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ON A DIFFERENCE EQUATION WITH MIN-MAX RESPONSE

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We investigate the global behavior of the (positive) solutions of the difference equation $x_{n+1} = \alpha_n + F(x_n,\ldots,x_{n-k})$, $n = 0,1,\ldots$, where (α_n) is a sequence of positive reals and F is a min-max function in the sense introduced here. Our results extend several results obtained in the literature.

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1. Introduction. Let k be a positive integer and let $\mathbb{R}_+$ be the set of all positive reals. We give the following definition.

Definition 1.1. A function $F : \mathbb{R}_+^{k+1} \to \mathbb{R}_+$ is called a min-max function if it satisfies the inequality

$$\frac{\wedge_{j=1}^{k+1} u_j}{\vee_{j=1}^{k+1} u_j} \le F(u_1,u_2,\ldots,u_{k+1}) \le \frac{\vee_{j=1}^{k+1} u_j}{\wedge_{j=1}^{k+1} u_j}, \tag{1.1}$$

for all $u_j > 0$, $j = 1,\ldots,k+1$, where, as usual, the symbol $\vee_{j=1}^{n} u_j$ stands for the maximum of the variables u_j, $j = 1,\ldots,n$, and $\wedge_{j=1}^{n} u_j$ stands for their minimum.

In Section 2, we give exact information on the form which a min-max function may have.

Simple examples of min-max functions are

$$F_1(u_1,u_2) := \frac{u_2}{u_1}, \qquad F_2(u_1,u_2) := \frac{u_1}{u_2} \tag{1.2}$$

which appear as the response functions, respectively, in the difference equation

$$y_{n+1} = \alpha + \frac{y_{n-1}}{y_n} \tag{1.3}$$

studied in [1] and in the difference equation

$$y_{n+1} = \alpha + \frac{y_n}{y_{n-1}} \tag{1.4}$$

studied in [2]. These two equations have completely different behavior; see Remark 3.6. Also in [13, 14], the second author considered the closely related equation

$$x_{n+1} = \alpha_n + \frac{x_{n-1}}{x_n}, \tag{1.5}$$

where (α_n) is either a periodic sequence (with period two) or a convergent sequence of nonnegative real numbers.

Motivated by the above-mentioned works, in this paper, we study the behavior of the difference equation

$$x_{n+1} = \alpha_n + F(x_n, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{1.6}$$

where the initial conditions $x_{-k}, \dots, x_0$ are positive real numbers, (α_n) is a sequence of positive real numbers, and F is a min-max function.

Since a min-max function takes the value 1 at the diagonal of the space $\mathbb{R}_+^{k+1}$, it follows that in case the sequence (α_n) converges to a certain α, the positive real number

$$K := \alpha + 1 \tag{1.7}$$

is the unique asymptotic equilibrium of (1.6).

Our purpose here is to discuss the boundedness and persistence of (1.6), as well as the attractivity of the asymptotic equilibrium $\alpha + 1$, where α is the limit of (α_n) whenever this exists. This follows immediately by Theorem 3.2, where we show that, if $1 < \liminf \alpha_n \le \limsup \alpha_n < +\infty$, then any solution (x_n) satisfies the relation

$$1 \le \frac{\limsup x_n}{\liminf x_n} \le \frac{\limsup \alpha_n - 1}{\liminf \alpha_n - 1}. \tag{1.8}$$

Thus, if the sequence (α_n) converges to some $\alpha (> 1)$, then any solution with positive initial values converges to the asymptotic equilibrium $K = \alpha + 1$. This generalizes [1, Theorem 5.2] and part of [2, Theorem 1]. For the case $\alpha_n = 1$, for all n (in Theorem 3.3), we show that any nonoscillatory solution converges to 2, while if F satisfies the additional (sufficient) conditions

$$u_i < \vee_{j \ne i} u_j \Longrightarrow F(u_1, u_2, \dots, u_{k+1}) < \frac{\vee_{j \ne i} u_j}{u_i}, \tag{1.9}$$

$$u_i > \wedge_{j \ne i} u_j \Longrightarrow F(u_1, u_2, \dots, u_{k+1}) > \frac{\wedge_{j \ne i} u_j}{u_i}, \tag{1.10}$$

then it is shown in Theorem 3.4 that all solutions converge to 2. Comparing this fact with the results in [1], we see that the pair of conditions (1.9)–(1.10) seems also to be necessary. Indeed, these conditions are not satisfied in case of (1.3) and, as it is shown in [1, Theorem 4.1], it has (nontrivial) solutions which are periodic with period 2.

In Theorem 3.5, we show that if $\alpha_n = \alpha < 1$, for all n, then there is a large class of equations of the form (1.6) which have unbounded (positive) solutions. This result extends [1, Theorem 3.1]. In the Section 4, we give two examples of difference equations with min-max response to illustrate our results.

Also the so-called $(2,2)$-type equation defined in [6] (where about 50 types of difference equations are presented) includes the equation

$$x_{n+1} = \frac{A_1 x_n + B_1 x_{n-1}}{A_2 x_n + B_2 x_{n-1}}. \tag{1.11}$$

Under appropriate choice of the parameters, (1.11) can be written as

$$x_{n+1} = \alpha + \frac{(\beta+\gamma)x_{n-1}}{\beta x_n + \gamma x_{n-1}}, \tag{1.12}$$

which is of the type (1.6). Thus in this paper, we push further the investigation originated in [6] for such a form of $(2,2)$-type difference equations.

For other closely related results, which mostly deal with difference equations and inequalities whose response is (or it can be transformed into) a min-max function, see, for instance, [7, 8, 9, 10, 11, 12, 13, 14] and the references cited therein.

2. On the min-max functions. In this section, we give a characterization of min-max functions. The result is incorporated in the following theorem.

Theorem 2.1. *A function $F:\mathbb{R}_+^{k+1}\to\mathbb{R}_+$ is a min-max function if and only if there are nonnegative real-valued functions $a_j(u_1,u_2,\dots,u_{k+1})$, $b_j(u_1,u_2,\dots,u_{k+1})$, $j=1,2,\dots,k+1$, such that*

$$\begin{aligned}\sum_{j=1}^{k+1} a_j(u_1,u_2,\dots,u_{k+1}) &= \sum_{j=1}^{k+1} b_j(u_1,u_2,\dots,u_{k+1}) = 1,\\ F(u_1,u_2,\dots,u_{k+1}) &= \frac{\sum_{j=1}^{k+1} a_j(u_1,u_2,\dots,u_{k+1})u_j}{\sum_{j=1}^{k+1} b_j(u_1,u_2,\dots,u_{k+1})u_j},\end{aligned} \tag{2.1}$$

for all $(u_1,u_2,\dots,u_{k+1})\in\mathbb{R}_+^{k+1}$.

Proof. The "*if*" part is easily proved by using the form of F and the conditions on the coefficients a_j, b_j.

To show the inverse, assume that $F(u_1,u_2,\dots,u_{k+1})$ is a min-max function and fix any element $(u_1,u_2,\dots,u_{k+1})\in\mathbb{R}_+^{k+1}$. We let

$$v := \wedge_{j=1}^{k+1} u_j, \qquad w := \vee_{j=1}^{k+1} u_j, \tag{2.2}$$

thus $v = u_{j_1}$ and $w = u_{j_2}$, for two indices $j_1, j_2 \in \{1,2,\dots,k+1\}$.

From the definition of the min-max functions, we know that the value $F(u_1, u_2,\dots,u_{k+1})$ lies in the interval $[v/w, w/v]$, thus there is a number $a\in[0,1]$ such that

$$F(u_1,u_2,\dots,u_{k+1}) = a\frac{w}{v} + (1-a)\frac{v}{w}. \tag{2.3}$$

Let

$$b := \frac{(1-a)v^2}{aw^2+(1-a)v^2}. \tag{2.4}$$

It is clear that b belongs to the interval $[0,1]$, and it depends on v, w (thus on $u_1,u_2,\dots,u_{k+1}$). By some simple calculations, we obtain

$$(bw+(1-b)v)\Big(a\frac{w}{v}+(1-a)\frac{v}{w}\Big) = aw+(1-a)v \tag{2.5}$$

and consequently we get

$$F(u_1,u_2,\dots,u_{k+1}) = a\frac{w}{v}+(1-a)\frac{v}{w} = \frac{aw+(1-a)v}{bw+(1-b)v}. \tag{2.6}$$

This proves the theorem since we can set $a_j(u_1,u_2,\dots,u_{k+1}) := 0$, if $j \neq j_1, j_2$, while $a_{j_1}(u_1,u_2,\dots,u_{k+1}) = 1-a$ and $a_{j_2}(u_1,u_2,\dots,u_{k+1}) = a$. Similar substitutions are used for the denominator. The proof is complete. □

Remark 2.2. The quotient of any two elements of the class of all $f:\mathbb{R}_+^{k+1}\to\mathbb{R}_+$ which satisfy an inequality of the form

$$\wedge_{j=1}^{k+1}u_j \le f(u_1,u_2,\dots,u_{k+1}) \le \vee_{j=1}^{k+1}u_j \tag{2.7}$$

produces a min-max function.

3. The main results.

Our first result refers to the boundedness of the solutions.

Theorem 3.1. *Consider (1.6), where F is a min-max function and the sequence (α_n) satisfies*

$$1 < C := \inf\alpha_n \le \sup\alpha_n =: B < +\infty. \tag{3.1}$$

Then any solution (x_n) with positive initial values satisfies the condition

$$\min\left\{\wedge_{j=1}^{k+1}x_j, \frac{LC}{L-1}\right\} \le x_n \le L, \tag{3.2}$$

for all $n = 1,2,\dots,$ where

$$L := \max\left\{\vee_{j=1}^{k+1}x_j, \frac{BC}{C-1}\right\}. \tag{3.3}$$

Also, if $\alpha_n = \alpha = 1$, for all n, then it holds that

$$M \le x_n \le \frac{M}{M-1}, \tag{3.4}$$

for all $n \ge 1$, where

$$M := \min\left\{\wedge_{j=1}^{k+1}x_j, \frac{\vee_{j=1}^{k+1}x_j}{\vee_{j=1}^{k+1}x_j-1}\right\}. \tag{3.5}$$

Proof. Let $n > k+1$. From (1.6), for all $j \ge 1$, we have

$$C < x_j \le \vee_{i=1}^{n}x_i. \tag{3.6}$$

Also, for all $j = k+2, k+3, \dots, n$, we get

$$x_j \le B + \frac{\vee_{i=j-k-1}^{j-1} x_i}{C} \le B + \frac{\vee_{i=1}^{n} x_i}{C}. \tag{3.7}$$

These facts imply that

$$\vee_{j=1}^{n} x_j \le \max\left\{\vee_{i=1}^{k+1} x_i, B + \frac{\vee_{i=1}^{n} x_i}{C}\right\}, \tag{3.8}$$

from which we get

$$x_n \le \vee_{i=1}^{n} x_i \le \max\left\{\vee_{i=1}^{k+1} x_i, \frac{BC}{C-1}\right\}, \tag{3.9}$$

and therefore,

$$C < x_m \le L, \tag{3.10}$$

for all $m = 1, 2, \dots$.

Next let $n > k+1$. From (3.10) and (1.6), it follows that for all $j = k+2, k+3, \dots, n$, it holds that

$$x_j \ge C + \frac{\wedge_{i=j-k-1}^{j-1} x_i}{L} \ge C + \frac{\wedge_{i=1}^{n} x_i}{L}. \tag{3.11}$$

Therefore, we have

$$x_j \ge \min\left\{\wedge_{i=1}^{k+1} x_i, C + \frac{\wedge_{i=1}^{n} x_i}{L}\right\}, \tag{3.12}$$

for all $j = 1, 2, \dots$. This implies that

$$\wedge_{i=1}^{n} x_i \ge \min\left\{\wedge_{i=1}^{k+1} x_i, C + \frac{\wedge_{i=1}^{n} x_i}{L}\right\} \tag{3.13}$$

and so

$$\wedge_{i=1}^{n} x_i \ge \min\left\{\wedge_{i=1}^{k+1} x_i, \frac{LC}{L-1}\right\}. \tag{3.14}$$

This gives

$$x_n \ge \wedge_{i=1}^{n} x_i \ge \min\left\{\wedge_{i=1}^{k+1} x_i, \frac{LC}{L-1}\right\}, \tag{3.15}$$

which, together with (3.10), proves the first result of the theorem.

Next assume that $\alpha_n = 1$, $n = 0,1,\dots$. To show inequality (3.4), we observe that

$$M \le x_n \le \frac{M}{M-1}, \tag{3.16}$$

for all $n = 1,2,\dots,k+1$. Also from (1.6), we get

$$\begin{aligned} x_{k+2} &\ge 1 + \frac{\wedge_{j=1}^{k+1} x_j}{\vee_{j=1}^{k+1} x_j} \ge 1 + \frac{M}{M/(M-1)} = M, \\ x_{k+2} &\le 1 + \frac{\vee_{j=1}^{k+1} x_j}{\wedge_{j=1}^{k+1} x_j} \le 1 + \frac{M/(M-1)}{M} = \frac{M}{M-1}. \end{aligned} \tag{3.17}$$

These arguments and the induction complete the proof. □

Theorem 3.2. *Consider (1.6), where F is a continuous min-max function and the sequence (α_n) satisfies the condition*

$$1 < \liminf \alpha_n \le \limsup \alpha_n < +\infty. \tag{3.18}$$

Then any (positive) solution (x_n) satisfies relation (1.8). Hence, if the sequence (α_n) converges to some $\alpha(>1)$, then (x_n) converges to (a constant, which, therefore, is equal to) $\alpha + 1 =: K$.

Proof. Let (x_n) be a solution. From Theorem 3.1, the solution is bounded, thus there are two-sided sequences, (y_m) (upper full limiting sequence) and (z_m) (lower full limiting sequence) of (x_n) (see, e.g., [3, 4, 5]), satisfying (1.6), for all integers m, and such that

$$\liminf x_n = z_0 \le z_m, \qquad y_m \le y_0 = \limsup x_n, \tag{3.19}$$

for all m. Let $a_0 := \liminf \alpha_n$ and $a^0 := \limsup \alpha_n$. Then from (1.6), we have

$$y_0 \le a^0 + \frac{y_0}{z_0}, \quad z_0 \ge a_0 + \frac{z_0}{y_0}. \tag{3.20}$$

Combining these two relations, we obtain (1.8). □

Theorem 3.3. *Consider (1.6), where $\alpha_n = 1$, $n = 0,1,\dots$, and F is a min-max function. Then every nonoscillatory (positive) solution converges to the equilibrium $K = 2$.*

Proof. Assume first that $x_n \ge 2$, for all $n \ge -k$. Set $u_n := x_n - 2$. From Theorem 2.1, we know that F may take the form (2.1), where the (nonnegative) functions a_j, b_j satisfy

$$\sum_{j=1}^{k+1} a_j(x_n,\dots,x_{n-k}) = \sum_{j=1}^{k+1} b_j(x_n,\dots,x_{n-k}) = 1. \tag{3.21}$$

Then we obtain

$$\begin{aligned} u_{n+1} &= \frac{\sum_{j=1}^{k+1} a_j(x_n,\ldots,x_{n-k})u_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} - \frac{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})u_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} \\ &\le \frac{\sum_{j=1}^{k+1} a_j(x_n,\ldots,x_{n-k})u_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} \le \frac{1}{2}\vee_{n-k}^{n} u_j. \end{aligned} \tag{3.22}$$

Our intention is to show that $\lim u_n = 0$. To this end, we can either use [7, Lemma 1] or proceed as follows.

Let (Y_m) be an upper full limiting sequence of (u_n) with $Y_m \le Y_0 = \limsup u_n$, for all integers m. Then, from the previous arguments, it follows that it satisfies the inequality

$$Y_0 \le \frac{1}{2}Y_0, \tag{3.23}$$

thus we have $Y_0 = 0$. This and the fact that $u_n \ge 0$ imply that $\lim x_n = 2$.

Next, assume that $x_n \le 2$, for all $n \ge -k$. Set $v_n := 2 - x_n$. From (1.3) and by using the form of the function F, we obtain

$$\begin{aligned} v_{n+1} &= \frac{\sum_{j=1}^{k+1} a_j(x_n,\ldots,x_{n-k})v_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} - \frac{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})v_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}}, \\ &\le \frac{\sum_{j=1}^{k+1} a_j(x_n,\ldots,x_{n-k})v_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} \le \frac{1}{M}\vee_{j=n-k}^{n} v_j, \end{aligned} \tag{3.24}$$

where $M(>1)$ is the number defined in Theorem 3.1. By using this fact and following the same procedure as in the first case, we derive that $\lim_{n\to\infty} v_n = 0$, which implies that $\lim x_n = 2$, as desired. □

Theorem 3.4. *Consider (1.6), where $\alpha_n = 1$, $n = 0,1,\ldots,$ and F is a continuous min-max function satisfying the properties (1.9) and (1.10). Then every (positive) solution converges to the equilibrium $K = 2$.*

Proof. Let (x_n) be a solution. Then by Theorem 3.1, (x_n) is bounded. Consider an upper full limiting sequence (y_m) and a lower full limiting sequence (z_m) of (x_n), as above. From (1.6), we have

$$y_0 \le 1 + \frac{y_0}{z_0}, \qquad z_0 \ge 1 + \frac{z_0}{y_0} \tag{3.25}$$

and therefore, we get

$$y_0 z_0 = y_0 + z_0. \tag{3.26}$$

This gives

$$\frac{1}{y_0} + \frac{1}{z_0} = 1. \tag{3.27}$$

If it happens that $y_0, z_0 > 2$, or $y_0, z_0 < 2$, then we should have $1/y_0, 1/z_0 < 1/2$ and $1/y_0, 1/z_0 > 1/2$, respectively. Both these arguments contradict (3.27). Therefore, we must have

$$z_0 \le 2 \le y_0. \tag{3.28}$$

Assume that there is some $j \in \{-k-1, \dots, -1\}$ such that $y_j < y_0$ and let j_0 be an index such that

$$y_{j_0} = \wedge_{j=-k-1}^{-1} y_j. \tag{3.29}$$

Then from (1.9), we get

$$y_{j_0} < \vee_{j \ne j_0} y_j \le y_0 \tag{3.30}$$

and so from (1.6) and condition (1.9), we have

$$y_0 = 1 + F(y_{-1}, \dots, y_{-k-1}) < 1 + \frac{y_0}{y_{j_0}} \le 1 + \frac{y_0}{z_0}. \tag{3.31}$$

This gives $y_0 z_0 < y_0 + z_0$, contradicting (3.26). Thus we have $y_j = y_0$, for all $j = -k-1, \dots, -1$, and therefore,

$$y_0 = 1 + F(y_{-1}, \dots, y_{-k-1}) = 1 + F(y_0, \dots, y_0) = 2. \tag{3.32}$$

Similarly, we can use condition (1.10) to obtain $z_0 = 2$. The proof is complete. □

Our final result refers to the case $\alpha \in [0, 1)$. We show that in this case, there are equations of the form (1.3) which admit unbounded solutions.

Theorem 3.5. *Consider the equation*

$$x_{n+1} = \alpha + \frac{\sum_{i=0}^{m} a_i x_{n-2i-1}}{\sum_{i=0}^{m} b_i x_{n-2i}}, \tag{3.33}$$

where $m \in \mathbb{N}$, $\alpha \in [0, 1)$, and where the coefficients a_j and b_j, $j = 0, \dots, m$, are nonnegative constants which satisfy the conditions

$$\sum_{i=0}^{m} a_i = \sum_{i=0}^{m} b_i. \tag{3.34}$$

Then there exist unbounded solutions of (3.33).

Proof. Obviously, without loss of the generality, we can assume that $\sum_{i=0}^{m} a_i = \sum_{i=0}^{m} b_i = 1$.

Assume that $\alpha \in (0,1)$. We choose the initial conditions such that

$$\begin{gathered} x_{-(2m+1)},\ldots,x_{-1} > \frac{1}{1-\alpha} > 1+\alpha, \\ \alpha < x_{-2m},\ldots,x_0 < 1. \end{gathered} \tag{3.35}$$

We set

$$D := \wedge_{i=0}^{m} x_{-(2i+1)} \tag{3.36}$$

and observe that

$$D > \frac{1}{1-\alpha}. \tag{3.37}$$

From (3.33), we have

$$\begin{aligned} x_1 &= \alpha + \frac{\sum_{i=0}^{m} a_i x_{-(2i+1)}}{\sum_{i=0}^{m} b_i x_{-2i}} > \alpha + \sum_{i=0}^{m} a_i x_{-(2i+1)} > \alpha + D, \\ x_2 &= \alpha + \frac{\sum_{i=0}^{m} a_i x_{1-(2i+1)}}{\sum_{i=0}^{m} b_i x_{1-2i}} < \alpha + \frac{1}{\sum_{i=0}^{m} b_i x_{1-2i}} \\ &= \alpha + \frac{1}{b_0 x_1 + b_1 x_{-1} + \cdots + b_m x_{-2m+1}} \le \alpha + \frac{1}{b_0(\alpha+D) + (1-b_0)(1/(1-\alpha))} \\ &\le \alpha + \frac{1}{b_0(\alpha + 1/(1-\alpha)) + (1-b_0)(1/(1-\alpha))} = \alpha + \frac{1}{b_0\alpha + 1/(1-\alpha)} \le 1, \\ x_3 &= \alpha + \frac{\sum_{i=0}^{m} a_i x_{2-(2i+1)}}{\sum_{i=0}^{m} b_i x_{2-2i}} > \alpha + \sum_{i=0}^{m} a_i x_{2-(2i+1)} \\ &\ge \alpha + \min\{x_1, x_{-1},\ldots,x_{-2m+1}\} \ge \alpha + \min\{x_1, x_{-1},\ldots,x_{-2m-1}\} \\ &= \alpha + \min\{x_1, \min\{x_{-1},\ldots,x_{-2m-1}\}\} \ge \alpha + D. \end{aligned} \tag{3.38}$$

Following the same procedure, we get

$$x_{2j+1} > \alpha + D, \qquad x_{2j+2} < 1, \tag{3.39}$$

for all $j = 0,1,\ldots,m$. By induction, we obtain

$$x_{(2m+2)j-(2s+1)} > \alpha j + D, \tag{3.40}$$

for all $j \in \mathbb{N}$ and $s = 0,1,\ldots,m$, as well as

$$\alpha < x_{2n} < 1, \quad n = -m, -(m-1), \ldots, -1, \ldots. \tag{3.41}$$

Inequality (3.40) implies the desired result in case $\alpha > 0$.

Assume that $\alpha = 0$. Choose $\varepsilon \in (0,1)$ and the initial conditions such that

$$\begin{gathered} x_{-(2m+1)},\ldots,x_{-1} > \frac{1}{1-\varepsilon}, \\ 0 < x_{-2m},\ldots,x_0 < 1-\varepsilon. \end{gathered} \tag{3.42}$$

From (3.33), we have

$$
\begin{aligned}
x_1 &= \frac{\sum_{i=0}^{m} a_i x_{-(2i+1)}}{\sum_{i=0}^{m} b_i x_{-2i}} > \frac{1/(1-\varepsilon)}{1-\varepsilon} = \frac{1}{(1-\varepsilon)^2} > \frac{1}{1-\varepsilon}, \\
x_2 &= \frac{\sum_{i=0}^{m} a_i x_{1-(2i+1)}}{\sum_{i=0}^{m} b_i x_{1-2i}} < \frac{1-\varepsilon}{b_0 x_1 + (1-b_0)(1/(1-\varepsilon))} \\
&\le \frac{1-\varepsilon}{b_0(1/(1-\varepsilon)^2) + (1-b_0)(1/(1-\varepsilon))} < 1-\varepsilon.
\end{aligned}
\tag{3.43}
$$

Following the same procedure, we get

$$
\begin{gathered}
x_{2j+1} > \frac{1}{(1-\varepsilon)^2} > \frac{1}{1-\varepsilon}, \\
x_{2j+2} < 1-\varepsilon,
\end{gathered}
\tag{3.44}
$$

for all $j = 0, 1, \ldots, m$. By induction, we obtain

$$
x_{(2m+2)j-(2s+1)} > \frac{1}{(1-\varepsilon)^{j+1}}, \tag{3.45}
$$

for all $j \in \mathbb{N}$ and $s = 0, 1, \ldots, m$, as well as

$$
0 < x_{2n} < 1-\varepsilon, \quad n = 1, 2, \ldots. \tag{3.46}
$$

From (3.45), the result follows. □

Remark 3.6. Equation (3.33) includes the special case (1.3). Thus for $\alpha \in (0,1)$, Theorem 3.5 applies and therefore, (1.3) has unbounded solutions with positive initial values. On the other hand, (3.33) does not include the case (1.4) and as proved in [2], for the same values of α, (1.4) has a global attractor.

Remark 3.7. By some modifications of the proof of Theorem 3.5, we can prove the following result.

Theorem 3.8. *Consider the equation*

$$
x_{n+1} = \alpha_n + \frac{\sum_{i=0}^{m} a_i x_{n-2i-1}}{\sum_{i=0}^{m} b_i x_{n-2i}}, \tag{3.47}
$$

where $m \in \mathbb{N}$, (α_n) is a sequence of positive real numbers such that $\lim_{n\to\infty} \alpha_n =: A \in [0,1)$, and where the coefficients a_j and b_j, $j = 0, \ldots, m$, are nonnegative constants which satisfy the conditions

$$
\sum_{i=0}^{m} a_i = \sum_{i=0}^{m} b_i. \tag{3.48}
$$

Then there exist unbounded solutions of (3.47).

4. Some illustrative examples

Example 4.1. Consider the difference equation

$$x_{n+1} = \alpha + \frac{\beta x_n + \gamma x_n^2 + \delta x_{n-1}^2}{\beta x_n + \gamma x_n x_{n-1} + \delta x_{n-1}^2}, \tag{4.1}$$

where all the coefficients are positive real numbers. The rational function on the right-hand side is a min-max function, since it can be written in the form

$$\frac{((\beta+\gamma x_n)/(\beta+\gamma x_n+\delta x_{n-1}))x_n + (\delta x_{n-1}/(\beta+\gamma x_n+\delta x_{n-1}))x_{n-1}}{(\beta/(\beta+\gamma x_n+\delta x_{n-1}))x_n + ((\gamma x_n+\delta x_{n-1})/(\beta+\gamma x_n+\delta x_{n-1}))x_{n-1}}. \tag{4.2}$$

Thus, from Theorems 3.2 and 3.4, we conclude that, for every fixed $\alpha \geq 1$, any solution of (4.1) converges to the equilibrium $\alpha+1$. Notice that conditions (1.9) and (1.10) are also satisfied.

Example 4.2. Consider the difference equation

$$x_{n+1} = \alpha + \frac{\sum_{j_i \in \{n,n-1,n-2\}} x_{j_1} x_{j_2} x_{j_3}}{x_n^3 + x_{n-1}^3 + x_{n-2}^3 + 6 x_n x_{n-1} x_{n-2}}, \tag{4.3}$$

where $\alpha \geq 0$. This is a third-order difference equation whose response on the right-hand side is a min-max function. Indeed, this can be written in the form

$$\frac{\sum_{j_i \in \{n,n-1,n-2\},\ j_1 \neq j_2 \neq j_3 \neq j_1} \left((x_{j_1}^2 + x_{j_1} x_{j_2} + x_{j_1} x_{j_3}) / (x_n + x_{n-1} + x_{n-2})^2 \right) x_{j_1}}{\sum_{j_i \in \{n,n-1,n-2\},\ j_1 \neq j_2 \neq j_3 \neq j_1} \left((x_{j_1}^2 + 2 x_{j_2} x_{j_3}) / (x_n + x_{n-1} + x_{n-2})^2 \right) x_{j_1}}. \tag{4.4}$$

Here, again, Theorems 3.2 and 3.4 apply and we conclude that in case $\alpha \geq 1$, any solution of (4.3) converges to $\alpha+1$.

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