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BOUNDEDNESS OF MULTILINEAR OPERATORS ON TRIEBEL-LIZORKIN SPACES

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The purpose of this paper is to study the boundedness in the context of Triebel-Lizorkin spaces for some multilinear operators related to certain convolution operators. The operators include Littlewood-Paley operator, Marcinkiewicz integral, and Bochner-Riesz operator.

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1. Introduction. Let T be a Calderon-Zygmund operator. A well-known result of Coifman et al. [6] states that the commutator $[b,T] = T(bf) - bTf$ (where $b \in \mathrm{BMO}$) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$; Chanillo [1] proves a similar result when T is replaced by the fractional integral operator. In [7, 9], Janson and Paluszyński extend these results to the Triebel-Lizorkin spaces and the case $b \in \mathrm{Lip}\beta$ (where $\mathrm{Lip}\beta$ is the homogeneous Lipschitz space). The main purpose of this paper is to discuss the boundedness of some multilinear operators related to certain convolution operators in the context of Triebel-Lizorkin spaces. In fact, we will establish the boundedness on the Triebel-Lizorkin spaces for some multilinear operators related to certain convolution operator only under certain conditions on the size of the operators. As applications, we obtain the boundedness of the multilinear operators related to the Marcinkiewicz integral, Littlewood-Paley operator, and Bochner-Riesz operator in the context of Triebel-Lizorkin spaces.

2. Preliminaries. Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, $M_p f = (M(f^p))^{1/p}$ for $p > 0$, and Q will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For a cube Q, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Triebel-Lizorkin space. The Lipschitz space $\dot{\wedge}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\wedge}_\beta} = \sup_{\substack{x,h\in\mathbb{R}^n \\ h\neq 0}} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty, \tag{2.1}$$

where Δ_h^k denotes the kth difference operator (see [9]).

The operators considered in this paper are following several sublinear operators.

Let m be a positive integer and let A be a function on $\mathbb{R}^n$. We denote

$$\mathbb{R}_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha. \tag{2.2}$$

Definition 2.1. Let $\varepsilon > 0$ and let ψ be a fixed function which satisfies the following properties:

(1) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(2) $|\psi(x+y)-\psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

The multilinear Littlewood-Paley operator is defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2}, \tag{2.3}$$

where

$$F_t^A(f)(x) = \int_{\mathbb{R}^n} \psi_t(x-y) \frac{\mathbb{R}_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy \tag{2.4}$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Denote $F_t(f) = \psi_t * f$. Also define

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2} \tag{2.5}$$

which is the Littlewood-Paley g function (see [10]).

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2} < \infty\}$. Then, for each fixed $x \in \mathbb{R}^n$, $F_t^A(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi(f)(x) = \|F_t(f)(x)\|, \qquad g_\psi^A(f)(x) = \|F_t^A(f)(x)\|. \tag{2.6}$$

Definition 2.2. Let $0 < \gamma \le 1$ and let Ω be homogeneous of degree zero on $\mathbb{R}^n$ such that $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x)-\Omega(y)| \le M|x-y|^\gamma$. The multilinear Marcinkiewicz integral operator is defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{2.7}$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{\mathbb{R}_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy. \tag{2.8}$$

Denote

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy. \tag{2.9}$$

Also define

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2} \tag{2.10}$$

which is the Marcinkiewicz integral (see [11]).

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then, it is clear that

$$\mu_\Omega(f)(x) = \|F_t(f)(x)\|, \qquad \mu_\Omega^A(f)(x) = \|F_t^A(f)(x)\|. \tag{2.11}$$

Definition 2.3. Let $B_t^\delta(\hat{f})(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$. Denote

$$B_{\delta,t}^A(f)(x) = \int_{\mathbb{R}^n} B_t^\delta(x-y)\frac{\mathbb{R}_{m+1}(A;x,y)}{|x-y|^m} f(y)dy, \tag{2.12}$$

where $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. The maximal multilinear Bochner-Riesz operator is defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)|. \tag{2.13}$$

Also define

$$B_*^\delta(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)| \tag{2.14}$$

which is the Bochner-Riesz operator (see [7, 8]).

Let H be the space $H = \{h : \|h\| = \sup_{t>0}|h(t)| < \infty\}$, then it is clear that

$$B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||, \qquad B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||. \tag{2.15}$$

More generally, we consider the following multilinear operators related to certain convolution operators.

Definition 2.4. Let $K(x,t)$ be defined on $\mathbb{R}^n \times [0,+\infty)$. Denote that

$$\begin{aligned} K_t f(x) &= \int_{\mathbb{R}^n} K(x-y,t)f(y)dy, \\ K_t^A f(x) &= \int_{\mathbb{R}^n} \frac{\mathbb{R}_{m+1}(A;x,y)}{|x-y|^m} K(x-y,t)f(y)dy. \end{aligned} \tag{2.16}$$

Let H be the normed space $H = \{h : \|h\| < \infty\}$. For each fixed $x \in \mathbb{R}^n$, $K_t f(x)$ and $K_t^A(f)(x)$ are viewed as a mapping from $[0,+\infty)$ to H. Then, the multilinear operators related to K_t is defined by

$$T_A f(x) = ||K_t^A(f)(x)||; \tag{2.17}$$

also define $Tf(x) = \|K_t f(x)\|$.

It is clear that Definitions 2.1, 2.2, and 2.3 are the particular examples of Definition 2.4. Note that when $m = 0$, T_A is just the commutator of K_t and A. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2, 3, 4, 5]). The main purpose of this paper is to consider the continuity of the multilinear operators on Triebel-Lizorkin spaces. We will prove the following theorems in Section 3.

Theorem 2.5. *Let g_ψ^A be the multilinear Littlewood-Paley operator as in Definition 2.1 and let $0 < \beta < \min(1,\varepsilon)$, $1 < p < \infty$, and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$. Then*

(a) *g_ψ^A is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)$,*

(b) *g_ψ^A is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1/p - 1/q = \beta/n$ and $1/p > \beta/n$.*

Theorem 2.6. *Let μ_Ω^A be the multilinear Marcinkiewicz integral operator as in Definition 2.2 and let $0 < \gamma \le 1$, $0 < \beta < \min(1/2, \gamma)$, $1 < p < \infty$, and $D^\alpha A \in \dot{\wedge}_\beta$ for $|\alpha| = m$. Then*

(a) *μ_Ω^A is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)$,*

(b) *μ_Ω^A is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1/p - 1/q = \beta/n$ and $1/p > \beta/n$.*

Theorem 2.7. *Let $B_{\delta,*}^A$ be the maximal multilinear Bochner-Riesz operator as in Definition 2.3 and let $\delta > (n-1)/2$, $0 < \beta < \min(1, \delta - (n-1)/2)$, $1 < p < \infty$, and $D^\alpha A \in \dot{\wedge}_\beta$ for $|\alpha| = m$. Then*

(a) *$B_{\delta,*}^A$ is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)$;*

(b) *$B_{\delta,*}^A$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1/p - 1/q = \beta/n$ and $1/p > \beta/n$.*

3. Main theorem and proof. First, we will establish the following theorem.

Theorem 3.1. *Let $0 < \beta < 1$, $1 < p < \infty$, and $D^\alpha A \in \dot{\wedge}_\beta$ for $|\alpha| = m$. Let K_t, T, and T_A be the same as in Definition 2.4. If T is bounded on $L^q(\mathbb{R}^n)$ for $q \in (1, +\infty)$ and T_A satisfies the size condition*

$$||K_t^A(f)(x) - K_t^A(f)(x_0)|| \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\wedge}_\beta} |Q|^{\beta/n} M(f)(x) \tag{3.1}$$

for any cube Q with $\operatorname{supp} f \subset (2Q)^c$ and $x \in Q$, then

(a) *T_A is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)$,*

(b) *T_A is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1/p - 1/q = \beta/n$ and $1/p > \beta/n$.*

To prove the theorem, we need the following lemmas.

Lemma 3.2 (see [9]). *For $0 < \beta < 1$ and $1 < p < \infty$,*

$$\begin{aligned} \|f\|_{\dot{F}_p^{\beta,\infty}} &\approx \left\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q|\, dx \right\|_{L^p} \\ &\approx \left\| \sup_{\cdot \in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c|\, dx \right\|_{L^p}. \end{aligned} \tag{3.2}$$

Lemma 3.3 (see [9]). *For $0 < \beta < 1$ and $1 \le p \le \infty$,*

$$\begin{aligned} \|f\|_{\dot{\wedge}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q|\, dx \\ &\approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p}. \end{aligned} \tag{3.3}$$

Lemma 3.4 (see [1]). *For $1 \le r < \infty$ and $\delta > 0$, let*

$$M_{\delta,r}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\delta r/n}} \int_Q |f(y)|^p dy \right)^{1/p}. \tag{3.4}$$

Suppose that $r < p < \delta/n$ and $1/q = 1/p - \delta/n$. Then $\|M_{\delta,r}(f)\|_{L^q} \le C\|f\|_{L^p}$.

Lemma 3.5 (see [9]). *Let $Q_1 \subset Q_2$. Then*

$$|f_{Q_1} - f_{Q_2}| \le C\|f\|_{\dot{\Lambda}_\beta} |Q_2|^{\beta/n}. \tag{3.5}$$

Lemma 3.6 (see [4]). *Let A be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|\mathbb{R}_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q}, \tag{3.6}$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Proof of Theorem 3.1. (a) Fix a cube $Q = Q(x_0, l)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}(1/\alpha!)(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $\mathbb{R}_m(A;x,y) = \mathbb{R}_m(\tilde{A};x,y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. For $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus \tilde{Q}}$,

$$\begin{aligned} K_t^A(f)(x) &= \int_{\mathbb{R}^n} \frac{\mathbb{R}_{m+1}(\tilde{A};x,y)}{|x-y|^m} K(x-y,t) f(y)dy \\ &= \int_{\mathbb{R}^n} \frac{\mathbb{R}_{m+1}(\tilde{A};x,y)}{|x-y|^m} K(x-y,t) f(y)dy \\ &\quad + \int_{\mathbb{R}^n} \frac{\mathbb{R}_m(\tilde{A};x,y)}{|x-y|^m} K(x-y,t) f_1(y)dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{K(x-y,t)(x-y)^\alpha}{|x-y|^m} D^\alpha \tilde{A}(y) f_1(y)dy, \end{aligned} \tag{3.7}$$

then

$$\begin{aligned} |T_A(f)(x) - T_{\tilde{A}}(f_2)(x_0)| &= \big| \|K_t^A(f)(x)\| - \|K_t^{\tilde{A}}(f_2)(x_0)\| \big| \\ &\le \left\| K_t\left(\frac{\mathbb{R}_m(\tilde{A};x,\cdot)}{|x-\cdot|^m} f_1 \right)(x) \right\| \\ &\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| K_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1 \right)(x) \right\| \\ &\quad + \|K_t^{\tilde{A}}(f_2)(x) - K_t^{\tilde{A}}(f_2)(x_0)\| \\ &= A(x) + B(x) + C(x). \end{aligned} \tag{3.8}$$

Thus,

$$\begin{aligned} &\frac{1}{|Q|^{1+\beta/n}} \int_Q |T_A f(x) - T_{\tilde{A}}(f)(x_0)| dx \\ &\quad \le \frac{1}{|Q|^{1+\beta/n}} \int_Q A(x)dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q B(x)dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q C(x)dx \\ &\quad := \text{I+II+III}. \end{aligned} \tag{3.9}$$

Now, we estimate I, II, and III, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, using Lemmas 3.3 and 3.6, we get

$$\begin{aligned} |\mathbb{R}_m(\tilde{A};x,y)| &\le C|x-y|^m \sum_{|\alpha|=m} \sup_{x\in\tilde{Q}} |D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}}| \\ &\le C|x-y|^m |Q|^{\beta/n} \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta}. \end{aligned} \tag{3.10}$$

Thus, by Holder's inequality and the L^r boundedness of T for $1<r<p$, we obtain

$$\begin{aligned} \mathrm{I} &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \frac{1}{|Q|}\int_Q |T(f_1)(x)|\,dx \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \|T(f_1)\|_{L^r} |Q|^{-1/r} \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \|f_1\|_{L^r} |Q|^{-1/r} \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} M_r(f)(\tilde{x}). \end{aligned} \tag{3.11}$$

Secondly, for $1<r<q$, using the inequality (see [9])

$$\|D^\alpha A-(D^\alpha A)_{\tilde{Q}} f\chi_{\tilde{Q}}\|_{L^r} \le C|Q|^{1/r+\beta/n}\|D^\alpha A\|_{\dot{\Lambda}_\beta} M_r(f)(x), \tag{3.12}$$

and similar to the proof of I, we obtain

$$\begin{aligned} \mathrm{II} &\le \frac{C}{|Q|^{1+\beta/n}} \sum_{|\alpha|=m} \|T((D^\alpha A-(D^\alpha A)_{\tilde{Q}})f\chi_{\tilde{Q}})\|_{L^r}|Q|^{1-1/r} \\ &\le C|Q|^{-\beta/n-1/r} \sum_{|\alpha|=m} \|(D^\alpha A-(D^\alpha A)_{\tilde{Q}})f\chi_{\tilde{Q}}\|_{L^r} \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} M_r(f)(\tilde{x}). \end{aligned} \tag{3.13}$$

For III, using the size condition of T_A, we have

$$\mathrm{III} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} M(f)(\tilde{x}). \tag{3.14}$$

Putting these estimates together, taking the supremum over all Q such that $\tilde{x}\in Q$, and using the L^p boundedness of M_r for $r<p$, we obtain

$$\|T_A(f)\|_{\dot{F}_p^{\beta,\infty}} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}. \tag{3.15}$$

This completes the proof of (a).

(b) By the same argument as in the proof of (a), we have

$$\frac{1}{|Q|}\int_Q |T_A(f)(x)-T_{\tilde{A}}(f_2)(x_0)|\,dx \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta}(M_{\beta,r}(f)+M_{\beta,1}(f)), \tag{3.16}$$

thus,

$$(T_A(f))^{\#} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} (M_{\beta,r}(f) + M_{\beta,1}(f)). \tag{3.17}$$

Now, using Lemma 3.4, we obtain

$$\begin{aligned} ||T_A(f)||_{L^q} &\le C||(T_A(f))^{\#}||_{L^q} \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} (||M_{\beta,r}(f)||_{L^q} + ||M_{\beta,1}(f)||_{L^q}) \le C\|f\|_{L^p}. \end{aligned} \tag{3.18}$$

This completes the proof of (b) and the theorem. □

To prove Theorems 2.5, 2.6, and 2.7, it suffices to verify that g_ψ^A, μ_Ω^A, and $B_{\delta,*}^A$ satisfy the size condition in the Theorem 3.1.

Suppose $\operatorname{supp} f \subset \tilde{Q}^c$ and $x \in Q = Q(x_0, l)$. Note that $|x_0 - y| \approx |x - y|$ for $y \in \tilde{Q}^c$.

For g_ψ^A, we write

$$\begin{aligned} &F_t^{\tilde{A}}(f)(x) - F_t^{\tilde{A}}(f)(x_0) \\ &\quad = \int_{\mathbb{R}^n \setminus \tilde{Q}} \left[\frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x_0-y)}{|x_0-y|^m} \right] \mathbb{R}_m(\tilde{A}; x, y) f(y)\, dy \\ &\qquad + \int_{\mathbb{R}^n \setminus \tilde{Q}} \frac{\psi_t(x_0-y) f(y)}{|x_0-y|^m} [\mathbb{R}_m(\tilde{A}; x, y) - \mathbb{R}_m(\tilde{A}; x_0, y)]\, dy \\ &\qquad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n \setminus \tilde{Q}} \left[\frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m} - \frac{\psi_t(x_0-y)(x_0-y)^\alpha}{|x_0-y|^m} \right] \\ &\qquad\qquad \times D^\alpha \tilde{A}(y) f(y)\, dy \\ &\quad = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned} \tag{3.19}$$

By the condition of ψ, we obtain

$$\begin{aligned} ||\mathrm{I}_1|| &\le C \int_{\mathbb{R}^n \setminus \tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{m+1}} |\mathbb{R}_m(\tilde{A}; x, y)| |f(y)| \\ &\qquad \times \left(\int_0^\infty \frac{t\, dt}{(t + |x_0-y|)^{2(n+1)}} \right)^{1/2} dy \\ &\quad + C \int_{\mathbb{R}^n \setminus \tilde{Q}} \frac{|x-x_0|^\varepsilon}{|x_0-y|^m} |\mathbb{R}_m(\tilde{A}; x, y)| |f(y)| \\ &\qquad \times \left(\int_0^\infty \frac{t\, dt}{(t + |x_0-y|)^{2(n+1+\varepsilon)}} \right)^{1/2} dy \\ &\le C \int_{\mathbb{R}^n \setminus \tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}} \right) |\mathbb{R}_m(\tilde{A}; x, y)| |f(y)|\, dy \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \\ &\qquad \times \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^{k+1}\tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}} \right) |f(y)|\, dy \end{aligned}$$

$$\begin{aligned}&\leq C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\sum_{k=1}^{\infty}(2^{-k}+2^{-k\varepsilon})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|\,dy\\&\leq C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\sum_{k=1}^{\infty}(2^{-k}+2^{-k\varepsilon})M(f)(x)\\&\leq C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M(f)(x).\end{aligned} \tag{3.20}$$

For I_2, by the formula (see [4])

$$\mathbb{R}_m(\tilde{A};x,y)-\mathbb{R}_m(\tilde{A};x_0,y)=\sum_{|\eta|<m}\frac{1}{\eta!}\mathbb{R}_{m-|\eta|}(D^{\eta}\tilde{A};x,x_0)(x-y)^{\eta} \tag{3.21}$$

and Lemma 3.6, we get

$$|\mathbb{R}_m(\tilde{A};x,y)-\mathbb{R}_m(\tilde{A};x_0,y)|\leq C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}|x-x_0|\,|x_0-y|^{m-1}. \tag{3.22}$$

Thus, similar to the proof of I_1,

$$\begin{aligned}||\text{I}_2||&\leq C\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|\mathbb{R}_m(\tilde{A};x,y)-\mathbb{R}_m(\tilde{A};x_0,y)|}{|x_0-y|^{m+n}}|f(y)|\,dy\\&\leq C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|\,dy\\&\leq C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M(f)(x).\end{aligned} \tag{3.23}$$

For I_3, by Lemma 3.5, we get

$$|D^{\alpha}A(y)-(D^{\alpha}A)_{\tilde{Q}}|\leq||D^{\alpha}A||_{\dot{\Lambda}_\beta}|x_0-y|^{\beta}. \tag{3.24}$$

Thus, similar to the proof of I_1, we obtain

$$\begin{aligned}||\text{I}_3||&\leq C\sum_{|\alpha|=m}\int_{\mathbb{R}^n\setminus\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}}\right)|f(y)|\,|D^{\alpha}\tilde{A}(y)|\,dy\\&\leq C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\sum_{k=1}^{\infty}(2^{k(\beta-1)}+2^{k(\beta-\varepsilon)})M(f)(x)\\&\leq C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M(f)(x).\end{aligned} \tag{3.25}$$

So,

$$||F_t^{\tilde{A}}(f)(x)-F_t^{\tilde{A}}(f)(x_0)||\leq C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M(f)(x). \tag{3.26}$$

For μ_Ω^A, we write

$$
\begin{aligned}
&\|F_t^{\tilde{A}}(f)(x)-F_t^{\tilde{A}}(f)(x_0)\| \\
&\quad\le\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)\mathbb{R}_m(\tilde{A};x,y)}{|x-y|^{m+n-1}}f(y)dy\right.\right. \\
&\qquad\left.\left.-\int_{|x_0-y|\le t}\frac{\Omega(x_0-y)\mathbb{R}_m(\tilde{A};x_0,y)}{|x_0-y|^{m+n-1}}f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\
&\qquad+C\sum_{|\alpha|=m}\left(\int_0^\infty\left|\int_{|x-y|\le t}\left(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}}\right.\right.\right. \\
&\qquad\qquad\left.-\int_{|x_0-y|\le t}\frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1}}\right) \\
&\qquad\qquad\left.\left.\times D^\alpha\tilde{A}(y)f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\
&\quad\le\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|>t}\frac{|\Omega(x-y)||\mathbb{R}_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}}|f(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2} \\
&\qquad+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\le t}\frac{|\Omega(x_0-y)||\mathbb{R}_m(\tilde{A};x_0,y)|}{|x_0-y|^{m+n-1}}|f(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2} \\
&\qquad+\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{\Omega(x-y)\mathbb{R}_m(\tilde{A};x,y)}{|x-y|^{m+n-1}}-\frac{\Omega(x_0-y)\mathbb{R}_m(\tilde{A};x_0,y)}{|x_0-y|^{m+n-1}}\right|\right.\right. \\
&\qquad\qquad\left.\left.\times|f(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2} \\
&\qquad+C\sum_{|\alpha|=m}\left(\int_0^\infty\left|\int_{|x-y|\le t}\left(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}}\right.\right.\right. \\
&\qquad\qquad\left.-\int_{|x_0-y|\le t}\frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1}}\right) \\
&\qquad\qquad\left.\left.\times D^\alpha\tilde{A}(y)f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\
&\quad:=J_1+J_2+J_3+J_4.
\end{aligned}
\tag{3.27}
$$

Thus

$$\begin{aligned} J_1 &\le C\int_{\mathbb{R}^n\setminus\tilde{Q}} \frac{|f(y)|\,|\mathbb{R}_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}}\left(\int_{|x-y|\le t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy \\ &\le C\int_{\mathbb{R}^n\setminus\tilde{Q}} \frac{|f(y)|\,|\mathbb{R}_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy \\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\wedge}_\beta}|Q|^{\beta/n}\sum_{k=1}^{\infty}2^{-k/2}|2^k\tilde{Q}|^{-1}\int_{2^k\tilde{Q}}|f(y)|dy \\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\wedge}_\beta}|Q|^{\beta/n}M(f)(x). \end{aligned} \tag{3.28}$$

Similarly, we have $J_2 \le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\wedge}_\beta}|Q|^{\beta/n}M(f)(x)$.

For J_3, by the inequality (see [11])

$$\left|\frac{\Omega(x-y)}{|x-y|^{m+n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{m+n-1}}\right| \le C\left(\frac{|x-x_0|}{|x_0-y|^{m+n}}+\frac{|x-x_0|^{\gamma}}{|x_0-y|^{m+n-1+\gamma}}\right), \tag{3.29}$$

we obtain

$$\begin{aligned} J_3 &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\wedge}_\beta}|Q|^{\beta/n}\int_{\mathbb{R}^n\setminus\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n}}+\frac{|x-x_0|^{\gamma}}{|x_0-y|^{n-1+\gamma}}\right) \\ &\qquad\qquad \times\left(\int_{|x_0-y|\le t,\,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}|f(y)|dy \\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\wedge}_\beta}|Q|^{\beta/n}\sum_{k=1}^{\infty}(2^{-k}+2^{-\gamma k})M(f)(x) \\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\wedge}_\beta}|Q|^{\beta/n}M(f)(x). \end{aligned} \tag{3.30}$$

For J_4, similar to the proof of J_1, J_2, and J_3, we obtain

$$\begin{aligned} J_4 &\le C\sum_{|\alpha|=m}\int_{\mathbb{R}^n\setminus\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x_0-y|^{n+1/2}}+\frac{|x-x_0|^{\gamma}}{|x_0-y|^{n+\gamma}}\right) \\ &\qquad \times|D^\alpha\tilde{A}(y)|\,|f(y)|dy \\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\wedge}_\beta}|Q|^{\beta/n} \\ &\qquad \times\sum_{k=1}^{\infty}\left(2^{k(\beta-1)}+2^{k(\beta-1/2)}+2^{k(\beta-\gamma)}\right)\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|dy \\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\wedge}_\beta}|Q|^{\beta/n}M(f)(x). \end{aligned} \tag{3.31}$$

For $B^{A}_{\delta,*}$, we write

$$\begin{aligned}
B^{\tilde{A}}_{\delta,t}(f)(x)-B^{\tilde{A}}_{\delta,t}(f)(x_0)
&=\int_{\mathbb{R}^n\setminus\tilde{Q}}\left[\frac{B^{\delta}_t(x-y)}{|x-y|^m}-\frac{B^{\delta}_t(x_0-y)}{|x_0-y|^m}\right]\mathbb{R}_m(\tilde{A};x,y)f(y)dy\\
&\quad+\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{B^{\delta}_t(x_0-y)}{|x_0-y|^m}[\mathbb{R}_m(\tilde{A};x,y)-\mathbb{R}_m(\tilde{A};x_0,y)]f(y)dy\\
&\quad-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n\setminus\tilde{Q}}\left(\frac{B^{\delta}_t(x-y)(x-y)^{\alpha}}{|x-y|^m}-\frac{B^{\delta}_t(x_0-y)(x_0-y)^{\alpha}}{|x_0-y|^m}\right)\\
&\qquad\times D^{\alpha}\tilde{A}(y)f(y)dy\\
&=L_1+L_2+L_3.
\end{aligned}\tag{3.32}$$

We consider the following two cases.

Case 1 ($0<t\le l$). In this case, notice that (see [8])

$$|B^{\delta}(z)|\le c(1+|z|)^{-(\delta+(n+1)/2)}.\tag{3.33}$$

We obtain

$$\begin{aligned}
||L_1||&\le Ct^{-n}\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|f(y)||\mathbb{R}_m(\tilde{A};x,y)|}{|x_0-y|^m}(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy\\
&\le C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_{\beta}}|Q|^{\beta/n}(t/l)^{\delta-(n-1)/2}\\
&\qquad\times\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|dy\\
&\le C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_{\beta}}|Q|^{\beta/n}M(f)(x),\\
||L_2||&\le Ct^{-n}\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|f(y)||\mathbb{R}_m(\tilde{A};x,y)-\mathbb{R}_m(\tilde{A};x_0,y)|}{|x_0-y|^m}\\
&\qquad\times(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy\\
&\le C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_{\beta}}|Q|^{\beta/n}(t/l)^{\delta-(n-1)/2}\\
&\qquad\times\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|dy\\
&\le C\sum_{|\alpha|=m}||D^{\alpha}A||_{\dot{\Lambda}_{\beta}}|Q|^{\beta/n}M(f)(x).
\end{aligned}\tag{3.34}$$

For L_3, similar to the proof of L_1, we get

$$\begin{aligned}||L_3|| &\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}(t/l)^{\delta-(n-1)/2} \\ &\qquad\times\sum_{k=1}^{\infty}2^{k(\beta-\delta+(n-1)/2)}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|\,dy \\ &\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M(f)(x).\end{aligned} \tag{3.35}$$

Case 2 ($t > l$)**.** In this case, we choose δ_0 such that $\beta+(n-1)/2 < \delta_0 < \min(\delta,(n+1)/2)$. Notice that (see [8])

$$|(\partial/\partial z)B^\delta(z)| \leq C(1+|z|)^{-(\delta+(n+1)/2)}. \tag{3.36}$$

Similar to the proof of Case 1, we obtain

$$\begin{aligned}||L_1|| &\leq Ct^{-n}\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|f(y)||\mathbb{R}_m(\tilde{A};x,y)|}{|x_0-y|^{m+1}} \\ &\qquad\times|x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\ &\quad+Ct^{-n-1}\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|f(y)||\mathbb{R}_m(\tilde{A};x,y)|}{|x_0-y|^{m}} \\ &\qquad\times|x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\ &\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}(l/t)^{(n+1)/2-\delta_0} \\ &\qquad\times\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta_0)}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|\,dy \\ &\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M(f)(x),\end{aligned}$$

$$\begin{aligned}||L_2|| &\leq Ct^{-n}\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|f(y)||\mathbb{R}_m(\tilde{A};x,y)-\mathbb{R}_m(\tilde{A};x_0,y)|}{|x_0-y|^{m}} \\ &\qquad\times(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\ &\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}(l/t)^{(n+1)/2-\delta_0} \\ &\qquad\times\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta_0)}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|\,dy \\ &\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M(f)(x),\end{aligned}$$

$$\begin{aligned}
\|L_3\| &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} (l/t)^{(n+1)/2-\delta_0} \\
&\qquad \times \sum_{k=1}^{\infty} 2^{k(\beta+(n-1)/2-\delta_0)} \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|\,dy \\
&\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M(f)(x).
\end{aligned} \tag{3.37}$$

These yield the desired results.

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