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EULER CASE FOR A GENERAL FOURTH-ORDER DIFFERENTIAL EQUATION

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We deal with an Euler case for a general fourth-order equation and under this case, we obtain the general formula for the asymptotic form of the solutions.

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1. Introduction. In this paper, we examine the asymptotic form of a fundamental set of solutions of the fourth-order differential equation

$$(p_0y'')''+(p_1y')'+\frac{1}{2}\sum_{j=0}^{1}\left[\{q_{2-j}y^{(i)}\}^{(j+1)}+\{q_{2-j}y^{(j+1)}\}^{(j)}\right]+p_2y=0 \tag{1.1}$$

as $x\to\infty$, where x is the independent variable and the prime denotes d/dx. The functions $p_i(x)$ $(0\le i\le 2)$ and $q_i(x)$ $(i=1,2)$ are defined on an interval $[a,\infty)$, are not necessarily real-valued, and are all nowhere zero in this interval. Our aims are to identify relations between q_0, q_1, p_0, p_1, and p_2 that represents an Euler case for (1.1) and to obtain the asymptotic forms of four linearly independent solutions under this case. Al-Hammadi [2] obtained an asymptotic formula of Liouville-Green type for (1.1) which extends those of Walker [9]. Also in [1], we consider (1.1) with $p_1=q_2=0$ and we give a complete analysis for the case where

$$p_2^{1/3}p_0=o\left(q_1^{4/3}\right)\quad(x\longrightarrow\infty). \tag{1.2}$$

A fourth-order equation similar to (1.1) has been considered previously by Walker [9, 10]. Eastham [4] considered an Euler case for (1.1) with $p_1=q_2=0$ and showed that this case represents a borderline between situations where all solutions have a certain exponential character as $x\to\infty$ and where only two solutions have this character. Al-Hammadi and Eastham [3] considered the case where the coefficients are small for large x.

The Euler case for (1.1) that has been referred to is given by

$$\begin{gathered}\frac{q_i'}{q_i}\sim\text{const}\,\frac{q_1}{q_0}\quad(i=1,2),\\ \frac{\left(p_1q_1^{-1/2}\right)'}{p_1q_1^{-1/2}}\sim\text{const}\,\frac{q_1}{p_0}\end{gathered} \tag{1.3}$$

as $x\to\infty$.

We will use the recent asymptotic theorem of Eastham [6, Section 2] to obtain the solutions of (1.1) under the above case. The main theorem for (1.1) is given in Section 4 with some discussion in Section 5.

2. A transformation of the differential equation. We write (1.1) in the standard way [7] as a first-order system

$$Y' = AY, \tag{2.1}$$

where the first component of Y is y and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2}q_1 p_0^{-1} & p_1^{-1} & 0 \\ -\frac{1}{2}q_2 & -p_1 + \frac{1}{4}q_1^2 p_o^{-1} & -\frac{1}{2}p_0^{-1}q_1 & 1 \\ -p_2 & -\frac{1}{2}q_2 & 0 & 0 \end{bmatrix}. \tag{2.2}$$

As in [1], we express A in its diagonal form

$$T^{-1}AT = \Lambda \tag{2.3}$$

and we therefore require the eigenvalues λ_j and the eigenvectors v_j $(1 \le j \le 4)$ of A.

The characteristic equation of A is given by

$$p_0\lambda^4 + q_1\lambda^3 + p_1\lambda^2 + q_2\lambda + p_2 = 0. \tag{2.4}$$

An eigenvector v_j of A corresponding to λ_j is

$$v_j = \left(1, \lambda_j, p_0\lambda_j^2 + \frac{1}{2}q_1\lambda_j, -\frac{1}{2}q_2 - p_2\lambda_j^{-1}\right)^t, \tag{2.5}$$

where the superscript t denotes the transpose. We assume at this stage that the λ_j are distinct, and we define the matrix T in (2.3) by

$$T = (v_1 \quad v_2 \quad v_3 \quad v_4). \tag{2.6}$$

Now from (2.2), we note that EA is symmetric, where

$$E = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.7}$$

Hence, be [5, Section 2(i)], the v_j have the orthogonality property

$$(Ev_k)^t v_j = 0 \quad (k \ne j). \tag{2.8}$$

We define the scalars m_j $(1 \le j \le 4)$ by

$$m_j = (Ev_j)^t v_j, \tag{2.9}$$

and the row vectors

$$r_j = (Ev_j)^t. \tag{2.10}$$

Hence, by [5, Section 2],

$$T^{-1} = \begin{bmatrix} m_1^{-1} r_1 \\ m_2^{-1} r_2 \\ m_3^{-1} r_3 \\ m_4^{-1} r_4 \end{bmatrix}, \tag{2.11}$$

$$m_j = 4p_0\lambda_j^3 + 3q_1\lambda^2 + 2p_1\lambda_j + q_2. \tag{2.12}$$

Now we define the matrix U by

$$U = (v_1 \quad v_2 \quad v_3 \quad \epsilon_1 \quad v_4) = TK, \tag{2.13}$$

where

$$\epsilon_1 = \frac{p_0 p_1}{q_1^2}. \tag{2.14}$$

The matrix K is given by

$$K = dg(1,1,1,\epsilon_1). \tag{2.15}$$

By (2.3) and (2.13), the transformation

$$Y = UZ \tag{2.16}$$

takes (2.1) into

$$Z' = (\Lambda - U^{-1}U')Z. \tag{2.17}$$

Now by (2.13),

$$U^{-1}U' = K^{-1}T^{-1}T'K + K^{-1}K', \tag{2.18}$$

where

$$K^{-1}K' = dg(0,0,0,\epsilon_1^{-1}\epsilon_1'), \tag{2.19}$$

and we use (2.15).

Now if we write

$$U^{-1}U' = \phi_{ij} \quad (1 \le i, j \le 4), \tag{2.20}$$

$$T^{-1}T' = \psi_{ij} \quad (1 \le i, j \le 4), \tag{2.21}$$

then by (2.18)–(2.21), we have

$$\begin{aligned} \phi_{ij} &= \psi_{ij} \quad (1 \le i, j \le 3), \\ \phi_{44} &= \psi_{44} + \epsilon_1^{-1}\epsilon_1', \\ \phi_{i4} &= \psi_{i4}\epsilon_1 \quad (1 \le i \le 3), \\ \phi_{4j} &= \epsilon_1^{-1}\psi_{4j} \quad (1 \le j \le 3). \end{aligned} \tag{2.22}$$

Now to work out ϕ_{ij} $(1 \le i, j \le 4)$, it suffices to deal with ψ_{ij} of the matrix $T^{-1}T'$. Thus by (2.10), (2.12), (2.6), and (2.11), we obtain

$$\psi_{ii} = \frac{1}{2}\frac{m_i'}{m_i} \quad (1 \le i \le 4), \tag{2.23}$$

and, for $i \ne j$, $1 \le i$, $j \le 4$,

$$\psi_{ij} = m_i^{-1}\left\{\lambda_j'\left(p_0\lambda_i^2 + \frac{1}{2}q_1\lambda_i\right) + \lambda_i\left(p_0\lambda_j^2 + \frac{1}{2}q_1\lambda_j\right)' + -\frac{1}{2}q_2' - \left(p_2\lambda_j^{-1}\right)'\right\}. \tag{2.24}$$

Now we need to work out (2.23) and (2.24) in some detail in terms of p_0, p_1, p_2, q_1, and q_2, then (2.22) in order to determine the form of (2.17).

3. The matrices Λ, $T^{-1}T'$, and $U^{-1}U'$. In our analysis, we impose a basic condition on the coefficients as follows.

(I) p_i $(0 \le i \le 2)$ and q_i $(i = 1, 2)$ are nowhere zero in some interval $[a, \infty)$, and

$$\begin{aligned} \frac{p_i}{q_{i+1}} &= o\left(\frac{q_{i+1}}{p_{i+1}}\right) \quad (i = 0, 1)\ (x \longrightarrow \infty), \\ \frac{q_1}{p_1} &= o\left(\frac{p_1}{q_2}\right). \end{aligned} \tag{3.1}$$

If we write

$$\epsilon_1 = \frac{p_0p_1}{q_1^2}, \qquad \epsilon_2 = \frac{q_1q_2}{p_1^2}, \qquad \epsilon_3 = \frac{p_2p_1}{q_2^2}, \tag{3.2}$$

then by (3.1) for $(1 \le i \le 3)$,

$$\epsilon_i = o(1) \quad (x \longrightarrow \infty). \tag{3.3}$$

Now as in [1], we can solve the characteristic equation (2.4) asymptotically as $x \to \infty$. Using (3.1) and (3.2), we obtain the distinct eigenvalues λ_j as

$$\lambda_1 = -\frac{p_2}{q_2}(1 + \delta_1), \tag{3.4}$$

$$\lambda_2 = -\frac{q_2}{p_1}(1 + \delta_2), \tag{3.5}$$

$$\lambda_3 = -\frac{p_1}{q_1}(1 + \delta_3), \tag{3.6}$$

$$\lambda_4 = -\frac{q_1}{p_0}(1 + \delta_4), \tag{3.7}$$

where

$$\delta_1 = 0(\epsilon_3), \qquad \delta_2 = 0(\epsilon_2) + 0(\epsilon_3), \qquad \delta_3 = 0(\epsilon_1) + 0(\epsilon_2), \qquad \delta_4 = 0(\epsilon_1). \tag{3.8}$$

Now by (3.1), the ordering of λ_j is such that

$$\lambda_j = o(\lambda_{j+1}) \quad (x \longrightarrow \infty,\ 1 \le j \le 3). \tag{3.9}$$

Now we work out m_j $(1 \le j \le 4)$ asymptotically as $x \to \infty$; hence by (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), and (3.8), (2.12) gives, for $1 \le j \le 4$,

$$m_1 = q_2\{1 + 0(\epsilon_3)\}, \tag{3.10}$$

$$m_2 = -q_2\{1 + 0(\epsilon_2) + 0(\epsilon_3)\}, \tag{3.11}$$

$$m_3 = \frac{p_1^2}{q_1}\{1 + 0(\epsilon_1) + 0(\epsilon_2)\}, \tag{3.12}$$

$$m_4 = -\frac{q_1^3}{p_0^2}\{1 + 0(\epsilon_1)\}. \tag{3.13}$$

Also by substituting λ_j $(j = 1,2,3,4)$ into (2.12) and using (3.4), (3.5), (3.6), and (3.7), respectively, and differentiating, we obtain

$$\begin{aligned}
m_1' &= q_2'\{1 + 0(\epsilon_3)\} + q_2\{0(\epsilon_3') + 0(\epsilon_3\delta_1') + 0(\epsilon_2'\epsilon_3^2) + 0(\epsilon_2'\epsilon_3^2\epsilon_3^3)\},\\
m_2' &= -q_2'\{1 + 0(\epsilon_2) + 0(\epsilon_3)\} + q_2\{0(\delta_2') + 0(\epsilon_2') + 0(\epsilon_1'\epsilon_2^2)\},\\
m_3' &= \left(\frac{p_1^2}{q_1}\right)'\{1 + 0(\epsilon_1) + 0(\epsilon_2)\} + \frac{p_1^2}{q_1}\{0(\delta_3') + 0(\epsilon_2') + 0(\epsilon_1')\},\\
m_4' &= -\left(\frac{q_1^3}{p_0^2}\right)'\{1 + 0(\epsilon_1)\} + \frac{q_1^3}{p_0^2}\{0(\delta_4') + 0(\epsilon_2'\epsilon_1^2) + 0(\epsilon_1')\}.
\end{aligned} \tag{3.14}$$

At this stage we also require the following conditions.

(II)

$$\frac{p_0'}{p_0}\epsilon_i,\ \frac{p_1'}{p_1}\epsilon_i,\ \frac{q_1'}{q_1}\epsilon_i,\ \frac{q_2'}{q_2}\epsilon_i,\ \frac{p_2'}{p_2}\epsilon_2,\ \frac{p_2'}{p_2}\epsilon_3 \in L(a,x) \quad (1 \le i \le 3). \tag{3.15}$$

Further, differentiating (3.2) for ϵ_i $(1 \le i \le 3)$, we obtain

$$\begin{aligned}
\epsilon_1' &= 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0\left(\frac{p_1'}{p_1}\epsilon_1\right) + 0\left(\frac{q_2}{q_2}\epsilon_1\right),\\
\epsilon_2' &= 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0\left(\frac{p_1'}{p_1}\epsilon_2\right),\\
\epsilon_3' &= 0\left(\frac{p_2'}{p_2}\epsilon_3\right) + 0\left(\frac{p_1'}{p_1}\epsilon_3\right) + 0\left(\frac{q_2'}{q_2}\epsilon_3\right).
\end{aligned} \tag{3.16}$$

For reference, we note that by substituting (3.4), (3.5), (3.6), and (3.7) into (2.4) and differentiating, we obtain

$$\begin{aligned}\delta_1' &= 0(\epsilon_3') + 0(\epsilon_2'\epsilon_3^2) + 0(\epsilon_2'\epsilon_3^3\epsilon_2^2),\\ \delta_2' &= 0(\epsilon_2') + 0(\epsilon_3') + 0(\epsilon_1'\epsilon_3^2),\\ \delta_3' &= 0(\epsilon_1') + 0(\epsilon_2') + 0(\epsilon_3'\epsilon_2^2),\\ \delta_4' &= 0(\epsilon_1') + 0(\epsilon_2'\epsilon_1^2) + 0(\epsilon_3'\epsilon_1^3\epsilon_2^2).\end{aligned} \tag{3.17}$$

Hence by (3.16) and (3.17), and (3.15),

$$\epsilon_j', \delta_j' \in L(a,\infty). \tag{3.18}$$

For the diagonal elements ψ_{ii} $(1 \le j \le 4)$ in (2.23), we can now substitute the estimates (3.10), (3.11), (3.12), (3.13), and (3.14) into (2.23). We obtain

$$\begin{aligned}\psi_{11} &= \frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0(\epsilon_3') + 0(\epsilon_3\delta_1') + 0(\epsilon_2'\epsilon_3^2) + 0(\epsilon_1'\epsilon_2^2\epsilon_3^3),\\ \psi_{22} &= \frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0(\delta_2') + 0(\epsilon_2') + 0(\epsilon_2'\epsilon_2^2),\\ \psi_{33} &= \frac{1}{2}\left[2\frac{p_1'}{p_1} - \frac{q_1'}{q_1}\right] + 0\left(\frac{p_1'}{p_1}\epsilon_1\right) + 0\left(\frac{p_1'}{p_1}\epsilon_2\right) + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right)\\ &\quad + 0(\delta_3') + 0(\epsilon_2') + 0(\epsilon_1'),\\ \psi_{44} &= \frac{1}{2}\left[3\frac{q_1'}{q_1} - 2\frac{p_0'}{p_0}\right] + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0(\delta_4') + 0(\epsilon_2'\epsilon_1^2) + 0(\epsilon_1').\end{aligned} \tag{3.19}$$

Now for the nondiagonal elements ψ_{ij} $(i \ne j,\ 1 \le i, j \le 4)$, we consider (2.24). Hence (2.24) gives, for $i = 1$ and $j = 2$,

$$\psi_{12} = m_1^{-1}\left\{\lambda_2'\left(p_0\lambda_1^2 + \frac{1}{2}q_1\lambda_1\right) + \lambda_1\left(p_0\lambda_2^2 + \frac{1}{2}q_1\lambda_2\right)' + -\frac{1}{2}q_2' - (p_2\lambda_2^{-1})'\right\}. \tag{3.20}$$

Now by (3.4), (3.5), (3.2), and (3.10), we have

$$m_1^{-1}\lambda_2'\left(p_0\lambda_1^2 + \frac{1}{2}q_1\lambda_1\right) = \frac{1}{2}\left[2\frac{q_2'}{q_2} - \frac{p_1'}{p_1}\right]\epsilon_2\epsilon_3\{1 + 0(\epsilon_3)\} + 0(\epsilon_2\epsilon_3\delta_2'), \tag{3.21}$$

$$\begin{aligned}m_1^{-1}\lambda_1\left(p_0\lambda_1^2 + \frac{1}{2}q_1\lambda_1\right)' &= 0(\epsilon_2\epsilon_3\delta_2') + 0(\epsilon_2^2\epsilon_1\epsilon_3) + \left[\frac{p_0'}{p_0} + 2\frac{q_2'}{q_2} - 2\frac{p_1'}{p_1}\right]\\ &\quad + 0(\epsilon_2\epsilon_3)\left[\frac{q_1'}{q_1} + \frac{q_2'}{q_2} - \frac{p_1'}{p_1}\right],\end{aligned} \tag{3.22}$$

$$-\frac{1}{2}q_2'm_1^{-1} = -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_3\right), \tag{3.23}$$

$$m_1^{-1}(p_2\lambda_2^{-1})' = 0\left(\frac{p_2'}{p_2}\epsilon_3\right) + 0\left(\frac{p_1'}{p_1}\epsilon_3\right) + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0(\epsilon_3\delta_2'). \tag{3.24}$$

Hence by (3.21), (3.22), (3.23), and (3.24), (3.20) gives

$$\psi_{12} = -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0\left(\frac{p_1'}{p_1}\epsilon_3\right) + 0\left(\frac{p_2'}{p_2}\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_2^2\epsilon_3\right) + 0(\epsilon_3\delta_2') + 0\left(\frac{q_1'}{q_1}\epsilon_2\epsilon_3\right). \tag{3.25}$$

Similar work can be done for the other elements ψ_{ij}; so we obtain

$$\begin{aligned}
\psi_{13} &= -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0\left(\frac{p_1'}{p_1}\epsilon_3\right) + 0\left(\frac{q_1'}{q_1}\epsilon_3\right) + 0(\epsilon_3\delta_3') + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_3\right) + 0\left(\frac{p_2'}{p_2}\epsilon_2\epsilon_3\right),\\
\psi_{14} &= -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0\left(\frac{q_1'}{q_1}\epsilon_1^{-1}\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1^{-1}\epsilon_3\right) + 0(\epsilon_1^{-1}\epsilon_3\delta_4')0\left(\frac{p_2'}{p_2}\epsilon_1\epsilon_2\epsilon_3\right),\\
\psi_{21} &= -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0(\delta_1') + 0\left(\epsilon_2\frac{p_2'}{p_2}\right) + 0\left(\epsilon_3\frac{p_2'}{p_2}\right)\\
&\quad + 0\left(\frac{q_1'}{q_1}\epsilon_2\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_2^2\epsilon_3^2\right),\\
\psi_{23} &= \left[\frac{1}{2}\frac{q_1'}{q_1} - \frac{p_1'}{p_1} + \frac{1}{2}\frac{q_2'}{q_2}\right] + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0\left(\frac{q_1'}{q_1}\epsilon_3\right) + 0\left(\frac{p_1'}{p_1}\epsilon_1\right)\\
&\quad + 0\left(\frac{p_1'}{p_1}\epsilon_2\right) + 0\left(\frac{p_1'}{p_1}\epsilon_3\right) + 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0(\delta_3') + 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0\left(\epsilon_2\epsilon_3\frac{p_2'}{p_2}\right),\\
\psi_{24} &= \epsilon_1^{-1}\left[\frac{1}{2}\frac{q_1'}{q_1} + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0\left(\frac{q_1'}{q_1}\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0\left(\frac{p_0'}{p_0}\epsilon_2\right)\right.\\
&\quad \left. + 0\left(\frac{p_0'}{p_0}\epsilon_3\right) + 0(\delta_4') + 0\left(\frac{q_2'}{q_2}\epsilon_1\right) + 0\left(\frac{p_2'}{p_2}\epsilon_1^2\epsilon_2\epsilon_3\right)\right],\\
\psi_{31} &= 0\left(\frac{p_2'}{p_2}\epsilon_2\right) + 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0(\delta_1'\epsilon_2) + 0\left(\frac{q_1'}{q_1}\epsilon_2\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_2^2\epsilon_3^2\right),\\
\psi_{32} &= 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0\left(\frac{p_1'}{p_1}\epsilon_2\right) + 0(\epsilon_2\delta_2') + 0\left(\epsilon_1\epsilon_2^2\frac{p_0'}{p_0}\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0\left(\epsilon_2\epsilon_3\frac{p_2'}{p_2}\right),\\
\psi_{34} &= \epsilon_1^{-1}\left[-\frac{1}{2}\frac{q_1'}{q_1} + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0\left(\frac{p_0'}{p_0}\epsilon_2\right)\right.\\
&\quad \left. + 0(\delta_4') + 0\left(\frac{q_1'}{q_1}\epsilon_1\epsilon_2\right) + 0\left(\frac{p_2'}{p_2}\epsilon_1^2\epsilon_2^2\epsilon_3\right)\right],\\
\psi_{41} &= \epsilon_1\left[0\left(\frac{q_1'}{q_1}\epsilon_2\epsilon_3\right) + 0\left(\frac{q_2'}{q_2}\epsilon_1\epsilon_2\right) + 0\left(\frac{p_2'}{p_2}\epsilon_1\epsilon_2\right) + 0\left(\delta_1'\epsilon_1\epsilon_2\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_2^2\epsilon_3^2\right)\right],\\
\psi_{42} &= 0\left(\frac{q_2'}{q_2}\epsilon_1\epsilon_2\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_2\right) + 0(\delta_2'\epsilon_1\epsilon_2) + 0\left(\frac{q_1'}{q_1}\epsilon_1^2\epsilon_2^2\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1^2\epsilon_2^2\right) + 0\left(\frac{p_2'}{p_2}\epsilon_1^2\epsilon_2\epsilon_3\right),\\
\psi_{43} &= \epsilon_1\left[-\frac{1}{2}\frac{q_1'}{q_1} + 0\left(\frac{p_1'}{p_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0(\delta_3'\epsilon_1) + 0\left(\frac{p_0'}{p_0}\epsilon_1\right)\right.\\
&\quad \left. + 0\left(\frac{p_2'}{p_2}\epsilon_1\epsilon_2^2\epsilon_3\right) + 0\left(\frac{q_2'}{q_2}\epsilon_1\epsilon_2\right)\right].
\end{aligned} \tag{3.26}$$

Now we need to work out (2.22) in order to determine the form (2.17). Now by (3.19), and (3.25) and (3.26), (2.22) will give

$$
\begin{aligned}
&\phi_{11} = \frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_1), \qquad \phi_{22} = \frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_2), \\
&\phi_{33} = \frac{p_1'}{p_1} - \frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_3), \qquad \phi_{44} = \frac{p_1'}{p_1} - \frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_4), \\
&\phi_{12} = \frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_5), \qquad \phi_{13} = -\frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_6), \\
&\phi_{14} = 0(\Delta_7), \qquad \phi_{21} = -\frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_8), \\
&\phi_{23} = \frac{1}{2}\left(\frac{q_1'}{q_1} + \frac{q_2'}{q_2}\right) - \frac{p_1'}{p_1} + 0(\Delta_9), \qquad \phi_{24} = \frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_{10}), \\
&\phi_{31} = 0(\Delta_{11}), \qquad \phi_{32} = 0(\Delta_{12}), \qquad \phi_{34} = -\frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_{13}), \\
&\phi_{41} = 0(\Delta_{14}), \qquad \phi_{42} = 0(\Delta_{15}), \qquad \phi_{43} = -\frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_{16}),
\end{aligned} \tag{3.27}
$$

where

$$
\Delta_i \in L(a,\infty) \quad (1 \le i \le 16) \tag{3.28}
$$

by (3.15) and (3.18).

Now by (3.27) and (3.28), we write the system (2.17) as

$$
Z' = (\Lambda + R + S)Z, \tag{3.29}
$$

where

$$
R = \begin{bmatrix} -\eta_1 & \eta_1 & \eta_1 & 0 \\ \eta_1 & -\eta_1 & \eta_2 - \eta_1 & -\eta_3 \\ 0 & 0 & -\eta_2 & \eta_3 \\ 0 & 0 & \eta_3 & -\eta_2 \end{bmatrix} \tag{3.30}
$$

with

$$
\eta_1 = \frac{1}{2}\frac{q_2'}{q_2}, \qquad \eta_2 = \frac{\left(p_1 q_1^{-1/2}\right)'}{p_1 q_1^{-1/2}}, \qquad \eta_3 = \frac{1}{2}\frac{q_1'}{q_1}, \tag{3.31}
$$

and $S \in L(a,\infty)$ by (3.28).

4. The Euler case.

Now we deal with (1.3) more generally, so we write (1.3) as

$$
\eta_k = \sigma_k \frac{q_1}{p_0}(1 + \varphi_k) \quad (1 \le k \le 3), \tag{4.1}
$$

where σ_k $(1 \le k \le 3)$ are nonzero constants, $\varphi_k(x) \to 0$ $(1 \le k \le 3,\ x \to \infty)$, and also at this stage we let

$$
\varphi_k' \in L(a,\infty) \quad (1 \le k \le 3). \tag{4.2}
$$

We note that by (4.1), the matrix Λ no longer dominates R and so Eastham's theorem [6, Section 2] is not satisfied which means that we have to carry out a second diagonalization of the system (3.29).

First we write

$$\Lambda + R = \lambda_4 \{S_1 + S_2\}, \tag{4.3}$$

and we need to work out the matrix $S_1 = \text{const}$ with the matrix $S_2(x) = o(1)$ as $x \to \infty$ using (3.4), (3.5), (3.6), and (3.7) and the Euler case (4.1). Hence after some calculations, we obtain

$$\begin{gathered} S_1 = \begin{pmatrix} \sigma_1 & -\sigma_1 & -\sigma_1 & 0 \\ -\sigma_1 & \sigma_1 & \sigma_1 - \sigma_2 & \sigma_3 \\ 0 & 0 & \sigma_2 & -\sigma_3 \\ 0 & 0 & -\sigma_3 & \sigma_2 \end{pmatrix}, \\ S_2(x) = \begin{pmatrix} u_1 & u_2 & u_2 & 0 \\ u_2 & u_3 & u_4 & u_5 \\ 0 & 0 & u_6 & -u_5 \\ 0 & 0 & -u_5 & -u_7 \end{pmatrix}, \end{gathered} \tag{4.4}$$

where

$$\begin{gathered} u_1 = \lambda_1 \lambda_4^{-1} - u_2, \qquad u_2 = -\sigma_1 (\varphi_1 - \delta_4)(1 + \delta_4)^{-1}, \\ u_3 = \lambda_2 \lambda_4^{-1} - u_2, \qquad u_4 = -u_2 + u_7, \\ u_5 = \sigma_3 (\varphi_3 - \delta_4)(1 + \delta_4)^{-1}, \qquad u_6 = \lambda_3 \lambda_4^{-1} - u_7, \\ u_7 = -\sigma_2 (\varphi_2 - \delta_4)(1 + \delta_4)^{-1}. \end{gathered} \tag{4.5}$$

It is clear that by (3.9) and (3.8), $S_2(x) \to 0$ as $x \to \infty$. Hence we diagonalize the constant matrix S_1. Now the eigenvalues of the matrix S_1 are given by

$$\alpha_1 = 0, \qquad \alpha_2 = 2\sigma_1, \qquad \alpha_3 = \sigma_2 + \sigma_3, \qquad \alpha_4 = \sigma_2 - \sigma_3. \tag{4.6}$$

Let

$$\sigma_2 \neq (\pm \sigma_3, \pm \sigma_3 + 2\sigma_1). \tag{4.7}$$

Hence by (4.7), the eigenvalues α_i $(1 \le i \le 4)$ are distinct. Thus we use the transformation

$$Z = T_1 W \tag{4.8}$$

in (3.29), where T_1 diagonalizes the constant matrix S_1. Then (3.29) transforms to

$$W' = (\Lambda_1 + M + T_1^{-1} S T_1) W, \tag{4.9}$$

where

$$\begin{gathered}\Lambda_1 = \lambda_4 T_1^{-1} S_1 T_1 = \operatorname{diag}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4) = \lambda_4 \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \\ M = \lambda_4 T_1^{-1} S_2 T_1, \\ T_1^{-1} S T_1 \in L(a, \infty).\end{gathered} \tag{4.10}$$

Now we can apply the asymptotic theorem of Eastham in [6, Section 2] to (4.9) provided only that Λ_1 and M satisfy the conditions in [6, Section 2].

We first require that the υ_j $(1 \le j \le 4)$ are distinct, and this holds because the α_j $(1 \le j \le 4)$ are distinct.

Second, we need to show that

$$\frac{M}{\upsilon_i - \upsilon_j} \longrightarrow 0 \quad (x \longrightarrow \infty) \tag{4.11}$$

for $i \ne j$ and $1 \le i, j \le 4$. Now

$$\frac{M}{\upsilon_i - \upsilon_j} = (\alpha_i - \alpha_j)^{-1} T_1^{-1} S_2 T_1 = o(1) \quad (x \longrightarrow \infty). \tag{4.12}$$

Thus (4.11) holds. Third, we need to show that

$$S_2' \in L(a, \infty). \tag{4.13}$$

Thus it suffices to show that

$$u_i'(x) \in L(a, \infty) \quad (1 \le i \le 8). \tag{4.14}$$

Now, by (3.4), (3.5), (3.6), (3.7), and (4.5),

$$\begin{gathered}u_1' = 0(\epsilon_1' \epsilon_2 \epsilon_3) + 0(\epsilon_2' \epsilon_1 \epsilon_3) + 0(\epsilon_3' \epsilon_1 \epsilon_2) + 0(\varphi') + 0(\delta_4'), \\ u_2' = 0(\varphi') + 0(\delta_4'), \\ u_3' = 0(\epsilon_1' \epsilon_2) + 0(\epsilon_2' \epsilon_1) + 0(\delta_2' \epsilon_1 \epsilon_2) + 0(\varphi_1') + 0(\delta_4'), \\ u_4' = 0(\varphi_1') + 0(\delta_4') + 0(\varphi_2'), \\ u_5' = 0(\varphi_3') + 0(\delta_4'), \\ u_6' = 0(\epsilon_1') + 0(\epsilon_1 \delta_3') + 0(\varphi_2') + 0(\delta_4'), \\ u_7' = 0(\varphi_2') + 0(\delta_4').\end{gathered} \tag{4.15}$$

Thus by (4.15), (3.18), and (4.2), (4.14) holds and consequently (4.13) holds. Now we state our main theorem for (1.1).

Theorem 4.1. *Let the coefficients q_1, q_2, and p_1 in (1.1) be in $C^{(2)}[a, \infty)$ and let p_0 and p_2 be $C^{(1)}[a, \infty)$. Let (3.1), (3.15), (4.1), (4.2), and (4.7) hold.*

Let

$$\begin{gathered}\operatorname{Re} I_j(x) \quad (j = 1, 2), \\ \operatorname{Re}[\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 - 2\eta_1 + 2\eta_2 \pm I_1 \pm I_2]\end{gathered} \tag{4.16}$$

be one sign in $[a,\infty)$, *where*

$$I_1 = \left[4\eta_1^2 + (\lambda_1 - \lambda_2)^2\right]^{1/2}, \quad I_2 = \left[4\eta_3^2 + (\lambda_3 - \lambda_4)^2\right]^{1/2}. \tag{4.17}$$

Then (1.1) has solutions

$$\begin{aligned} y_1 &\sim q_2^{-1/2} \exp\left(\frac{1}{2}\int_a^x [\lambda_1 + \lambda_2 + I_1]dt\right), \\ y_2 &= o\left\{q_1^{1/2} \exp\left(\frac{1}{2}\int_a^x [\lambda_1 + \lambda_2 - I_1]dt\right)\right\}, \\ y_k &= o\left\{q_1^{-1/2} p_1^{-1} \exp\left(\frac{1}{2}\int_a^x [\lambda_3 + \lambda_4 + (-1)^{k+1} I_2]dt\right)\right\} \quad (k = 3,4). \end{aligned} \tag{4.18}$$

Proof. Before applying the theorem in [6, Section 2], we show that the eigenvalues μ_k of $\Lambda_1 + M$ satisfy the dichotomy condition [8]. As in [1], the dichotomy condition holds if

$$(\mu_j - \mu_k) = f + g \quad (j \neq k,\ 1 \le j,\ k \le 4), \tag{4.19}$$

where f has one sign in $[a,\infty)$ and g belongs to $L[a,\infty)$ [6, (1.5)]. Now since the eigenvalues of $\Lambda_1 + M$ are the same as the eigenvalues of $\Lambda + R$, hence by (2.3) and (3.23),

$$\begin{aligned} \mu_k &= \frac{1}{2}(\lambda_3 + \lambda_2 - 2\eta_1) + \frac{1}{2}(-1)^{k+1} I_1 \quad (k = 1,2), \\ \mu_k &= \frac{1}{2}(\lambda_3 + \lambda_4 - 2\eta_2) + \frac{(-1)^{k+1}}{2} I_2 \quad (k = 3,4). \end{aligned} \tag{4.20}$$

Thus by (4.20) and (4.16), (4.19) holds. Since (4.9) satisfies all the conditions for the asymptotic result [6, Section 2], it follows that, as $x \to \infty$, (4.9) has four linearly independent solutions

$$W_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t)dt\right) \tag{4.21}$$

with e_k being the coordinate vector with kth component unity and other components being zero. Now we transform back to Y by means of (2.16) and (4.8), where T_1 in (4.8) is given by

$$T_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \tag{4.22}$$

We obtain

$$Y_k(x) = U T_1 W_k(x) \quad (1 \le k \le 4). \tag{4.23}$$

Now using (2.13), (2.14), (2.15), (4.20), (4.21), (4.22), and (3.31) in (4.23) and carrying out the integration of $(1/2)(q_2'/q)$ and $(q_1^{1/2} p_1^{-1})'/q_1^{1/2} p_1^{-1}$, for $1 \le k \le 4$, we obtain (4.18). □

5. Discussion. (i) In the familiar case, the coefficients which are covered by Theorem 4.1 are $p_i(x) = c_i x^{\alpha_j}$ $(i = 0, 1, 2)$ and $q_i(x) = c_{i+2} x^{\alpha_{i+2}}$ $(i = 1, 2)$ with real constants α_i and c_i $(0 \le i \le 4)$. Then the Euler case (4.2) is given by

$$\alpha_0 - \alpha_3 = 1. \tag{5.1}$$

The values of σ_k $(1 \le k \le 3)$ in (4.1) are given by

$$\sigma_1 = \frac{1}{2}\alpha_4 c_0 c_3^{-1}, \qquad \sigma_2 = \left(\alpha_1 - \frac{1}{2}\alpha_3\right) c_0 c_3^{-1}, \qquad \sigma_3 = \frac{1}{2}\alpha_3 c_0 c_3^{-1}. \tag{5.2}$$

Also in this example, $\varphi_k(x) = 0$ in (4.1).

(ii) Also the theorem covered the class of the coefficients

$$\begin{gathered} p_0 = c_0 x^{\alpha_0} e^{x^b}, \qquad p_1 = c_1 x^{\alpha_1} e^{(1/4)x^b}, \qquad p_2 = c_2 x^{\alpha_2} e^{-3x^b}, \\ q_1 = c_3 x^{\alpha_3} e^{x^b}, \qquad q_2 = c_4 x^{\alpha_4} e^{-x^b} \end{gathered} \tag{5.3}$$

with real constants c_i, α_i $(0 \le i \le 4)$ and $b(> 0)$.

The Euler case (4.1) is given by

$$\alpha_3 - \alpha_0 = b - 1. \tag{5.4}$$

The values of σ_k $(1 \le k \le 4)$ in (4.1) are given by

$$\sigma_1 = \frac{1}{2} b c_0 c_3^{-1}, \qquad \sigma_2 = \frac{1}{2}\sigma_1, \qquad \sigma_3 = -\sigma_1. \tag{5.5}$$

Also

$$\varphi_1 = -\alpha_4 b^{-1} x^{-b}, \qquad \varphi_2 = 4b^{-1}\left(\frac{1}{2}\alpha_3 - \alpha_1\right) x^{-b}, \qquad \varphi_3 = b^{-1}\alpha_3 x^{-b}. \tag{5.6}$$

References

[1] A. S. A. Al-Hammadi, *Asymptotic theory for a class of fourth-order differential equations*, Mathematika **43** (1996), no. 1, 198–208.

[2] ______, *Asymptotic formulae of Liouville-Green type for a general fourth-order differential equation*, Rocky Mountain J. Math. **28** (1998), no. 3, 801–812.

[3] A. S. A. Al-Hammadi and M. S. P. Eastham, *Higher-order differential equations with small oscillatory coefficients*, J. London Math. Soc. (2) **40** (1989), no. 3, 507–518.

[4] M. S. P. Eastham, *Asymptotic theory for a critical class of fourth-order differential equations*, Proc. Roy. Soc. London Ser. A **383** (1982), no. 1785, 465–476.

[5] ______, *On the eigenvectors for a class of matrices arising from quasiderivatives*, Proc. Roy. Soc. Edinburgh Sect. A **97** (1984), 73–78.

[6] ______, *The asymptotic solutions of linear differential systems*, Mathematika **32** (1985), no. 1, 131–138.

[7] W. N. Everitt and A. Zettl, *Generalized symmetric ordinary differential expressions. I. The general theory*, Nieuw Arch. Wisk. (3) **27** (1979), no. 3, 363–397.

[8] N. Levinson, *The asymptotic nature of solutions of linear systems of differential equations*, Duke Math. J. **15** (1948), 111–126.

[9] P. W. Walker, *Asymptotica of the solutions to* $[(ry'')' - py']' + qy = \sigma y$, J. Differential Equations **9** (1971), 108–132.

[10] ______, *Asymptotics for a class of fourth order differential equations*, J. Differential Equations **11** (1972), 321–334.

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