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ON THE PUTNAM-FUGLEDE THEOREM

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We extend the Putnam-Fuglede theorem and the second-degree Putnam-Fuglede theorem to the nonnormal operators and to an elementary operator under perturbation by quasinilpotents. Some asymptotic results are also given.

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1. Introduction.

Let H be a complex Hilbert space and let $B(H)$ be the Banach algebra consisting of all the bounded linear operators on H. For the normal operators, we have the following well-known Putnam-Fuglede (PF) theorem [7].

Theorem 1.1. *If N, M are normal operators in $B(H)$, and if $X \in B(H)$ such that $NX = XM$, then $N^*X = XM^*$.*

Putnam [7] also obtained another important result that we call the second-degree PF (SPF) theorem.

Theorem 1.2. *If N, M are normal operators in $B(H)$, and if $X \in B(H)$ such that $N(NX - XM) = (NX - XM)M$, then $NX = XM$.*

If we let $\mathbf{A} = (N_1, N_2)$ and $\mathbf{B} = (M_1, M_2)$ denote tuples of commuting operators in $B(H)$, and define the elementary operators $\Delta_{(\mathbf{A},\mathbf{B})}$ and $\Delta_{(\mathbf{A}^*,\mathbf{B}^*)} \in B(B(H))$ by

$$\begin{aligned} \Delta_{(\mathbf{A},\mathbf{B})}(X) &= N_1XN_2 - M_1XM_2, \\ \Delta_{(\mathbf{A}^*,\mathbf{B}^*)}(X) &= N_1^*XN_2^* - M_1^*XM_2^*, \end{aligned} \tag{1.1}$$

then an extension of the classical PF theorem, Theorem 1.1, is obtained as follows (see [4, 5]).

Theorem 1.3. *If the operators $N_i, M_i \in B(H)$, $i = 1, 2$, are normal, then $\Delta_{(\mathbf{A},\mathbf{B})}(X) = 0$ for some $X \in B(H)$ implies $\Delta_{(\mathbf{A}^*,\mathbf{B}^*)}(X) = 0$.*

Let $\mathbf{A} = (N_1, N_2)$ and $\mathbf{B} = (M_1, M_2)$. For $n = 2, 3, \ldots$, we define the high-order elementary operator $\Delta^{(n)}_{(\mathbf{A},\mathbf{B})}$ by

$$\Delta^{(n)}_{(\mathbf{A},\mathbf{B})}(X) = \Delta_{(\mathbf{A},\mathbf{B})}\left(\Delta^{(n-1)}_{(\mathbf{A},\mathbf{B})}(X)\right), \quad X \in B(H). \tag{1.2}$$

2. Putnam-Fuglede theorem under perturbation by quasinilpotents

Theorem 2.1. *Let A, B be normal operators, and let C, D be quasinilpotents such that $AC = CA$, $BD = DB$. If $(A + C)X = X(B + D)$ for some $X \in B(H)$, then $AX = XB$.*

Proof. If $(A+C)X = X(B+D)$, then $AX - XB = -(CX - XD)$. For any $N, M \in B(H)$, denote by δ_{NM} the linear operator on $B(H)$:

$$\delta_{NM}(X) = NX - XM; \tag{2.1}$$

then $\delta_{AB}(X) = -\delta_{CD}(X)$, so

$$\delta_{AB}^{(n)}(X) = (-1)^n \delta_{CD}^{(n)}(X). \tag{2.2}$$

Since $\sigma(\delta_{CD}) = \sigma(C) - \sigma(D) = \{0\}$ (see [6]), we have $\sqrt[n]{\|\delta_{CD}^{(n)}\|} \to 0$. But

$$\sqrt[n]{\left\|\delta_{AB}^{(n)}(X)\right\|} \le \sqrt[n]{\left\|\delta_{CD}^{(n)}\right\|}\sqrt[n]{\|X\|}, \tag{2.3}$$

so $\sqrt[n]{\|\delta_{AB}^{(n)}(X)\|} \to 0$. The theorem follows by a result of Anderson and Foiaş [1] which says that if A, B are normal operators, and $\sqrt[n]{\|\delta_{AB}^{(n)}(X)\|} \to 0$, then $AX - XB = 0$. □

Remark 2.2. With the operators A and B being normal, it follows from Theorem 2.1 that $(A+C)X = X(B+D) \Rightarrow (A^* + C)X = X(B^* + D)$. It is, however, not true in general that $(A+C)^*X = X(B+D)^*$ (see [9]).

We give now a simple application of Theorem 2.1.

Corollary 2.3. *Let N be a normal operator and let C be a quasinilpotent that commutes with N. If f is a polynomial of degree n such that $f(N+C) = 0$, then $f^{(k)}(N)C^k = 0$ for $k = 0, 1, \ldots, n$. So C is nilpotent of order at most n. Moreover, if f has no multiple root, then $C = 0$.*

Proof. It is easy to see that

$$f(N+C) = f(N) + f'(N)C + \frac{f''(N)}{2!}C^2 + \cdots + \frac{f^{(n)}(N)}{n!}C^n. \tag{2.4}$$

Applying Theorem 2.1 to (2.4), we have $f(N) = 0$ and

$$f'(N)C + \frac{f''(N)}{2!}C^2 + \cdots + \frac{f^{(n)}(N)}{n!}C^n = 0, \tag{2.5}$$

or

$$\left(f'(N) + \frac{f''(N)}{2!}C + \cdots + \frac{f^{(n)}(N)}{n!}C^{n-1}\right)C = 0. \tag{2.6}$$

Applying Theorem 2.1 again to (2.6) yields $f'(N)C = 0$ and

$$\left(\frac{f''(N)}{2!} + \cdots + \frac{f^{(n)}(N)}{n!}C^{n-2}\right)C^2 = 0. \tag{2.7}$$

So we have $(f''(N)/2!)C^2 = 0, \ldots, (f^{(n)}(N)/n!)C^n = 0$.

If f has no multiple root, then it follows from $f(N) = 0$ that $f'(N)$ is invertible. As $f'(N)C = 0$, we know immediately that $C = 0$. □

Lemma 2.4. *Let $C, M \in B(H)$. If C is quasinilpotent, then the only solution $X \in B(H)$ of $X = CXM$ is $X = 0$.*

Proof. If $X = CXM$, we have, for $n = 2, 3, \ldots$, $X = C^n X M^n$, so

$$\|X\| \le \|C^n\| \|X\| \|M^n\| \le \|C^n\| \|X\| \|M\|^n. \tag{2.8}$$

But with C being quasinilpotent, it follows that

$$\sqrt[n]{\|C^n\| \|M\|^n} = \sqrt[n]{\|C^n\|} \|M\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.9}$$

Thus $\|C^n\| \|M\|^n \to 0$, so $X = 0$ by (2.8). □

Lemma 2.5. *Let N be a normal operator and let C, D be quasinilpotents such that N, C, D mutually commute. If $M \in B(H)$, and $(N+C)X(N+C) = MXD$ for some $X \in B(H)$, then $NXN = 0$.*

Proof. Suppose that $X \in B(H)$ such that $(N+C)X(N+C) = MXD$. If the kernel $\operatorname{Ker}(N) \neq \{0\}$, then letting P be the project from H to $\operatorname{Ker}(N)$, we have $NPXN = 0$, $NXPN = 0$. Therefore, to prove $NXN = 0$, it is sufficient to prove $NP^\perp XP^\perp N = 0$. Thus we can assume that $\operatorname{Ker}(N) = \{0\}$. Let

$$N = \int_{\sigma(N)} \lambda \, dE_\lambda \tag{2.10}$$

be the spectral decomposition of N. Define $\Delta_\epsilon = \{z \mid |z| \le \epsilon\}$, $\Delta_\epsilon^c = \mathbf{C} \setminus \Delta_\epsilon$, and $T_\epsilon = E(\Delta_\epsilon^c) T|_{E(\Delta_\epsilon^c)H}$ for any $T \in B(H)$, then we have

$$(N_\epsilon + C_\epsilon) X_\epsilon (N_\epsilon + C_\epsilon) = M_\epsilon X_\epsilon D_\epsilon, \tag{2.11}$$

but N_ϵ is invertible, so

$$(N_\epsilon + C_\epsilon)^{-1} = N_\epsilon^{-1} + C_\epsilon^o, \tag{2.12}$$

where C_ϵ^o is also quasinilpotent, and

$$X_\epsilon = (N_\epsilon + C_\epsilon)^{-1} M_\epsilon X_\epsilon D_\epsilon (N_\epsilon + C_\epsilon)^{-1}. \tag{2.13}$$

Because $D_\epsilon (N_\epsilon + C_\epsilon)^{-1}$ is quasinilpotent, by Lemma 2.4, we have $X_\epsilon = 0$. Letting $\epsilon \to 0$, we have $X = 0$, so $NXN = 0$. This completes the proof. □

Lemma 2.6. *Let N be a normal operator and let C be quasinilpotent such that $NC = CN$. If $(N+C)X(N+C) = X$ for some $X \in B(H)$, then $NXN = X$.*

Proof. If $\operatorname{Ker}(N) \neq \{0\}$, then let P be the project $H \mapsto \operatorname{Ker}(N)$. If $(N+C)X(N+C) = X$ for some $X \in B(H)$, then $P(N+C)X(N+C) = PX$, so $CPX(N+C) = PX$, but since C is quasinilpotent, by Lemma 2.4, we have $PX = 0$. The same way shows that $XP = 0$. Therefore, we may assume $\operatorname{Ker}(N) = \{0\}$.

Let $N=\int_{\sigma(N)}\lambda dE_\lambda$ be the spectral decomposition of N. Define Δ_ϵ, Δ_ϵ^c, and T_ϵ to be the same as in Lemma 2.5. Then

$$(N_\epsilon+C_\epsilon)X_\epsilon(N_\epsilon+C_\epsilon)=X_\epsilon \tag{2.14}$$

or

$$(N_\epsilon+C_\epsilon)X_\epsilon=X_\epsilon(N_\epsilon+C_\epsilon)^{-1}=X_\epsilon(N_\epsilon^{-1}+C_\epsilon^o), \tag{2.15}$$

where C_ϵ^o is quasinilpotent. So by Theorem 2.1, $N_\epsilon X_\epsilon=X_\epsilon N_\epsilon^{-1}$, or $X_\epsilon=N_\epsilon X_\epsilon N_\epsilon$. Letting $\epsilon\to 0$, we have $NXN=X$. □

Using the same technique as in the proof of Lemma 2.6, we are able to obtain the following theorem.

Theorem 2.7. *Let N,M be normal operators and let C,D be quasinilpotents such that $NC=CN$ and $MD=DM$. If $(N+C)X(N+C)=(M+D)X(M+D)$ for some $X\in B(H)$, then $NXN=MXM$.*

Proof. If $\operatorname{Ker}(N)\neq\{0\}$, then let P be the project: $H\mapsto\operatorname{Ker}(N)$. If $(N+C)X(N+C)=(M+D)X(M+D)$ for some $X\in B(H)$, then $P(N+C)X(N+C)=P(M+D)X(M+D)$, that is, $CPX(N+C)=(M+D)PX(M+D)$. Since C is quasinilpotent, by Lemma 2.5, we have $MPXM=0$. The same method shows that $MXPM=0$. Therefore, we can assume that $\operatorname{Ker}(N)=\{0\}$.

Let $N=\int_{\sigma(N)}\lambda dE_\lambda$ be the spectral decomposition of N. Define Δ_ϵ, Δ_ϵ^c, and T_ϵ to be the same as in Lemma 2.5. Then

$$(N_\epsilon+C_\epsilon)X_\epsilon(N_\epsilon+C_\epsilon)=(M_\epsilon+D_\epsilon)X_\epsilon(M_\epsilon+D_\epsilon). \tag{2.16}$$

If we write $(N_\epsilon+C_\epsilon)^{-1}=N_\epsilon^{-1}+C_\epsilon^o$, where C_ϵ^o is quasinilpotent, then the above equation becomes

$$X_\epsilon=(N_\epsilon^{-1}+C_\epsilon^o)(M_\epsilon+D_\epsilon)X_\epsilon(M_\epsilon+D_\epsilon)(N_\epsilon^{-1}+C_\epsilon^o) \tag{2.17}$$

or

$$X_\epsilon=(N_\epsilon^{-1}M_\epsilon+F_\epsilon)X_\epsilon(N_\epsilon^{-1}M_\epsilon+F_\epsilon), \tag{2.18}$$

where F_ϵ is quasinilpotent. Applying Lemma 2.6 to the equation yields $X_\epsilon=N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon$ or $N_\epsilon X_\epsilon N_\epsilon=M_\epsilon X_\epsilon M_\epsilon$. Letting $\epsilon\to 0$, we have $NXN=MXM$. □

More generally, using Berberian's trick, we obtain the PF theorem under perturbation by quasinilpotents for the elementary operators.

Theorem 2.8. *Let N_1, N_2, M_1, M_2 be normal operators and let C_1, C_2, D_1, D_2 be quasinilpotents such that N_i,M_i,C_i,D_i mutually commute for $i=1,2$. If $(N_1+C_1)X(N_2+C_2)=(M_1+D_1)X(M_2+D_2)$ for some $X\in B(H)$, then $N_1XN_2=M_1XM_2$.*

Proof. Let

$$\tilde{T}=\begin{pmatrix}T_1 & \\ & T_2\end{pmatrix},\qquad \tilde{X}=\begin{pmatrix}0 & X\\ 0 & 0\end{pmatrix},\tag{2.19}$$

where $T=N,M,C,D$; then $\tilde{N}$, $\tilde{M}$ are normal, and $\tilde{C}$, $\tilde{D}$ are quasinilpotents in $B(H\oplus H)$. If $(N_1+C_1)X(N_2+C_2)=(M_1+D_1)X(M_2+D_2)$, then $(\tilde{N}+\tilde{C})\tilde{X}(\tilde{N}+\tilde{C})=(\tilde{M}+\tilde{D})\tilde{X}(\tilde{M}+\tilde{D})$, so $\tilde{N}\tilde{X}\tilde{N}=\tilde{M}\tilde{X}\tilde{M}$ by Theorem 2.7, that is, $N_1XN_2=M_1XM_2$. □

3. Second-degree PF theorem. First we will extend Theorem 1.2 to the more general case.

Theorem 3.1. *Let N_1, N_2, M_1, M_2 be normal operators such that $N_1M_1=M_1N_1$, $N_2M_2=M_2N_2$. If $N_1(N_1XN_2-M_1XM_2)N_2=M_1(N_1XN_2-M_1XM_2)M_2$ for some $X\in B(H)$, then $N_1XN_2-M_1XM_2=0$.*

Proof. First we will prove that if N,M are normal operators, then $N(NXN-MXM)N=M(NXN-MXM)M$ implies $NXN=MXM$.

If $\operatorname{Ker}(N)\neq\{0\}$, then letting P be the project $H\mapsto\operatorname{Ker}(N)$, we have $PN(NXN-MXM)N=PM(NXN-MXM)M$. That is, $0=-M^2PXM^2$ or $M(M(PXM^2)-(PXM^2)0)=(M(PXM^2)-(PXM^2)0)0$. By the SPF theorem (Theorem 1.2), $MPXM^2=0$. By the same way, we have $MPXM=0$. Similarly, $MXPM=0$. So we may assume that $\operatorname{Ker}(N)=\{0\}$.

Let T_ϵ be the same as in Lemma 2.5. If $X\in B(H)$ such that

$$N(NXN-MXM)N=M(NXN-MXM)M,\tag{3.1}$$

then

$$N_\epsilon(N_\epsilon X_\epsilon N_\epsilon-M_\epsilon X_\epsilon M_\epsilon)N_\epsilon=M_\epsilon(N_\epsilon X_\epsilon N_\epsilon-M_\epsilon X_\epsilon M_\epsilon)M_\epsilon\tag{3.2}$$

or

$$X_\epsilon-N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon=N_\epsilon^{-1}M_\epsilon(X_\epsilon-N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon)N_\epsilon^{-1}M_\epsilon.\tag{3.3}$$

Since $N_\epsilon^{-1}M_\epsilon$ is normal, by [2], we have

$$X_\epsilon-N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon=0\tag{3.4}$$

or

$$N_\epsilon X_\epsilon N_\epsilon=M_\epsilon X_\epsilon M_\epsilon.\tag{3.5}$$

Letting $\epsilon\to 0$, we have $NXN=MXM$.

In general, let

$$\tilde{N}=\begin{pmatrix}N_1 & \\ & N_2\end{pmatrix},\qquad \tilde{M}=\begin{pmatrix}M_1 & \\ & M_2\end{pmatrix},\qquad \tilde{X}=\begin{pmatrix}0 & X\\ 0 & 0\end{pmatrix}.\tag{3.6}$$

If

$$N_1(N_1XN_2 - M_1XM_2)N_2 = M_1(N_1XN_2 - M_1XM_2)M_2, \tag{3.7}$$

then

$$\tilde{N}(\tilde{N}\tilde{X}\tilde{N} - \tilde{M}\tilde{X}\tilde{M})\tilde{N} = \tilde{M}(\tilde{N}\tilde{X}\tilde{N} - \tilde{M}\tilde{X}\tilde{M})\tilde{M}; \tag{3.8}$$

so $\tilde{N}\tilde{X}\tilde{N} = \tilde{M}\tilde{X}\tilde{M}$, that is, $N_1XN_2 = M_1XM_2$. □

Let $\mathbf{A} = (N_1, N_2)$, $\mathbf{B} = (M_1, M_2)$ be tuples of commuting operators in $B(H)$. We say that $(\mathbf{A}, \mathbf{B})$ has the SPF theorem if for any $X \in B(H)$ and for some $n \geq 2$ such that $\Delta^{(n)}_{(\mathbf{A},\mathbf{B})}(X) = 0$, we have $\Delta_{(\mathbf{A},\mathbf{B})}(X) = 0$.

Theorem 3.2. *Let $N, M, D \in B(H)$ such that N commutes with D and M. If N is invertible and D is quasinilpotent, then $((N,N),(M,D))$ has the SPF theorem.*

Proof. If

$$N(NXN - MXD)N = M(NXN - MXD)D, \tag{3.9}$$

then

$$X - N^{-1}MXN^{-1}D = N^{-1}M(X - N^{-1}MXN^{-1}D)N^{-1}D. \tag{3.10}$$

Note that $N^{-1}D$ is quasinilpotent; so by applying Lemma 2.4 to $X - N^{-1}MXN^{-1}D$, we have $X - N^{-1}MXN^{-1}D = 0$, that is, $NXN - MXD = 0$. □

Theorem 3.3. *Let $N, M \in B(H)$ such that N commutes with M. If M is invertible and $\|N\|\|M^{-1}\| \leq 1$, then $((N,N),(M,M))$ has the SPF theorem.*

Proof. If (3.1) holds for some $X \in B(H)$, then

$$NM^{-1}XNM^{-1} - X = NM^{-1}(NM^{-1}XNM^{-1} - X)NM^{-1}. \tag{3.11}$$

Since $\|N\|\|M^{-1}\| \leq 1$, by [2], we have $NM^{-1}XNM^{-1} - X = 0$, that is, $NXN = MXM$. □

The next theorem establishes the relationship between the SPF theorem and the PF theorem under perturbation by nilpotents.

Theorem 3.4. *Let $N_i, M_i \in B(H)$ and let C_i, D_i be nilpotents such that C_i, D_i, N_i, M_i mutually commute for $i = 1,2$. If $((N_1,N_2),(M_1,M_2))$ has the SPF theorem, then $(N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2)$ implies that $N_1XN_2 = M_1XM_2$.*

Proof. If

$$(N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2), \tag{3.12}$$

then by expanding both sides of the equation and moving M_1XM_2 to the left-hand side and moving all the terms in the left-hand side to the right-hand side except N_1XN_2, we have

$$N_1XN_2 - M_1XM_2 = S(X), \tag{3.13}$$

where S is a linear operator on $B(H)$ defined by

$$S(X) = -N_1XC_2 - C_1XN_2 - C_1XC_2 + M_1XD_2 + D_1XM_2 + D_1XD_2. \tag{3.14}$$

It is clear that $S^{(2)}(X) = S(S(X))$ consists of 6^2 terms like

$$(-1)^l N_1^{m_1} M_1^{n_1} C_1^{s_1} D_1^{t_1} X N_2^{m_2} M_2^{n_2} C_2^{s_2} D_2^{t_2}, \quad \text{where } s_1 + t_1 + s_2 + t_2 \geq 2, \dots, \tag{3.15}$$

$S^{(n)}(X)$ consists of 6^n terms like $(-1)^l N_1^{m_1} M_1^{n_1} C_1^{s_1} D_1^{t_1} X N_2^{m_2} M_2^{n_2} C_2^{s_2} D_2^{t_2}$, where $s_1 + t_1 + s_2 + t_2 \geq n$.

Since C_1, C_2, D_1, D_2 are all nilpotents, we have n_0 such that $C_1^{n_0} = D_1^{n_0} = C_2^{n_0} = D_2^{n_0} = 0$. Thus for each term of $S^{(4n_0+1)}(X)$, as $s_1 + t_1 + s_2 + t_2 \geq 4n_0 + 1$, we have at least one integer among s_1, s_2, t_1, t_2 greater than n_0, so every term of $S^{(4n_0+1)}(X)$ is 0. Therefore, $S^{(4n_0+1)}(X) = 0$. But

$$\Delta_{((N_1,N_2),(M_1,M_2))}{}^{(4n_0+1)}(X) = S^{(4n_0+1)}(X) = 0, \tag{3.16}$$

and $((N_1,N_2),(M_1,M_2))$ has the SPF theorem; so it follows that

$$\Delta_{((N_1,N_2),(M_1,M_2))}(X) = 0, \tag{3.17}$$

or $N_1XN_2 = M_1XM_2$. □

By Theorems 3.3 and 3.4, it is easy to see the following.

Theorem 3.5. *Let $N, M \in B(H)$ and let C, D be nilpotents such that N, M, C, D mutually commute. If M is invertible and $\|N\|\|M^{-1}\| \leq 1$, then $(N+C)X(N+C) = (M+D)X(M+D)$ implies $NXN = MXM$.*

Moreover, if the strict inequality in Theorem 3.5 holds, then Theorem 3.5 is true even for the quasinilpotent operators.

Theorem 3.6. *Let $N, M \in B(H)$ and let C, D be quasinilpotents such that N, M, C, D mutually commute. If M is invertible and $\|N\|\|M^{-1}\| < 1$, then $(N+C)X(N+C) = (M+D)X(M+D)$ implies $X = 0$.*

Proof. If D is quasinilpotent and M is invertible, then $M+D$ is invertible. If $(N+C)X(N+C) = (M+D)X(M+D)$ for some $X \in B(H)$, then

$$(N+C)(M+D)^{-1}X(N+C)(M+D)^{-1} = X \tag{3.18}$$

or

$$(NM^{-1}+F)X(NM^{-1}+F) = X, \tag{3.19}$$

where F is quasinilpotent. By [3],

$$\sigma(\Delta_{((NM^{-1}+F,NM^{-1}+F),(I,I))}) = \sigma(NM^{-1})\sigma(NM^{-1}) - 1. \tag{3.20}$$

Since $\|N\|\|M^{-1}\| < 1$, 0 is not in

$$\sigma(\Delta_{((NM^{-1}+F,NM^{-1}+F),(I,I))}), \tag{3.21}$$

and therefore $\Delta_{((NM^{-1}+F,NM^{-1}+F),(I,I))}$ is invertible. It follows from the equation

$$\Delta_{((NM^{-1}+F,NM^{-1}+F),(I,I))}(X) = 0 \tag{3.22}$$

that $X = 0$. □

The following results show that even if $((A,A),(B,B))$ has the SPF theorem, we still do not know if $((A^2,A^2),(B^2,B^2))$ has the SPF theorem.

Theorem 3.7. *Let $A,B \in B(H)$. Let ω be an nth root of 1, but $\omega^k \neq 1$ for k such that $1 \le k \le n-1$. If for any k such that $0 \le k \le n-1$, $((A,A),(B,\omega^k B))$ has the SPF theorem, then $((A^n,A^n),(B^n,B^n))$ has the SPF theorem too.*

Proof. By induction, we can prove that

$$\begin{aligned}&\Delta_{((A^n,A^n),(B^n,B^n))}(X)\\ &\quad = \Delta_{((A,A),(B,B))}(\Delta_{((A,A),(B,\omega B))}(\cdots(\Delta_{((A,A),(B,\omega^{(n-1)}B))}(X))\cdots)).\end{aligned} \tag{3.23}$$

Now if

$$\Delta^{(2)}_{((A^n,A^n),(B^n,B^n))}(X) = 0, \tag{3.24}$$

then

$$\Delta^{(2)}_{((A,A),(B,B))}(\Delta^{(2)}_{((A,A),(B,\omega B))}(\cdots(\Delta^{(2)}_{((A,A),(B,\omega^{(n-1)}B))}(X))\cdots)) = 0. \tag{3.25}$$

Since $((A,A),(B,B))$ has the SPF theorem, it follows that

$$\Delta_{((A,A),(B,B))}(\Delta^{(2)}_{((A,A),(B,\omega B))}(\cdots(\Delta^{(2)}_{((A,A),(B,\omega^{(n-1)}B))}(X))\cdots)) = 0. \tag{3.26}$$

or

$$\Delta^{(2)}_{((A,A),(B,\omega B))}(\Delta_{((A,A),(B,B))}(\cdots(\Delta^{(2)}_{((A,A),(B,\omega^{(n-1)}B))}(X))\cdots)) = 0, \tag{3.27}$$

and therefore

$$\Delta_{((A,A),(B,\omega B))}(\Delta_{((A,A),(B,B))}(\cdots(\Delta^{(2)}_{((A,A),(B,\omega^{(n-1)}B))}(X))\cdots)) = 0. \tag{3.28}$$

Proceeding in this way, we have finally

$$\Delta_{((A,A),(B,B))}(\Delta_{((A,A),(B,\omega B))}(\cdots(\Delta_{((A,A),(B,\omega^{(n-1)}B))}(X))\cdots)) = 0, \tag{3.29}$$

that is, by (3.23),

$$\Delta_{((A^n,A^n),(B^n,B^n))}(X) = 0. \tag{3.30}$$

□

The following result says that the converse of Theorem 3.8 is also true.

Theorem 3.8. *Let $A,B \in B(H)$. Let ω be an nth root of 1, but $\omega^k \neq 1$ for k such that $1 \le k \le n-1$. If A or B is invertible and $((A^n,A^n),(B^n,B^n))$ has the SPF theorem, then for any k such that $0 \le k \le n-1$, $((A,A),(B,\omega^k B))$ has the SPF theorem too.*

Proof. It is sufficient to prove that if (A^n,B^n) has the SPF theorem and B is invertible, then $((A,A),(B,B))$ has the SPF theorem. Now if

$$A(AXA-BXB)A = B(AXA-BXB)B, \tag{3.31}$$

then

$$A^n(AXA-BXB)A^n = B^n(AXA-BXB)B^n \tag{3.32}$$

or

$$A^n(A^nXA^n-B^nXB^n)A^n = B^n(A^nXA^n-B^nXB^n)B^n; \tag{3.33}$$

so (3.24) holds. Since $((A^n,A^n),(B^n,B^n))$ has the SPF theorem, we have (3.30). It follows from (3.23) that (3.29) holds. From (3.31), we see that

$$\Delta^{(2)}_{((A,A),(B,B))}\big(\Delta_{((A,A),(B,\omega^2 B))}\big(\cdots\big(\Delta_{((A,A),(B,\omega^{(n-1)}B))}(X)\big)\cdots\big)\big) = 0. \tag{3.34}$$

Note that

$$\Delta_{((A,A),(B,B))}(Y) - \Delta_{((A,A),(B,\omega B))}(Y) = (\omega-1)BYB. \tag{3.35}$$

Since B is invertible, (3.29) and (3.34) will give

$$\Delta_{((A,A),(B,B))}\big(\Delta_{((A,A),(B,\omega^2 B))}\big(\cdots\big(\Delta_{((A,A),(B,\omega^{(n-1)}B))}(X)\big)\cdots\big)\big) = 0. \tag{3.36}$$

From (3.31), we see also that

$$\Delta^{(2)}_{((A,A),(B,B))}\big(\Delta_{((A,A),(B,\omega^3 B))}\big(\cdots\big(\Delta_{((A,A),(B,\omega^{(n-1)}B))}(X)\big)\cdots\big)\big) = 0; \tag{3.37}$$

then (3.36) and (3.37) yields

$$\Delta_{((A,A),(B,B))}\big(\Delta_{((A,A),(B,\omega^3 B))}\big(\cdots\big(\Delta_{((A,A),(B,\omega^{(n-1)}B))}(X)\big)\cdots\big)\big) = 0. \tag{3.38}$$

Proceeding in this way, we have finally

$$\Delta_{((A,A),(B,B))}\big(\Delta_{((A,A),(B,\omega^{(n-1)}B))}(X)\big) = 0. \tag{3.39}$$

Now (3.31) and (3.39) will give the desired equation: $AXA-BXB = 0$. □

Theorem 3.9. *If C, D are nilpotents such that $CD = DC$ but $C^2 \neq D^2$, then $((C,C),(D,D))$ does not have the SPF theorem.*

Proof. It is not difficult to see that

$$\Delta^{(n)}_{((C,C),(D,D))}(I) = \sum_{k=0}^{n} (-1)^k C_n^k C^{2n-2k} D^{2k}, \tag{3.40}$$

where I is the identity operator.

If C, D are nilpotents, then there exists an n_0 such that $C^{n_0} = 0$, $D^{n_0} = 0$. For any k such that $1 \leq k \leq n_0$, at least one of $2n_0 + 2 - 2k$ and $2k$ is greater than n_0. So by (3.40), we have

$$\Delta^{(n_0+1)}_{((C,C),(D,D))}(I) = 0. \tag{3.41}$$

But $\Delta_{((C,C),(D,D))}(I) = C^2 - B^2 \neq 0$. This completes the proof. □

4. Asymptotic PF theorem and compact operators. We now give a theorem about the compact operators, which generalizes the relative result in [2].

Theorem 4.1. *Let $\mathbf{A} = (N_1, N_2)$ and $\mathbf{B} = (M_1, M_2)$ be tuples of commuting normal operators in $B(H)$. If $X \in B(H)$ such that $\Delta^{(n)}_{(\mathbf{A},\mathbf{B})}(X)$ is compact for some $n \geq 2$, then $\Delta_{(\mathbf{A},\mathbf{B})}(X)$ is compact too.*

Proof. Let $K(H)$ be the ideal of $B(H)$ consisting of all compact operators on H, let $B(H)/K(H)$ be the Calkin algebra, and let π be the Calkin map from $B(H)$ to $B(H)/K(H)$. It is clear that

$$\pi\left(\Delta^{(n)}_{((N_1,N_2),(M_1,M_2))}(X)\right) = \Delta^{(n)}_{((\pi(N_1),\pi(N_2)),(\pi(M_1),\pi(M_2)))}\big(\pi(X)\big). \tag{4.1}$$

If $\Delta^{(n)}_{((N_1,N_2),(M_1,M_2))}(X)$ is compact, then $\pi(\Delta^{(n)}_{((N_1,N_2),(M_1,M_2))}(X)) = 0$. It follows that

$$\Delta^{(n)}_{((\pi(N_1),\pi(N_2)),(\pi(M_1),\pi(M_2)))}\big(\pi(X)\big) = 0. \tag{4.2}$$

Since $\pi(N_i)$, $\pi(M_i)$ are normal, for $i = 1, 2$, applying Theorem 3.1, we have

$$\Delta_{((\pi(N_1),\pi(N_2)),(\pi(M_1),\pi(M_2)))}\big(\pi(X)\big) = 0. \tag{4.3}$$

Therefore, $\Delta_{((N_1,N_2),(M_1,M_2))}(X)$ is compact. □

The following theorem is an asymptotic version of the SPF theorem. It generalizes the corresponding result in [10].

Theorem 4.2. *Let $\mathbf{A} = (N_1, N_2)$ and $\mathbf{B} = (M_1, M_2)$ be tuples of commuting normal operators in $B(H)$. Let K be any positive real number and let n be an integer greater than 1. Then for every neighborhood U of 0 in $B(H)$ (under uniform, strong or weak topology), a neighborhood V of 0 under the same topology is obtained such that if $\Delta^{(n)}_{(\mathbf{A},\mathbf{B})}(X) \in V$ and $\|X\| \leq K$, then $\Delta_{(\mathbf{A},\mathbf{B})}(X) \in U$.*

Proof. We first consider the following particular case: $N_1 = N_2 = N$, $M_1 = M_2 = M$. Assume that $\|N\|$ and $\|M\|$ are not greater than 1 (if not, we can replace N and M by $N/(\|N\|+\|M\|)$ and $M/(\|N\|+\|M\|)$, resp.).

Let $K > 0$ and let U be any neighborhood of 0 in $B(H)$ under uniform (or strong or weak) topology. Let U_{ij}, $i,j = 1,2,3,4$, be neighborhoods of 0 in $B(H)$ under the same topology such that

$$\sum_{i=1}^{4}\sum_{j=1}^{4} U_{ij} \subset U. \tag{4.4}$$

Suppose that N, M have the following spectral decomposition:

$$N = \int_{\sigma(N)} \lambda\, dE_\lambda, \qquad M = \int_{\sigma(M)} \lambda\, dF_\lambda. \tag{4.5}$$

For any $\epsilon > 0$, define $\Delta_\epsilon = \{z \mid |z| \le \epsilon\}$, $\Delta_\epsilon^c = \mathbf{C} \setminus \Delta_\epsilon$, and

$$\begin{aligned} H_1(\epsilon) &= E(\Delta_\epsilon)F(\Delta_\epsilon)H, \\ H_2(\epsilon) &= E(\Delta_\epsilon)F(\Delta_\epsilon^c)H, \\ H_3(\epsilon) &= E(\Delta_\epsilon^c)F(\Delta_\epsilon)H, \\ H_4(\epsilon) &= E(\Delta_\epsilon^c)F(\Delta_\epsilon^c)H. \end{aligned} \tag{4.6}$$

Then H can be written as $H = H_1(\epsilon)\oplus H_2(\epsilon)\oplus H_3(\epsilon)\oplus H_4(\epsilon)$. Under this decomposition, we have

$$\begin{aligned} N &= \begin{pmatrix} N_1(\epsilon) & & & \\ & N_2(\epsilon) & & \\ & & N_3(\epsilon) & \\ & & & N_4(\epsilon) \end{pmatrix}, \\ M &= \begin{pmatrix} M_1(\epsilon) & & & \\ & M_2(\epsilon) & & \\ & & M_3(\epsilon) & \\ & & & M_4(\epsilon) \end{pmatrix}, \end{aligned} \tag{4.7}$$

where $\|N_1(\epsilon)\|$, $\|N_2(\epsilon)\|$, $\|M_1(\epsilon)\|$, $\|M_3(\epsilon)\|$ are not greater than ϵ, and $N_3(\epsilon)$, $N_4(\epsilon)$, $M_2(\epsilon)$, and $M_4(\epsilon)$ are invertible.

Let $X = ((X_{ij}(\epsilon)))_{i,j=1,2,3,4}$ and let Z denote the set

$$Z = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,3),(3,1),(3,2),(4,1)\}. \tag{4.8}$$

If $(k,l) \in Z$, then at least one operator in each pair of (N_k,N_l), (M_k,M_l) has norm less than ϵ. Hence

$$\|N_k(\epsilon)X_{kl}(\epsilon)N_l(\epsilon)-M_k(\epsilon)X_{kl}(\epsilon)M_l(\epsilon)\|\longrightarrow 0 \quad \text{as } \epsilon\longrightarrow 0. \tag{4.9}$$

Therefore, we are able to choose a fixed number $\epsilon_0>0$ such that for each pair $(k,l)\in Z$,

$$\left(\delta_{ij}(k,l)\Delta_{((N_i(\epsilon_0),N_j(\epsilon_0)),(M_i(\epsilon_0),M_j(\epsilon_0)))}(X_{ij}(\epsilon_0))\right)_{4\times 4}\in U_{kl}, \tag{4.10}$$

where $\delta_{ij}(k,l)$ equals 1 if $i=k$, $j=l$ and 0 otherwise. Set $V_{kl}=U_{kl}$.

For the sake of simplicity, we will omit ϵ_0 in the notations of each component in the decompositions of H, N, M, X.

It is easy to see that $\Delta^{(n)}_{(\mathbf{A},\mathbf{B})}(X)$ has the following decomposition:

$$\Delta^{(n)}_{((N,N),(M,M))}(X)=\left(\Delta^{(n)}_{((N_i,N_j),(M_i,M_j))}(X_{ij})\right)_{4\times 4}. \tag{4.11}$$

If (k,l) is not in Z, then at least one pair of (N_k,N_l) and (M_k,M_l) has two invertible operators. We assume that N_k and N_l are invertible (we can follow the same way if M_k, M_l are invertible).

Let

$$O_{kl}=\left\{o_{kl}:(\delta_{ij}(k,l)o_{ij})_{4\times 4}\in U_{ij}\right\}. \tag{4.12}$$

Then O_{kl} is a neighborhood of 0 in $B(H_l,H_k)$.

Since N_k, N_l are invertible, we can see that

$$\Delta^{(n)}_{((N_k,N_l),(M_k,M_l))}(X_{kl})=N_k^n\Delta^{(n)}_{((I_k,I_l),(N_k^{-1}M_k,N_l^{-1}M_l))}(X_{kl})N_l^n, \tag{4.13}$$

where I_k, I_l are identities on H_k, H_l. It follows from the asymptotic PF theorem in [2] that there is the neighborhood P_{kl} of 0 in $B(H_l,H_k)$ such that for $\|X_{kl}\|\le K$, if

$$\Delta^{(n)}_{((I_k,I_l),(N_k^{-1}M_k,N_l^{-1}M_l))}(X_{kl})\in P_{kl}, \tag{4.14}$$

then

$$\Delta_{((I_k,I_l),(N_k^{-1}M_k,N_l^{-1}M_l))}(X_{kl})\in N_k^{-1}O_{kl}N_l^{-1}. \tag{4.15}$$

Set $V_{kl}=N_k^nP_{kl}N_l^n$. If

$$\Delta^{(n)}_{((N_k,N_l),(M_k,M_l))}(X_{kl})\in V_{kl}, \tag{4.16}$$

then

$$\Delta_{((N_k,N_l),(M_k,M_l))}(X_{kl})\in O_{kl}. \tag{4.17}$$

Let

$$V=\left\{(v_{ij})_{4\times 4}:v_{ij}\in V_{ij}\right\}. \tag{4.18}$$

Then V is a neighborhood of 0. If $\|X\| \le K$ and $\Delta^{(n)}_{(\mathbf{A},\mathbf{B})}(X) \in V$, then for each pair (k,l), $\|X_{kl}\| \le K$ and (4.16) holds; so it follows that (4.17) holds, that is,

$$\left(\delta_{ij}(k,l)\Delta_{((N_k,N_l),(M_k,M_l))}(X_{kl})\right)_{4\times 4} \in U_{kl}, \tag{4.19}$$

but

$$\Delta_{(\mathbf{A},\mathbf{B})}(X) = \sum_{k=1}^{4}\sum_{l=1}^{4}\left(\delta_{ij}(k,l)\Delta_{((N_k,N_l),(M_k,M_l))}(X_{kl})\right)_{4\times 4}, \tag{4.20}$$

which is in U by (4.4).

In general, let

$$\tilde{N} = \begin{pmatrix} N_1 & \\ & N_2 \end{pmatrix}, \qquad \tilde{M} = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix}. \tag{4.21}$$

Then $\tilde{N}$, $\tilde{M}$ are normal in $B(H \oplus H)$. Let

$$\tilde{U} = \left\{ \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} : u_i \in U,\ i = 1,2,3,4 \right\}. \tag{4.22}$$

$\tilde{U}$ is a neighborhood of 0 in $B(H \oplus H)$. So there is a neighborhood $\tilde{V}$ of 0 in $B(H \oplus H)$ such that if $\|\tilde{X}\| \le K$, $\Delta^{(n)}_{(\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\tilde{X}) \in \tilde{V}$, then $\Delta_{(\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\tilde{X}) \in \tilde{U}$. Let

$$\tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \qquad V = \left\{ v : \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in \tilde{V} \right\}. \tag{4.23}$$

V is a neighborhood of 0 in $B(H)$. If $\|X\| \le K$, $\Delta^{(n)}_{(\mathbf{A},\mathbf{B})}(X) \in V$, then $\|\tilde{X}\| \le K$ and $\Delta^{(n)}_{(\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\tilde{X}) \in \tilde{V}$; so $\Delta_{(\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\tilde{X}) \in \tilde{U}$, which means that

$$\begin{pmatrix} 0 & \Delta_{(\mathbf{A},\mathbf{B})}(X) \\ 0 & 0 \end{pmatrix} \in \tilde{U} \tag{4.24}$$

or $\Delta_{(\mathbf{A},\mathbf{B})}(X) \in U$. □

Using the same technique, we are able to generalize the asymptotic PF theorems obtained by Moore [6] and Rogers [8].

Theorem 4.3. *Let N_1, N_2, M_1, M_2, k be the same as in Theorem 4.2. Then for any neighborhood U of 0 in $B(H)$ (under uniform, strong or weak topology), a neighborhood V of 0 under the same topology is obtained such that if $N_1^* X N_2^* - M_1^* X M_2^* \in V$ and $\|X\| \le K$, then $NXN - MXM \in U$.*

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