - \frac{1}{p} < \frac{2m}{d}. \tag{2.54}$$

*Then for any fixed $\varphi \in (0,\pi/2)$,*

$$||(\lambda I - A)^{-m}v||_p \le C(1+|\lambda|)^{(d/2)(1/q-1/p)-m}\|v\|_q \quad \forall\lambda \notin \operatorname{Int}\Sigma_{\varphi},\ v \in L_q. \tag{2.55}$$

**Theorem 2.14.** *Let $m \in \mathbb{N} \cup \{0\}$ and $1 \le q \le p \le \infty$ be such that*

$$\frac{1}{q} - \frac{1}{p} < \frac{2m+1}{d}. \tag{2.56}$$

*Then for any fixed $\varphi \in (0, \pi/2)$,*

$$\|(\lambda I - A)^{-(m+1)} v\|_{p;1} \le C(1+|\lambda|)^{(d/2)(1/q-1/p)-(m+1/2)} \|v\|_q \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ v \in L_q. \tag{2.57}$$

The proofs of Theorems 2.13 and 2.14 are similar to that of Theorem 2.12. They are based on using (2.28) and (2.37).

It remains to show resolvent estimates in pairs of Hölder spaces.

**Theorem 2.15.** *For any fixed $\varphi \in (0, \pi/2)$, $\xi \in [0,1]$, and $\eta \in [0, \xi]$,*

$$|(\lambda I - A)^{-1} v|_\xi \le C(1+|\lambda|)^{(\xi-\eta)/2-1} |v|_\eta \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ v \in \dot{\mathscr{C}}^\eta. \tag{2.58}$$

*In the case $\eta = 0$, the result is still valid for all $v \in L_\infty$, and in the case $\xi = \eta = 1$, it is valid for all $v \in \dot{W}^1_\infty$.*

**Proof.** In the case $\eta = 0$, (2.58) holds for all $v \in L_\infty$ at least if $\xi = 0$ or $\xi = 1$. This follows directly from (2.11). By interpolation, we get immediately for $\xi \in [0,1]$,

$$|(\lambda I - A)^{-1} v|_\xi \le C(1+|\lambda|)^{\xi/2-1} |v|_0 \quad \forall v \in L_\infty, \tag{2.59}$$

which shows the claim in the case $\eta = 0$.

Next note that the result also holds in the case $\xi = \eta = 1$, for all $v \in \dot{W}^1_\infty$, as follows from (2.14) with $\alpha = 0$. In particular, (2.58) with $\xi = \eta = 1$ is true for all $v \in \dot{\mathscr{C}}^1$.

Moreover, with this in mind and using (2.59) with $\xi = 0$ (which clearly can be considered for all $v \in \dot{\mathscr{C}}$), we get by interpolation (see, e.g., Triebel [35, Theorem 1.3.3]), for $\xi \in (0,1)$,

$$|(\lambda I - A)^{-1} v|_\xi \le C(1+|\lambda|)^{-1} \|v\|_{(\mathscr{C}, \dot{W}^1_\infty)_\xi} \quad \forall v \in \dot{\mathscr{C}}^\xi, \tag{2.60}$$

whence applying (2.3) yields, if $\xi \in (0,1)$,

$$|(\lambda I - A)^{-1} v|_\xi \le C(1+|\lambda|)^{-1} |v|_\xi \quad \forall v \in \dot{\mathscr{C}}^\xi. \tag{2.61}$$

The last result can also be thought of as a refined version of a well-known resolvent estimate in Hölder norms (cf. Campanato [8] and Cannarsa et al. [9]). Note that, in view of the above comment, (2.61) also holds for $\xi = 0$ and $\xi = 1$.

On the other hand, applying (2.14) with $\alpha = 0$ and (2.59) with $\xi = 1$ (which clearly can be considered for all $v \in \dot{\mathscr{C}}$) and using again an interpolation argument combined with (2.3), we get, for $0 \le \eta \le 1$,

$$|(\lambda I - A)^{-1} v|_1 \le C(1+|\lambda|)^{(1-\eta)/2-1} |v|_\eta \quad \forall v \in \dot{\mathscr{C}}^\eta. \tag{2.62}$$

(This is a direct consequence of (2.59) with $\xi = 1$ in the case $\eta = 0$ and of (2.14) with $\alpha = 0$ in the case $\eta = 1$.)

Finally, we get by interpolation from (2.61) and (2.62), for $\xi \in [0,1]$, $\eta \in [0,\xi]$,

$$|(\lambda I - A)^{-1} v|_{\xi} \le C(1+|\lambda|)^{(\xi-\eta)/2-1} |v|_{\eta} \quad \forall v \in \dot{\mathscr{C}}^{\eta}. \tag{2.63}$$

So the proof is complete. □

**3. Resolvent estimates of the operator** $A_h$**.** In this section, we present sectorial resolvent estimates for the finite-element operator $A_h$. These estimates will look similar to those involving the operator $A$. All of them will hold uniformly with respect to $h \in (0,h_0]$.

We start by recalling some facts used below and connected with the application of the finite-element method. First of all we state the well-known inverse property (cf. Ciarlet [11]), for $j = 0,1$ and $1 \le q \le p \le \infty$,

$$\|\chi\|_{p;j;\tau} \le Ch^{-j-d(1/q-1/p)} \|\chi\|_{q;0;\tau} \quad \text{for } h \in (0,h_0],\ \tau \in \mathscr{T}_h,\ \chi \in S_h. \tag{3.1}$$

Let further $P_h$ be the orthogonal (in the sense of $L_2$) projection onto $S_h$ for which, given a function $v$,

$$(P_h v, \chi) = (v, \chi) \quad \forall \chi \in S_h. \tag{3.2}$$

For our subsequent purposes, we state the uniform boundedness of $P_h$ in $L_p$-norm (cf. Descloux [15] and Douglas Jr. et al. [16]):

$$\|P_h v\|_p \le C\|v\|_{p;\Omega_h} \le C\|v\|_p \quad \forall 1 \le p \le \infty,\ v \in L_p. \tag{3.3}$$

Note that using (3.1), (3.3) with $p = \infty$, and well-known properties of the standard Lagrange interpolant $I_h$ yields (cf. [7])

$$|P_h v|_1 \le C|v|_{1;\Omega_h} \le C|v|_1 \quad \text{for } v \in \dot{W}^1_{\infty}. \tag{3.4}$$

Moreover, an interpolation argument (see, e.g., Triebel [35, Theorem 1.3.3]), applied to (3.4) and (3.3) with $p = \infty$ (the last one clearly holds for all $v \in \dot{\mathscr{C}}^0$), implies for any fixed $\xi \in [0,1)$, with the aid of (2.3) (for $\xi = 0$ this is in fact a direct consequence of (3.3) with $p = \infty$),

$$|P_h v|_{\xi} \le C|v|_{\xi} \quad \text{for } v \in \dot{\mathscr{C}}^{\xi}. \tag{3.5}$$

Note in passing that (3.4) is an extension of (3.5) to the case $v \in W^1_{\infty}$.

Next, let $R_h$ be the standard Ritz projection onto $S_h$ for which, given a function $v$,

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi) \quad \forall \chi \in S_h. \tag{3.6}$$

In what follows we will use the following stability estimate for $R_h$ in $W^1_p$-norm, for all $1 \le p \le \infty$ and $v \in \dot{W}^1_{\infty}$:

$$\|R_h v\|_{p;1} \le C\|v\|_{p;1;\Omega_h} + Ch|v|_{1;\Omega\setminus\Omega_h}. \tag{3.7}$$

For a proof of this in the case $p = \infty$, see [7]. Note however that the techniques in [7] can be slightly modified (using also the ideas developed in [27]) in order to obtain (3.7) as stated for $2 \le p \le \infty$.

Applying now (3.7) with $p = \infty$ to $(I_h v - v)$, we find for all $v \in \dot{W}^2_\infty$,

$$\begin{aligned} |R_h v - v|_{1;\Omega_h} &\le |R_h(v - I_h v)|_1 + |(I_h v - v)|_{1;\Omega_h} \\ &\le C|I_h v - v|_{1;\Omega_h} + Ch|v|_{1;\Omega\setminus\Omega_h} \le Ch|v|_2. \end{aligned} \tag{3.8}$$

Using the fact that

$$P_h v - R_h v = P_h(v - R_h v) \tag{3.9}$$

and taking (3.4) into account, this yields as well

$$|P_h v - R_h v|_1 \le C|R_h v - v|_{1;\Omega_h} \le Ch|v|_2 \quad \forall v \in \dot{W}^2_\infty. \tag{3.10}$$

Another estimate for $P_h - R_h$ will be contained in the following assertion.

**Lemma 3.1.** *For any fixed* $1 < p < \infty$,

$$\|(P_h - R_h)A^{-1}v\|_p \le Ch^2|v|_p \quad \forall v \in L_p. \tag{3.11}$$

**Proof.** Let throughout the proof $p$ and $q$ be fixed, subject to $1 < p, q < \infty$.

Applied to $(I - I_h)w$, (3.7) yields

$$\|R_h(I - I_h)w\|_{q;1;\Omega_h} \le C\|(I - I_h)w\|_{q;1;\Omega_h} + Ch|w|_{1;\Omega\setminus\Omega_h} \quad \forall w \in \dot{W}^1_\infty. \tag{3.12}$$

With the aid of this estimate we obtain, assuming $q$ sufficiently large and applying the inequality $|w|_1 \le C\|w\|_{q;2}$,

$$\|\nabla R_h(I - I_h)w\|_{q;\Omega_h} \le Ch\|w\|_{q;2} \quad \forall w \in \dot{W}^2_q. \tag{3.13}$$

Using this and the above argument, we get by elliptic regularity, with $q$ sufficiently large,

$$\|\nabla(R_h - I)A^{-1}\psi\|_{q;\Omega_h} = \|\nabla(R_h - I)(I - I_h)A^{-1}\psi\|_{q;\Omega_h} \le Ch\|A^{-1}\psi\|_{q;2} \le Ch\|\psi\|_q. \tag{3.14}$$

Now, in order to apply a duality argument, we will be based on the following identity, considered for suitable pairs of $v$ and $\psi$:

$$\begin{aligned} ((R_h - I)A^{-1}v, \psi) &= (v, (R_h - I)A^{-1}\psi) \\ &= (\nabla(I - I_h)A^{-1}v, \nabla(R_h - I)A^{-1}\psi) \\ &= (\nabla(I - I_h)A^{-1}v, \nabla(R_h - I)A^{-1}\psi)_{\Omega_h} \\ &\quad - (\nabla A^{-1}v, \nabla A^{-1}\psi)_{\Omega\setminus\Omega_h}. \end{aligned} \tag{3.15}$$

Using now (3.14) and the above reasonings, this leads in a standard way to the estimate

$$\|(R_h - I)A^{-1}v\|_p \le Ch^2\|v\|_p, \tag{3.16}$$

which therefore will be valid at least for $p$ sufficiently close to 1 (the last term on the right-hand side of (3.15) is estimated as desired in the same way as in the proof of [20, Lemma A.4]). With the aid of the equality in the first line of (3.15), we conclude, by duality, that (3.16) holds as well for all $p$ sufficiently large, and further it is easily seen, by interpolation, that (3.16) is valid in fact for all $1 < p < \infty$.

It thus remains to combine (3.9), (3.3), and (3.16). □

We proceed now to show resolvent estimates for the operator $A_h$. The first two results presented below are in fact found in [7]. We state them here, however, in view of their significance and for subsequent reference. For the second assertion, we also propose a new proof, which is given, from our point of view, in a more straightforward way than in [7] and is based on using (2.14) with $\alpha = 1$. The latter fact is of particular interest because (2.14) with $\alpha = 1$ will be an essential ingredient in showing resolvent estimates of the operator $A_h$ in Hölder norms.

**Theorem 3.2.** *For any fixed $\varphi \in (0, \pi/2)$,*

$$\left|(\lambda I - A_h)^{-1}\chi\right|_0 \le C(1+|\lambda|)^{-1}|\chi|_0 \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ \chi \in S_h. \tag{3.17}$$

**Theorem 3.3.** *For any fixed $\varphi \in (0, \pi/2)$,*

$$\left|A_h(\lambda I - A_h)^{-1}\chi\right|_0 \le C(1+|\lambda|)^{-1/2}|\chi|_1 \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ \chi \in S_h. \tag{3.18}$$

The proof of (3.18) is given in fact only for $|\lambda| \le \omega_0 h^{-2}$, with $\omega_0 > 0$ sufficiently small, because just this case needs to be settled. It was remarked in [7] that, in the opposite case $|\lambda| > \omega_0 h^{-2}$, (3.18) is obtained in a trivial way. We will therefore emphasize below on the case $|\lambda| \le \omega_0 h^{-2}$, with $\omega_0 > 0$ sufficiently small.

**Proof of Theorem 3.3 for $|\lambda| \le \omega_0 h^{-2}$.** We use, for $\chi \in S_h$, the identity

$$A_h(\lambda I - A_h)^{-1}\chi = P_h A(\lambda I - A)^{-1}\chi + \lambda A_h(\lambda I - A_h)^{-1}(P_h - R_h)(\lambda I - A)^{-1}\chi, \tag{3.19}$$

both sides of which are well defined for all $\lambda \notin \operatorname{Int}\Sigma_\varphi$. Applying (3.3) with $p = \infty$ and (2.12), we find

$$\left|P_h A(\lambda I - A)^{-1}\chi\right|_0 \le C(1+|\lambda|)^{-1/2}|\chi|_1. \tag{3.20}$$

Next, by (3.10) and (2.14) with $\alpha = 1$, we get

$$\left|(P_h - R_h)(\lambda I - A)^{-1}\chi\right|_1 \le Ch\left|(\lambda I - A)^{-1}\chi\right|_2 \le Ch(1+|\lambda|)^{-1/2}|\chi|_1. \tag{3.21}$$

Now, given a linear bounded operator $B_h : S_h \to S_h$, denote

$$|||B_h|||_{S_h;1\to 0} := \sup_{\chi \in S_h}\left(|B_h\chi|_0/|\chi|_1\right). \tag{3.22}$$

Inserting (3.21) and (3.20) into (3.19), we find

$$\begin{aligned} &\left|\left|\left|A_h(\lambda I - A_h)^{-1}\right|\right|\right|_{S_h;1\to 0} \\ &\quad \le C(1+|\lambda|)^{-1/2} + C_1 h(1+|\lambda|)^{1/2}\left|\left|\left|A_h(\lambda I - A_h)^{-1}\right|\right|\right|_{S_h;1\to 0}. \end{aligned} \tag{3.23}$$

Clearly, if $C_1 h(1+|\lambda|)^{1/2} \le 1/2$, this implies

$$\left|\left|\left|A_h(\lambda I - A_h)^{-1}\right|\right|\right|_{S_h;1\to 0} \le C(1+|\lambda|)^{-1/2}, \tag{3.24}$$

so that the result follows for $|\lambda| \le \omega_0 h^{-2}$ if $\omega_0$ is sufficiently small. □

The above estimate (3.17) implies similar estimates for the whole Lebesgue scale $L_p$, $1 \le p \le \infty$.

**Theorem 3.4.** *For any fixed* $\varphi \in (0, \pi/2)$,

$$\|(\lambda I - A_h)^{-1}\chi\|_p \le C(1+|\lambda|)^{-1}\|\chi\|_p \quad \forall 1 \le p \le \infty,\ \lambda \notin \operatorname{Int}\Sigma_\varphi,\ \chi \in S_h. \tag{3.25}$$

**Proof.** The proof is immediate by duality and interpolation. □

Our next result gives a resolvent estimate of $A_h$ in $W^1_\infty$-norm. Note that the below proof of this assertion will be essentially based on the use of the estimate (2.14) with $\alpha = 1$.

**Theorem 3.5.** *For any fixed* $\varphi \in (0, \pi/2)$,

$$|(\lambda I - A_h)^{-1}\chi|_1 \le C(1+|\lambda|)^{-1}|\chi|_1 \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ \chi \in S_h. \tag{3.26}$$

**Proof.** We use the identity (cf. [7]), for $\chi \in S_h$,

$$(\lambda I - A_h)^{-1}\chi = P_h(\lambda I - A)^{-1}\chi + A_h(\lambda I - A_h)^{-1}(P_h - R_h)(\lambda I - A)^{-1}\chi, \tag{3.27}$$

where both sides are well defined for $\lambda \notin \operatorname{Int}\Sigma_\varphi$. By (3.4) and (2.14) with $\alpha = 0$, we have

$$|P_h(\lambda I - A)^{-1}\chi|_1 \le C(1+|\lambda|)^{-1}|\chi|_1. \tag{3.28}$$

Next, denoting

$$G := A_h(\lambda I - A_h)^{-1}(P_h - R_h)(\lambda I - A)^{-1}\chi, \tag{3.29}$$

we find by (3.1), (3.18), (3.10), and (2.14) with $\alpha = 1$,

$$\begin{aligned} |G|_1 &\le Ch^{-1}|G|_0 \\ &\le Ch^{-1}(1+|\lambda|)^{-1/2}|(P_h - R_h)(\lambda I - A)^{-1}\chi|_1 \\ &\le C(1+|\lambda|)^{-1/2}|(\lambda I - A)^{-1}\chi|_2 \le C(1+|\lambda|)^{-1}|\chi|_1. \end{aligned} \tag{3.30}$$

The claim thus follows by combining (3.27), (3.28), and (3.30). □

Now we can show estimates in pairs of Hölder norms.

**Theorem 3.6.** *For any fixed $\varphi \in (0, \pi/2)$, $\xi \in [0,1]$, and $\eta \in [0, \xi]$,*

$$|(\lambda I - A_h)^{-1}\chi|_\xi \le C(1+|\lambda|)^{(\xi-\eta)/2-1}|\chi|_\eta \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ \chi \in S_h. \tag{3.31}$$

**Proof.** First of all, by (3.4), (3.7) with $p = \infty$, and (2.58) with $\xi = 1$, we have

$$|(P_h - R_h)(\lambda I - A)^{-1}\chi|_1 \le C|(\lambda I - A)^{-1}\chi|_1 \le C(1+|\lambda|)^{-(\eta+1)/2}|\chi|_\eta. \tag{3.32}$$

With $G$ defined as above (see (3.29)), using further (2.17) with $A_h$ substituted for $A$, (3.26), and (3.32) yields

$$\begin{aligned} |G|_1 &\le |(P_h - R_h)(\lambda I - A)^{-1}\chi|_1 + |\lambda(\lambda I - A_h)^{-1}(P_h - R_h)(\lambda I - A)^{-1}\chi|_1 \\ &\le C|(P_h - R_h)(\lambda I - A)^{-1}\chi|_1 \\ &\le C(1+|\lambda|)^{-(\eta+1)/2}|\chi|_\eta. \end{aligned} \tag{3.33}$$

Combining now (3.5) (if $\xi = 1$, we use instead (3.4)) and (2.58), this yields

$$|P_h(\lambda I - A)^{-1}\chi|_\xi \le C|(\lambda I - A)^{-1}\chi|_\xi \le C(1+|\lambda|)^{(\xi-\eta)/2-1}|\chi|_\eta. \tag{3.34}$$

At the same time, applying (3.18) and (3.32) allows one to get as well

$$|G|_0 \le C(1+|\lambda|)^{-1/2}|(P_h - R_h)(\lambda I - A)^{-1}\chi|_1 \le C(1+|\lambda|)^{-\eta/2-1}|\chi|_\eta. \tag{3.35}$$

It follows from (3.33) and (3.35) that

$$|G|_\xi \le C|G|_1^\xi |G|_0^{1-\xi} \le C(1+|\lambda|)^{(\xi-\eta)/2-1}|\chi|_\eta. \tag{3.36}$$

Finally, the claim is obtained by combining (3.27), (3.34), and (3.36). □

We also derive a finite-element analogue of Theorem 2.9 involving (Hölder, Lebesgue) pairs.

**Theorem 3.7.** *For any fixed $\varphi \in (0, \pi/2)$, $\xi \in [0,1]$, and $q \in (d/(2-\xi), \infty]$,*

$$|(\lambda I - A_h)^{-1}\chi|_\xi \le C(1+|\lambda|)^{\xi/2+d/(2q)-1}\|\chi\|_q \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ \chi \in S_h. \tag{3.37}$$

**Proof.** Let throughout the proof $G$ be just the same as in (3.29).

Assume first that $q \in (d, \infty]$. By (3.32) and (2.34) with $\xi = 1$, we obtain

$$|(P_h - R_h)(\lambda I - A)^{-1}\chi|_1 \leq C(1 + |\lambda|)^{d/(2q)-1/2}\|\chi\|_q, \tag{3.38}$$

and a similar argument shows that

$$|P_h(\lambda I - A)^{-1}\chi|_1 \leq C(1 + |\lambda|)^{d/(2q)-1/2}\|\chi\|_q. \tag{3.39}$$

Inserting now (3.38) into (3.33), we find

$$|G|_1 \leq C(1 + |\lambda|)^{d/(2q)-1/2}\|\chi\|_q. \tag{3.40}$$

So, combining (3.27), (3.39), and (3.40) shows the claim at least for $\xi = 1$.

Let further $\xi = 0$ and $q \in (d/2, \infty]$. From now on we assume that $d \geq 2$ (the below reasonings can be easily modified to the case $d = 1$). If, in addition, $q = \infty$, the result is directly obtained by (3.17). We may thus consider only the case $q \in (d/2, \infty)$. Let $p$ be a fixed number such that $p > q$ and $p \in (d, \infty)$. Clearly, $p$ can be chosen such that $1/q - 1/p < 1/d$. Then, using the inverse inequality $|A_h\chi|_0 \leq Ch^{-1}|\chi|_1$, we get, since $p > d$ and the result is already proved for $\xi = 1$,

$$\begin{aligned} |G|_0 &\leq Ch^{-1}|(\lambda I - A_h)^{-1}(P_h - R_h)(\lambda I - A)^{-1}\chi|_1 \\ &\leq Ch^{-1}(1 + |\lambda|)^{d/(2p)-1/2}\|(P_h - R_h)(\lambda I - A)^{-1}\chi\|_p. \end{aligned} \tag{3.41}$$

Combining further this with (3.11), (2.13), and (2.17) yields

$$|G|_0 \leq Ch(1 + |\lambda|)^{d/(2p)-1/2}\|\chi\|_p. \tag{3.42}$$

Defining now $r \in [1, p)$ by $1/r := 1/p + 1/d$ and using the inverse inequality (3.1) with $j = 0$, it follows from the last estimate that

$$|G|_0 \leq C(1 + |\lambda|)^{d/(2p)-1/2}\|\chi\|_r. \tag{3.43}$$

At the same time, as a direct consequence of (3.36) with $\xi = \eta = 0$, we have

$$|G|_0 \leq C(1 + |\lambda|)^{-1}\|\chi\|_\infty. \tag{3.44}$$

An interpolation argument, applied to (3.43) and (3.44), thus leads to the estimate

$$|G|_0 \leq C(1 + |\lambda|)^{d/(2q)-1}\|\chi\|_q. \tag{3.45}$$

Moreover, by (3.3) and (2.28), both with $p = \infty$, we have

$$|P_h(\lambda I - A)^{-1}\chi|_0 \leq C|(\lambda I - A)^{-1}\chi|_0 \leq C(1 + |\lambda|)^{d/(2q)-1}\|\chi\|_q. \tag{3.46}$$

Altogether (3.27), (3.45), and (3.46) show the claim for $\xi = 0$.

It remains to get the result in the case $\xi \in (0,1)$. In order to do this, note that we can write any fixed $q \in (d/(2-\xi),\infty]$ in the form

$$\frac{1}{q} = \frac{2-\xi}{d} - \varepsilon, \quad \varepsilon > 0. \tag{3.47}$$

Assuming that $\varepsilon < 1/d$ and defining then $q_0$ and $q_1$ as in (2.33), we have by (3.37) with $\xi = 1$ and $\xi = 0$ (already shown) and by (3.3),

$$\begin{aligned} |(\lambda I - A_h)^{-1} P_h v|_1 &\le C(1+|\lambda|)^{d/(2q_1)-1/2} \|P_h v\|_{q_1} \\ &\le C(1+|\lambda|)^{d/(2q_1)-1/2} \|v\|_{q_1}, \end{aligned} \tag{3.48}$$

$$|(\lambda I - A_h)^{-1} P_h v|_0 \le C(1+|\lambda|)^{d/(2q_0)-1} \|v\|_{q_0}. \tag{3.49}$$

Using these estimates, we find, by interpolation,

$$|(\lambda I - A_h)^{-1} P_h v|_\xi \le C(1+|\lambda|)^{\xi/2+d/(2q)-1} \|v\|_q, \tag{3.50}$$

whence the result with $0 < \xi < 1$ follows by taking $v = \chi$.

If the above $\varepsilon$ does not satisfy the restriction $\varepsilon < 1/d$, this situation can be considered after all by using an interpolation argument (as it is done in the proof of Theorem 2.8).

So the proof is complete. □

In the case of purely Lebesgue pairs we have the following result.

**Theorem 3.8.** *For any fixed $\varphi \in (0,\pi/2)$ and $1 \le q \le p \le \infty$ such that $1/q - 1/p < 2/d$,*

$$\|(\lambda I - A_h)^{-1}\chi\|_p \le C(1+|\lambda|)^{(d/2)(1/q-1/p)-1} \|\chi\|_q \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ \chi \in S_h. \tag{3.51}$$

**Proof.** In the case $p = \infty$, $1/q < 2/d$, the result is obtained straightforwardly by (3.37) with $\xi = 0$. By duality, the claim also follows for $1 - 1/p < 2/d$, $q = 1$. Finally, the general case can then be shown by interpolation. □

The following results unite and generalize in some sense the assertions of Theorems 3.7 and 3.8.

**Theorem 3.9.** *Let $\xi \in [0,1]$, $q \in [1,\infty]$, and $m \in \mathbb{N} \cup \{0\}$ be such that*

$$\frac{1}{q} < \frac{2m+2-\xi}{d}. \tag{3.52}$$

*Then, for any fixed $\varphi \in (0,\pi/2)$,*

$$|(\lambda I - A_h)^{-(m+1)}\chi|_\xi \le C(1+|\lambda|)^{\xi/2+d/(2q)-(m+1)} \|\chi\|_q \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ \chi \in S_h. \tag{3.53}$$

**Theorem 3.10.** *Let* $1 \le q \le p \le \infty$ *and* $m \in \mathbb{N} \cup \{0\}$ *be such that*

$$\frac{1}{q} - \frac{1}{p} < \frac{2m}{d}. \tag{3.54}$$

*Then, for any fixed* $\varphi \in (0, \pi/2)$,

$$\|(\lambda I - A_h)^{-m}\chi\|_p \le C(1+|\lambda|)^{(d/2)(1/q-1/p)-m}\|\chi\|_q \quad \forall \lambda \notin \operatorname{Int}\Sigma_\varphi,\ \chi \in S_h. \tag{3.55}$$

The proofs of these results are based on an induction argument and are similar to their continuous analogues (see the proofs of Theorems 2.12 and 2.13).

In conclusion, we note that using the above results, combined with more or less standard reasonings, allows one to obtain finite-element analogues of Theorems 2.10, 2.11, and 2.14. We omit the corresponding statements.

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