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THE STRUCTURE OF A SUBCLASS OF AMENABLE BANACH ALGEBRAS

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We give sufficient conditions that allow contractible (resp., reflexive amenable) Banach algebras to be finite-dimensional and semisimple algebras. Moreover, we show that any contractible (resp., reflexive amenable) Banach algebra in which every maximal left ideal has a Banach space complement is indeed a direct sum of finitely many full matrix algebras. Finally, we characterize Hermitian $*$-algebras that are contractible.

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1. Introduction. The purpose of this note is to establish the structure of some class of amenable Banach algebras. Let $\mathcal{A}$ be a Banach algebra over the complex field $\mathbb{C}$. We define a Banach left $\mathcal{A}$-module $\mathcal{X}$ to be a Banach space which is also a unital left $\mathcal{A}$-module such that the linear map $\mathcal{A}\times\mathcal{X}\to\mathcal{X}$, $(a,x)\to ax$, is continuous. Right modules are defined analogously. A Banach $\mathcal{A}$-bimodule is a Banach space with a structural $\mathcal{A}$-bimodule such that the linear map $\mathcal{A}\times\mathcal{X}\times A\to\mathcal{X}$, $(a\times x\times b)\to axb$, is jointly continuous, where $\mathcal{A}\times\mathcal{X}\times\mathcal{A}$ carries the Cartesian product topology. A submodule $\mathcal{Y}$ of a Banach ⟨left, right, bi-⟩ $\mathcal{A}$-module $\mathcal{X}$ is a closed subspace of $\mathcal{X}$ with the structural Banach ⟨left, right, or bi-⟩ $\mathcal{A}$-module. A Banach left $\mathcal{A}$-module morphism $\theta:\mathcal{X}\to\mathcal{Y}$ is a continuous linear map between two left Banach $\mathcal{A}$-modules such that $\theta(ax)=a\theta(x)$ for all $a\in\mathcal{A}$ and all $x\in\mathcal{X}$. A Banach right $\mathcal{A}$-module morphism and a Banach $\mathcal{A}$-bimodule morphism are defined analogously. For each Banach ⟨left, bi-⟩ module $\mathcal{X}$ on $\mathcal{A}$, the dual $\mathcal{X}^*$ is naturally a Banach ⟨left, bi-⟩ $\mathcal{A}$-bimodule with the module actions defined by $\langle aT(x)=T(xa)$, $aT(x)=T(xa)$, and $Ta(x)=T(ax)\rangle$, for all $a\in\mathcal{A}$, $T\in\mathcal{X}^*$, and $x\in\mathcal{X}$, where $T(x)$ denotes the evaluation of T at x. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are Banach ⟨left, or bi-⟩ $\mathcal{X}$-modules and $\theta:\mathcal{X}\to\mathcal{Y}$, $\beta:\mathcal{Y}\to\mathcal{Z}$ are ⟨left, bi-⟩ module morphisms, then the sequence

$$\Sigma:0\longrightarrow\mathcal{X}\longrightarrow\mathcal{Y}\longrightarrow\mathcal{Z}\longrightarrow 0 \tag{1.1}$$

is exact if θ is one-to-one, $\Im\beta=\mathcal{Z}$, and $\Im\theta=\ker\beta$. The exact sequence Σ is admissible if β has a continuous right inverse, equivalently, $\ker\beta$ has a Banach space complement in $\mathcal{Y}$. The admissible exact sequence splits if the right inverse of β is Banach ⟨left, bi-⟩ module, equivalently, $\ker\beta$ is a Banach space complement in $\mathcal{Y}$ which is an $\mathcal{A}$-submodule.

A derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a linear operator $D:\mathcal{A}\to\mathcal{X}$ which satisfies $D(ab)=D(a)b+aD(b)$, for all $a,b\in\mathcal{A}$. Recall that for any $x\in\mathcal{X}$, the mapping $\delta_x:\mathcal{A}\to\mathcal{X}$ defined by $\delta_x(a)=ax-xa$, $a\in\mathcal{A}$, is a continuous derivation, called an inner derivation. A Banach algebra $\mathcal{A}$ is said to be contractible if for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$, each continuous derivation from $\mathcal{A}$ into $\mathcal{X}$ is inner. We say that $\mathcal{A}$ is amenable whenever every continuous derivation from $\mathcal{A}$ into $\mathcal{X}^*$ is inner for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$. Obviously, every contractible Banach algebra is an amenable Banach algebra and the converse is true in the finite-dimension case. It is well known that a finite-dimensional algebra is semisimple if and only if it is isomorphic to a finite Cartesian product of a family of full matrix algebras. Using Theorem 2.1, it is easy to check that a finite Cartesian product of a family of full matrix algebras is contractible.

The purpose of this note is to contribute to the study of the following questions, raised, respectively, in [2], [3, page 817], and [5, page 212].

Question 1.1. Is every contractible Banach algebra semisimple?

Question 1.2. Is every reflexive amenable Banach algebra finite-dimensional and semisimple?

Question 1.3. Is every contractible Banach algebra finite-dimensional?

Recall that a Banach algebra is called a reflexive Banach algebra if it is reflexive as a Banach space. In this note, we will present two situations in which a contractible Banach algebra is finite-dimensional. First, we will give a partial answer to the above questions, where we assume that each maximal left ideal is complemented as a Banach space. This result improves [5, Proposition IV.4.3] for contractible Banach algebras and [3, Corollary 2.3] for reflexive amenable Banach algebras, where the authors suppose only that all of their primitive ideals have finite codimensions. Second, we will show that a Hermitian Banach *-algebra is contractible if and only if it is a finite-dimensional semisimple algebra.

2. Preliminaries.

In this section, we recall some facts about the structure of contractible and amenable Banach algebras. Let $\mathcal{A}$ be a Banach algebra over the complex field $\mathbb{C}$ and let $\mathcal{A}^{**}$ be the bidual of $\mathcal{A}$ with the usual multiplication defined by $\psi\cdot\phi(f)=\psi(f)\phi(f)$ for all $\psi,\phi\in\mathcal{A}^{**}$ and $f\in\mathcal{A}^*$. Consider on $\mathcal{A}^{**}$ the Banach $\mathcal{A}$-bimodule structure defined by $aT=\eta(a)T$, $Ta=T\eta(a)$ with $\eta:\mathcal{A}\to\mathcal{A}^{**}$ the canonical map. Notice that if a Banach algebra $\mathcal{A}$ has a bounded approximate identity, then its bidual $\mathcal{A}^{**}$ has an identity. It is a fact that a contractible Banach algebra has an identity and an amenable Banach algebra admits bounded right, left, bilateral approximate identities. Of course, a reflexive amenable Banach algebra must be unital. We denote the identity element of $\mathcal{A}$ by 1 and we write $\mathcal{A}\hat{\otimes}\mathcal{A}$ for the completed projective tensorial product (see [4]). The Banach space $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule if we define

$$a(b\otimes c)=ab\otimes c,\quad (b\otimes c)a=b\otimes ca,\quad a,b,c\in\mathcal{A}. \tag{2.1}$$

For a unital Banach algebra $\mathcal{A}$, a diagonal of $\mathcal{A}$ is an element $d\in\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $ad=da$, for all $a\in\mathcal{A}$, and $\pi(d)=1$, where $\pi:\mathcal{A}\hat{\otimes}\mathcal{A}\to\mathcal{A}$ is the canonical Banach $\mathcal{A}$-bimodule morphism. For such a Banach algebra $\mathcal{A}$, a virtual diagonal of $\mathcal{A}$ is an element

$d \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that

$$ad = da, \quad \forall a \in \mathcal{A}, \qquad \pi^{**}(d) = 1, \tag{2.2}$$

where $\pi^{**} : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \to \mathcal{A}^{**}$ is the bidual Banach $\mathcal{A}$-module morphism of π. In the following theorems, we present characterizations of contractible (resp., amenable) Banach algebras. We recall, respectively, [1, Theorem 6.1] and [6, Theorem 1.3].

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra. The following are equivalent:*
(1) *$\mathcal{A}$ is contractible;*
(2) *$\mathcal{A}$ has a diagonal.*

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra. The following are equivalent:*
(1) *$\mathcal{A}$ is amenable;*
(2) *$\mathcal{A}$ has a virtual diagonal.*

We choose as a basis of the algebra $\mathbb{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices the set of elementary matrices e_{ij}. Consider $d = \sum_{i,j}^{n} \delta_{ij} e_{ij} \otimes e_{ji} \in \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})$. Then $Md = dM$, for all $M \in \mathbb{M}_n(\mathbb{C})$, and $\pi(d) = 1$, where $\pi : \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is the canonical morphism. It follows that $\mathbb{M}_n(\mathbb{C})$ is contractible.

Next, the following propositions hold.

Proposition 2.3. *Let $\mathcal{A}$ be a $\langle$contractible, amenable$\rangle$ Banach algebra. Then, if $\theta : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism from $\mathcal{A}$ into another Banach algebra $\mathcal{B}$ with dense range, then $\mathcal{B}$ is $\langle$contractible, amenable$\rangle$. In particular, if $\mathcal{I}$ is a closed two-sided ideal of a $\langle$contractible, amenable$\rangle$ Banach algebra $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is $\langle$contractible, amenable$\rangle$ too.*

Proof. Assume that $\mathcal{A}$ is contractible. Let $\mathcal{X}$ be a Banach $\mathcal{B}$-bimodule. Consider on $\mathcal{X}$ the structure of $\mathcal{A}$-bimodule defined by $a \cdot x = \theta(a)x$ and $x \cdot a = x\theta(x)$. Since θ is continuous, $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. Now, let $D : \mathcal{B} \to \mathcal{X}$ be a continuous derivation. It is easy to see that $D \circ \theta$ is a continuous derivation from $\mathcal{A}$ to the Banach $\mathcal{A}$-bimodule $\mathcal{X}$, and thus it is inner. Therefore, there exists $x \in \mathcal{X}$ such that $D(\theta(a)) = a \cdot x - x \cdot a = \theta(a)x - x\theta(a)$ for all $a \in \mathcal{A}$. Since $\theta(\mathcal{A})$ is dense in $\mathcal{B}$, we have $D(b) = bx - xb$ for all $b \in \mathcal{B}$. It follows that D is inner and $\mathcal{B}$ is contractible. If $\mathcal{A}$ is amenable, we will consider a continuous derivation $D : \mathcal{B} \to \mathcal{X}^*$ from $\mathcal{B}$ to the dual of the bimodule $\mathcal{X}$ and we use the same way to prove that $\mathcal{B}$ is amenable. □

Proposition 2.4 [1, Theorems 2.3 and 2.5]**.** *Let $\mathcal{A}$ be an amenable Banach algebra and let*

$$\Sigma : 0 \longrightarrow \mathcal{X}^* \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0 \tag{2.3}$$

be an admissible short exact sequence of Banach (left, right, or bi-) modules with $\mathcal{X}^*$ a dual of $\mathcal{X}$. Then Σ splits.

Proposition 2.5 [1, Theorem 6.1]**.** *Let $\mathcal{A}$ be a contractible Banach algebra and let*

$$\Sigma : 0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0 \tag{2.4}$$

be an admissible short exact sequence of Banach (left, right, or bi-) modules. Then Σ splits.

Remark 2.6. Notice that for each closed two-sided ideal $\mathcal{I}$ of a reflexive Banach algebra, $\mathcal{I}$ and the quotient $\mathcal{A}/\mathcal{I}$ are reflexive Banach algebras too.

Proposition 2.7. *Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra and assume that $\mathcal{I}$ is a closed ⟨left, two-sided⟩ ideal of $\mathcal{A}$ which has a Banach space complement. Then there exists a closed ⟨left, two-sided⟩ ideal $\mathcal{J}$ of $\mathcal{A}$ such that*

$$\mathcal{A} = \mathcal{I} + \mathcal{J}. \tag{2.5}$$

Proof. Let $\mathcal{A}$ be an amenable Banach algebra and let $\mathcal{I}$ be a closed ⟨left, two-sided⟩ ideal of $\mathcal{I}$ which has a Banach space complement. Then the short exact sequence $\Sigma : 0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0$ is admissible. If $\mathcal{A}$ is reflexive, then the space $\mathcal{I}$ will be the same, and so it will be the dual of the Banach ⟨left, bi-⟩ $\mathcal{A}$-module $\mathcal{I}^*$. By Proposition 2.4, Σ splits and $\mathcal{I}$ has a Banach space complement which is a ⟨left, two-sided⟩ ideal. When $\mathcal{A}$ is contractible, by Proposition 2.5, we have the result. □

3. Main results

Theorem 3.1. *Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra. Assume that each maximal left ideal of $\mathcal{A}$ is complemented as a Banach space in $\mathcal{A}$. Then there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that*

$$\mathcal{A} \cong \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_{n_k}(\mathbb{C}). \tag{3.1}$$

Proof. By Section 2, the algebra $\mathcal{A}$ has an identity $1_{\mathcal{A}}$. Let $(\mathcal{M}_i)_{i\in I}$ be the family of all maximal left ideals. Since $\mathcal{M}_i$ is complemented as a Banach space for each i, there exists a left ideal $\mathcal{J}_i$ such that $\mathcal{A} = \mathcal{M}_i \oplus \mathcal{J}_i$. Notice that

$$\operatorname{Rad}(\mathcal{A}) = \bigcap_i \mathcal{M}_i \tag{3.2}$$

is the Jacobson radical of $\mathcal{A}$ and

$$\bigoplus_i \mathcal{J}_i \subseteq \operatorname{Soc}(\mathcal{A}), \tag{3.3}$$

where $\operatorname{Soc}(\mathcal{A})$ is the socle of the algebra $\mathcal{A}$, that is, it is the sum of all minimal left ideals of $\mathcal{A}$ and it coincides with the sum of all minimal right ideals of $\mathcal{A}$. Recall that every minimal left ideal of $\mathcal{A}$ is of the form $\mathcal{A}e$, where e is a minimal idempotent, that is, $e^2 = e \neq 0$ and $e\mathcal{A}e = \mathbb{C}e$. On the other hand, for each finite family of minimal idempotents $(e_k)_{k\in K}$, we have

$$\mathcal{A} = \bigoplus_{k\in K} \mathcal{A}e_k \bigoplus \bigcap_{k\in K} \mathcal{A}(1_{\mathcal{A}} - e_k). \tag{3.4}$$

It follows from (3.3) and (3.4) that $\operatorname{Soc}(\mathcal{A})$ is dense in $\mathcal{A}/\operatorname{Rad}(\mathcal{A})$. This shows that $\mathcal{A}/\operatorname{Rad}(\mathcal{A})$ is finite-dimensional. Therefore

$$\mathcal{A} = \operatorname{Rad}(\mathcal{A}) \bigoplus \operatorname{Soc}(\mathcal{A}). \tag{3.5}$$

If $\mathrm{Rad}(\mathcal{A}) \neq \{0\}$, this would mean that $\mathrm{Rad}(\mathcal{A})$ has an identity, which is impossible. So, $\mathcal{A} = \mathrm{Soc}(\mathcal{A})$, and then it is a finite direct sum of certain full matrix algebras. □

Corollary 3.2. *Every commutative ⟨contractible, reflexive amenable⟩ Banach algebra $\mathcal{A}$ is finite-dimensional and semisimple.*

Corollary 3.3. *Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra such that every irreducible representation of $\mathcal{A}$ is finite-dimensional. Then $\mathcal{A}$ is finite-dimensional and semisimple.*

Proof. It is easy to check that every primitive ideal of a Banach algebra is finite-codimensional if and only if each of its maximal left ideals is finite-codimensional. So, the corollary follows. □

It should be emphasized that the following result appears in [9] or [5, Corollary in page 212].

Corollary 3.4. *Every ⟨contractible, reflexive amenable⟩ C^*-algebra $\mathcal{A}$ is finite-dimensional and semisimple.*

Proof. Suppose that $\mathcal{A}$ is a contractible or reflexive amenable C^*-algebra. Let $\mathcal{M}$ be a maximal left ideal. By [7, Theorems 5.3.5 and 5.2.4], the space $\mathcal{A}/\mathcal{M}$ is a Hilbert space. It follows that the short exact sequence

$$\Sigma : 0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{M} \longrightarrow 0 \tag{3.6}$$

is admissible, and thus $\mathcal{M}$ has a Banach space complement. By Theorem 3.1, $\mathcal{A}$ is isomorphic to a finite direct sum of full matrix algebras.

Remark 3.5. Recall that a simple algebra is an algebra which has no proper ideals other than the zero ideal. To show that every ⟨contractible, reflexive amenable⟩ Banach algebra is finite-dimensional and semisimple, it suffices to prove that every ⟨contractible, reflexive amenable⟩ simple contractible Banach algebra is finite-dimensional. Indeed, let $\mathcal{A}$ be a contractible Banach algebra. Let $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$. Then the algebra $\mathcal{A}/\mathcal{P}$ is a ⟨contractible, reflexive amenable⟩ Banach algebra. Put $\mathcal{B} = \mathcal{A}/\mathcal{P}$ and consider some maximal two-sided ideal $\mathcal{M}$ of $\mathcal{B}$. Since $\mathcal{B}/\mathcal{M}$ is a ⟨contractible, reflexive amenable⟩ simple Banach algebra, it is finite-dimensional. There exists then a closed two-sided ideal $\mathcal{J}$ such that $\mathcal{B} = \mathcal{M} \oplus \mathcal{J}$. Recall that in a primitive algebra, every nonzero ideal is essential, that is, it has a nonzero intersection with every nonzero ideal of the algebra. It follow that $\mathcal{M} = 0$, and so $\mathcal{B}$ is finite-dimensional. Using Corollary 3.2, $\mathcal{A}$ must be a finite-dimensional and semisimple algebra. This completes the proof. □

Proposition 3.6. *Let $\mathcal{A}$ be a ⟨contractible, reflexive amenable⟩ simple contractible Banach algebra having a maximal left ideal complemented as a Banach space. Then $\mathcal{A}$ is finite-dimensional.*

Proof. If $\mathcal{A}$ is an infinite-dimensional simple algebra, then $\mathrm{Soc}(\mathcal{A}) = 0$. Moreover, if $\mathcal{A}$ is ⟨contractible, reflexive amenable⟩ with a maximal left ideal complemented as a Banach space, then $\mathcal{A}$ has a nontrivial minimal left ideal. This is a contradiction. □

Now, assume that $\mathcal{A}$ is a unital Banach $*$-algebra which admits at least one state τ. Then there exists a $*$-representation π_τ of $\mathcal{A}$ on a Hilbert space H_τ, with a cyclic vector ζ of norm 1 in H_τ such that $\tau(a) = \langle \pi_\tau(a)\zeta, \zeta\rangle$, for all a $\in \mathcal{A}$, $\langle\cdot,\cdot\rangle$ being the inner product in H_τ.

Theorem 3.7. *A Hermitian Banach $*$-algebra $\mathcal{A}$ is contractible if and only if there are $n_1, n_2, \dots, n_k \in \mathbb{N}$ such that (3.1) holds.*

Proof. It suffices to show the "only if" part. Suppose that a Hermitian Banach algebra $\mathcal{A}$ is contractible. Let $\mathrm{T}(\mathcal{A})$ be the set of all states of $\mathcal{A}$ and let $R^*(\mathcal{A})$ be the $*$-radical of $\mathcal{A}$, that is, the intersection of the kernels of all $*$-representations of $\mathcal{A}$ on Hilbert spaces. Since $\mathcal{A}$ is Hermitian and has an identity, $\mathrm{T}(\mathcal{A}) \neq \emptyset$, and so $R^*(\mathcal{A}) \neq \mathcal{A}$. Put $\pi = \bigoplus_{\tau\in\mathrm{T}(\mathcal{A})}\pi_\tau$ and $H = \bigoplus_{\tau\in\mathrm{T}(\mathcal{A})} H_\tau$. Then π is a $*$-representation of $\mathcal{A}$ on H. Consider

$$||\pi(a)|| = \sup_{\tau\in\mathrm{T}(\mathcal{A})} ||\pi_\tau(a)||. \tag{3.7}$$

Then $\|\cdot\|$ is a C^*-norm on $\pi(A)$. Let $\mathcal{B}$ denote the closure of $(\pi(\mathcal{A}), \|\cdot\|)$. Moreover, $\pi : \mathcal{A} \to \mathcal{B}$ is a continuous mapping into a C^*-algebra $\mathcal{B}$ such that $\ker(\pi) = R^*(\mathcal{A})$. As $\mathcal{A}$ is contractible, $\mathcal{B}$ is also contractible. Using Corollary 3.4, the algebra $\mathcal{B}$ has to be finite-dimensional. Notice that $\mathcal{A}/R^*(\mathcal{A})$ is isometric with the $*$-subalgebra $\pi(\mathcal{A})$ of $\mathcal{B}$. Thus, it follows that $\mathcal{A}/R^*(\mathcal{A})$ is finite-dimensional. Since $R^*(\mathcal{A})$ is a finite-codimensional closed two-sided $*$-ideal, there exists a closed two-sided ideal $\mathcal{K}$ such that

$$\mathcal{A} = R^*(\mathcal{A}) \oplus \mathcal{K}. \tag{3.8}$$

Next, note that $\|\pi(a)\|^2 = \sup\{\tau(a^*a),\ \tau \in \mathrm{T}(\mathcal{A})\} \geq |a^*a|_\sigma$, where $|a|_\sigma$ is the spectral radius of $a \in \mathcal{A}$. By Pták [8], we obtain $\|\pi(a)\|^2 \geq |a|_\sigma^2$. So, if $a \in R^*(\mathcal{A})$, then $|a|_\sigma = 0$. Therefore, every element of $R^*(\mathcal{A})$ is quasinilpotent. Notice that in general $\mathrm{Rad}(\mathcal{A}) \subseteq R^*(\mathcal{A})$. Since $R^*(\mathcal{A})$ is a closed two-sided $*$-ideal, we have $R^*(\mathcal{A}) = \mathrm{Rad}(\mathcal{A})$, and so $\mathcal{A}$ is finite-dimensional and semisimple. □

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