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# ON A DIRECT CONSTRUCTION OF INVERSE SCATTERING PROBLEMS FOR INTEGRABLE NONLINEAR EVOLUTION EQUATIONS IN THE TWO-SPATIAL DIMENSION

**H. H. CHEN and J. E. LIN**

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We present a method to construct inverse scattering problems for integrable nonlinear evolution equations in the two-spatial dimension. The temporal component is the adjoint of the linearized equation and the spatial component is a partial differential equation with respect to the spatial variables. Although this idea has been known for the one-spatial dimension for some time, it is the first time that this method is presented for the case of the higher-spatial dimension. We present this method in detail for the Veselov-Novikov equation and the Kadomtsev-Petviashvili equation.

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**1. Introduction.** The search for the methods to construct inverse scattering problems for integrable nonlinear evolution equations has been an interesting subject, see [5, 9, 14] for example, since the discovery of the soliton solutions for the Korteweg-de Vries equation [1, 2, 6, 7, 10, 11]. It was known that the adjoint of the linearized equation could be used as the temporal component to construct an inverse scattering problem for integrable equations in the case of the *one-spatial dimension* [3]. In this note, we extend this concept to the case of the two-spatial dimension. Let the spatial variables be $x$ and $y$. The spatial component is assumed to be of the form $L\phi_y = R\phi$, where $L$ and $R$ are assumed to be partial differential operators with respect to $x$. Requiring the temporal equation and the spatial equation to be compatible, we obtain an explicit form for the spatial component as a special solution to the compatibility condition. To find the spatial component is the main work of this note. A consequence of this process is a natural derivation of a Miura-type transformation that can be used to generate Lax pairs for the integrable equations. We present it in details for the Veselov-Novikov equation [13], which is also known as the Nizhnik-Veselov-Novikov equation [12], and the Kadomtsev-Petviashvili equation [8].

**2. Veselov-Novikov equation.**

$$u_t = u_{xxx} + u_{yyy} + 3(uv)_x + 3(uw)_y, \tag{2.1}$$

where $u$ is a function of $t$, $x$, and $y$, $v$ and $w$ depend on $u$ with $v_y = u_x$, $w_x = u_y$, and the subscript denotes the partial differentiation, for example, $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $a_{1,x} = \partial a_1/\partial x$, and so forth.

The adjoint of the linearized equation of (2.1) is

$$\phi_t = \phi_{xxx} + \phi_{yyy} + 3v\phi_x + 3\Psi_x + 3w\phi_y + 3\partial_x^{-1}\partial_y(u\phi_y), \tag{2.2}$$

where

$$\Psi_y = u\phi_x. \tag{2.3}$$

We now propose

$$L\phi_y = R\phi, \tag{2.4}$$

where $L$ and $R$ are partial differential operators with respect to $x$,

$$\begin{aligned} L &= e_0 + e_1\partial_x + e_2\partial_x^2 + \cdots + e_m\partial_x^m, \\ R &= r_0 + r_1\partial_x + r_2\partial_x^2 + \cdots + r_n\partial_x^n, \end{aligned} \tag{2.5}$$

and where $e_i$, $0 \le i \le m$, and $r_j$, $0 \le j \le n$, are functions depending on $u$ and its partial derivatives but not explicitly on $t$, $x$, or $y$.

We request that (2.2) and (2.4) be compatible. Since (2.2) is symmetric with respect to $x$ and $y$, we also request that (2.4) be symmetric with respect to $x$ and $y$. Interchanging $x$ and $y$ in (2.4), we get

$$(e_0^* + e_1^*\partial_y + e_2^*\partial_y^2 + \cdots + e_m^*\partial_y^m)\phi_x = (r_0^* + r_1^*\partial_y + r_2^*\partial_y^2 + \cdots + r_n^*\partial_y^n)\phi, \tag{2.6}$$

where $e_i^*$ and $r_j^*$ are obtained from $e_i$ and $r_j$, $0 \le i \le m$, $0 \le j \le n$, by interchanging $x$ and $y$, respectively. In order for (2.6) to be the same as (2.4), we need $m = 1$ and $n = 1$. Thus (2.4) becomes

$$(e_0 + e_1\partial_x)\phi_y = (r_0 + r_1\partial_x)\phi. \tag{2.7}$$

In order to find $e_0$, $e_1$, $r_0$, and $r_1$, we rewrite (2.7) as

$$\phi_y = (e_0 + e_1\partial_x)^{-1}(r_0 + r_1\partial_x)\phi = (b_0 + b_{-1}\partial_x^{-1} + b_{-2}\partial_x^{-2} + \cdots)\phi. \tag{2.8}$$

Equating the coefficients of $\partial_x^{(n)}\phi$, $n \le 1$, on both sides of

$$(e_0 + e_1\partial_x)(b_0 + b_{-1}\partial_x^{-1} + b_{-2}\partial_x^{-2} + \cdots)\phi = (r_0 + r_1\partial_x)\phi, \tag{2.9}$$

we get

$$e_1 b_0 = r_1, \tag{2.10}$$

$$e_1 b_{0,x} + e_0 b_0 + e_1 b_{-1} = r_0, \tag{2.11}$$

$$e_1 b_{-1,x} + e_0 b_{-1} + e_1 b_{-2} = 0, \tag{2.12}$$

$$e_1 b_{-h,x} + e_0 b_{-h} + e_1 b_{-h-1} = 0, \quad h \ge 1. \tag{2.13}$$

Now back to (2.3), $\Psi$ must be expressible in terms of $\phi$. Therefore, we propose

$$\Psi = (a_k\partial_x^k + a_{k-1}\partial_x^{k-1} + \cdots)\phi. \tag{2.14}$$

Substituting it into (2.3), we get

$$\Psi_y = a_{k,y}\partial_x^k\phi + a_k\partial_x^k\phi_y + a_{k-1,y}\partial_x^{k-1}\phi + a_{k-1}\partial_x^{k-1}\phi_y + \cdots = u\phi_x. \tag{2.15}$$

Substituting $\phi_y$ by (2.8) into (2.15) and equating the coefficients of $\partial_x^{(n)}\phi$ on both sides of (2.15), we get $k = 1$. Thus

$$\Psi = (a_1\partial_x + a_0 + a_{-1}\partial_x^{-1} + \cdots)\phi. \tag{2.16}$$

Furthermore, equating the coefficients of $\partial_x^{(n)}\phi$ on both sides of (2.15) with $k = 1$ for $n \le 1$, we get, for the coefficient of $\phi_x$,

$$a_{1,y} + a_1 b_0 = u, \tag{2.17}$$

for the coefficient of $\phi$,

$$a_1 b_{0,x} + a_1 b_{-1} + a_{0,y} + a_0 b_0 = 0, \tag{2.18}$$

for the coefficient of $\partial_x^{-1}\phi$,

$$a_1 b_{-1,x} + a_1 b_{-2} + a_0 b_{-1} + a_{-1,y} + a_{-1} b_0 = 0, \tag{2.19}$$

for the coefficient of $\partial_x^{-2}\phi$,

$$a_1 b_{-2,x} + a_1 b_{-3} + a_0 b_{-2} - a_{-1} b_{0,x} + a_{-1} b_{-1} + a_{-2,y} + a_{-2} b_0 = 0, \tag{2.20}$$

and for the coefficient of $\partial_x^{-h}\phi$,

$$a_1 b_{-h,x} + a_1 b_{-h-1} + a_0 b_{-h} + (\text{terms linear in } a_{-1},\dots,a_{-h+1}) + a_{-h,y} + a_{-h} b_0 = 0, \tag{2.21}$$

for $h \ge 2$.

We now demand that (2.2) and (2.8) be compatible, that is, $\phi_{txy} = \phi_{ytx}$.

Differentiating both sides of (2.2) with respect to $x$ and $y$, we get

$$\begin{aligned}\phi_{txy} = {} & \phi_{xxxxy} + \phi_{yyyxy} + 3v_{xy}\phi_x + 3v_x\phi_{xy} + 3v_y\phi_{xx} + 3v\phi_{xxy} \\ & + 3\Psi_{xxy} + 3w_{xy}\phi_y + 3w_x\phi_{yy} + 3w_y\phi_{xy} \\ & + 3w\phi_{xyy} + 3u_{yy}\phi_y + 6u_y\phi_{yy} + 3u\phi_{yyy}.\end{aligned} \tag{2.22}$$

Differentiating both sides of (2.8) with respect to $t$ and $x$, we get

$$\phi_{ytx} = \sum_{j\le 0}(b_{j,tx}\partial_x^j\phi + b_{j,t}\partial_x^{j+1}\phi + b_{j,x}\partial_x^j\phi_t + b_j\partial_x^{j+1}\phi_t). \tag{2.23}$$

By equating the coefficients of $\partial_x^{(n)}\phi$ on both sides of $\phi_{txy} = \phi_{ytx}$ after we use (2.2), (2.8), and (2.16) to express both sides in terms of $\partial_x^{(n)}\phi$, we get, for the coefficient of $\partial_x^4\phi$,

$$b_0 = b_0, \tag{2.24}$$

for the coefficient of $\partial_x^3\phi$,

$$u = -b_{0,x} + b_0 a_1, \tag{2.25}$$

for the coefficient of $\partial_x^2\phi$,

$$2b_{0,xx} + b_{-1,x} + v_y + 2u_x = b_{0,x}a_1 + 2b_0 a_{1,x} + b_0 a_0 + b_{-1}a_1, \tag{2.26}$$

and for the coefficient of $\partial_x\phi$,

$$\begin{aligned} &4b_{0,xxx} + 6b_{-1,xx} + 3b_{-2,x} + b_{0,yyy} + 4b_{0,yy}b_0 + 3(b_{0,y})^2 + 6b_{0,y}b_0^2 + 3v_{xy} \\ &\quad + 3vb_{0,x} + 3u_{xx} + 3w_y b_0 + 3wb_{0,y} \\ &= b_{0,t} + 3b_{0,x}(a_{1,x} + a_0) + b_0 b_{0,yy} + 3b_{0,y}b_0^2 + 3b_0(a_{1,xx} + 2a_{0,x} + a_{-1}) \\ &\quad + 3b_{-1,x}a_1 + 3b_{-1}(a_{1,x} + a_0) + 3b_{-2}a_1. \end{aligned} \tag{2.27}$$

From (2.17) and (2.25), we get $a_{1,y} = -b_{0,x}$. Let $q = -\partial_x^{-1}a_1$, then

$$a_1 = -q_x, \qquad b_0 = q_y, \tag{2.28}$$

and, from (2.25),

$$u = -q_{xy} - q_x q_y. \tag{2.29}$$

Substituting (2.29) into (2.1), we get

$$\begin{aligned} 0 &= u_t - u_{xxx} - u_{yyy} - 3(uv)_x - 3(uw)_y \\ &= M(q_t - q_{xxx} - q_{yyy} - 3q_{xx}q_x - q_x^3 - 3vq_x - 3q_{yy}q_y - q_y^3 - 3wq_y), \end{aligned} \tag{2.30}$$

where $M = -q_y\partial_x - q_x\partial_y - \partial_{xy}^2$.

Thus (2.29) is a Miura-type transformation relating the Veselov-Novikov equation, (2.1), to the modified Veselov-Novikov equation

$$q_t - q_{xxx} - q_{yyy} - 3q_{xx}q_x - q_x^3 - 3vq_x - 3q_{yy}q_y - q_y^3 - 3wq_y = 0. \tag{2.31}$$

From (2.18), we get

$$b_{-1} = \left(\frac{-1}{a_1}\right)(a_1 b_{0,x} + a_{0,y} + a_0 b_0). \tag{2.32}$$

Note that $a_1 \neq 0$ from (2.17).

Substituting (2.32) into (2.26) and using (2.28) and (2.29), we get

$$(q_x\partial_{xy}^2 + (-q_{xx} + q_x^2)\partial_y + q_x q_y \partial_x + q_x q_{xy} - q_y q_{xx})(a_0 - q_x^2) = 0. \tag{2.33}$$

We take

$$a_0 = q_x^2 \tag{2.34}$$

as it is an obvious solution to the above equation. Therefore, from (2.32), we have

$$b_{-1} = q_{xy} + q_x q_y = -u. \tag{2.35}$$

Now, from (2.19), we have

$$b_{-2} = \left(\frac{-1}{a_1}\right)(a_1 b_{-1,x} + a_0 b_{-1} + a_{-1,y} + a_{-1} b_0). \tag{2.36}$$

Substituting it into (2.27) and using (2.28), (2.29), (2.31), (2.34), and (2.35), we get

$$[\partial_y + \partial_x\left(\left(\frac{1}{q_x}\right)(\partial_y + q_y)\right)]a_{-1} = 0. \tag{2.37}$$

Thus we take

$$a_{-1} = 0 \tag{2.38}$$

as an obvious solution. Therefore, (2.36) becomes

$$b_{-2} = -q_{xxy} - q_{xx} q_y + q_x^2 q_y. \tag{2.39}$$

We are now ready to solve (2.7). Normalizing $e_1 = 1$, we have, from (2.10),

$$r_1 = b_0 = q_y. \tag{2.40}$$

From (2.12) and (2.39), we have

$$\begin{aligned} e_0(q_{xy} + q_x q_y) &= -(q_{xy} + q_x q_y)_x - (-q_{xxy} - q_{xx} q_y + q_x^2 q_y) \\ &= -q_x(q_{xy} + q_x q_y). \end{aligned} \tag{2.41}$$

Hence $e_0 = -q_x$, since $q_{xy} + q_x q_y = -u \neq 0$. From (2.11), we have

$$r_0 = q_{xy} + (-q_x) q_y + q_{xy} + q_x q_y = 2q_{xy}. \tag{2.42}$$

Hence, (2.7) becomes

$$(\partial_x - q_x)\phi_y = (q_y \partial_x + 2q_{xy})\phi. \tag{2.43}$$

Thus (2.2) and (2.43) form a pair of inverse scattering problems for (2.1).

**Remark 2.1.** Since $v_y = u_x$, $w_x = u_y$, and $u = -q_{xy} - q_x q_y$, we have

$$\begin{aligned} v &= -q_{xx} - \partial_y^{-1}(q_{xx} q_y) - \left(\frac{1}{2}\right) q_x^2, \\ w &= -q_{yy} - \partial_x^{-1}(q_x q_{yy}) - \left(\frac{1}{2}\right) q_y^2. \end{aligned} \tag{2.44}$$

Substituting $u$, $v$, and $w$ by their expressions in terms of $q$ into (2.2), and using (2.43), we get an inverse scattering problem for the modified Veselov-Novikov equation, (2.31).

**Remark 2.2.** From (2.13), we have

$$b_{-h,x} - q_x b_{-h} + b_{-h-1} = 0, \quad h \geq 1. \tag{2.45}$$

Multiplying it by $a_1$,

$$a_1 b_{-h,x} - a_1 q_x b_{-h} + a_1 b_{-h-1} = 0. \tag{2.46}$$

Since $a_0 = -a_1 q_x$,

$$a_1 b_{-h,x} + a_0 b_{-h} + a_1 b_{-h-1} = 0. \tag{2.47}$$

Now, suppose $a_{-1} = \cdots = a_{-h+1} = 0$, then from (2.21) and (2.47),

$$(a_{-h})_y + a_{-h} b_0 = 0, \tag{2.48}$$

that is,

$$(\partial_y + q_y) a_{-h} = 0. \tag{2.49}$$

Thus we can take $a_{-h} = 0$. Since we have shown in (2.38) that we can take $a_{-1} = 0$, hence we can take $a_{-h} = 0$, $h \geq 1$. Note that this provides a consistency between (2.13) and (2.21). Also, (2.14) becomes $\Psi = (a_1 \partial_x + a_0)\phi = -q_x \phi_x + q_x^2 \phi$.

**Remark 2.3.** From (2.47),

$$\begin{aligned} b_{-h-1} &= -b_{-h,x} + q_x b_{-h} = (-\partial_x + q_x) b_{-h} \\ &= (-\partial_x + q_x)^h b_{-1} = -(-\partial_x + q_x)^h u, \quad \text{for } h \geq 1. \end{aligned} \tag{2.50}$$

This provides a recursive relation among the coefficients of (2.8).

**Remark 2.4.** Let $q = \ln \Psi$; we get from (2.29) and (2.31) that

$$\Psi_{xy} + u\Psi = 0, \qquad \Psi_t = \Psi_{xxx} + 3v\Psi_x + \Psi_{yyy} + 3w\Psi_y. \tag{2.51}$$

This is the inverse scattering problem for (2.1) given in [13].

**3. Kadomtsev-Petviashvili equation.** We now turn to the Kadomtsev-Petviashvili equation [8]

$$u_t = \partial_x^{-1} u_{yy} - u_{xxx} - 6uu_x. \tag{3.1}$$

The adjoint of its linearized equation is

$$\phi_t = \partial_x^{-1}\phi_{yy} - \phi_{xxx} - 6u\phi_x. \tag{3.2}$$

Just like in the case of the Veselov-Novikov equation, we propose (2.4) where $L$ and $R$ are partial differential operators with respect to $x$ and the coefficients depend on $u$ and its partial derivatives but not explicitly on $x$, $y$, or $t$.

We demand that (3.2) and (2.4) be compatible. In order to do so, we rewrite (2.4) as $\phi_y = L^{-1}R\phi$, where, formally,

$$L^{-1}R = a_m\partial_x^m + a_{m-1}\partial_x^{m-1} + \cdots + a_0 + a_{-1}\partial_x^{-1} + \cdots = \sum_{k\le m} a_k\partial_x^k. \tag{3.3}$$

Thus

$$\phi_y = \sum_{k\le m} a_k\partial_x^k\phi. \tag{3.4}$$

We now demand that (3.2) and (3.4) be compatible, that is, $\phi_{txy} = \phi_{ytx}$.

Differentiating both sides of (3.2) with respect to $x$ and $y$, we get

$$\begin{aligned}\phi_{txy} = {} & \sum_{k\le m} a_{k,yy}\partial_x^k\phi + 2\sum_{k\le m} a_{k,y}\partial_x^k\phi_y + \sum_{k\le m} a_k\partial_x^k\phi_{yy} - 6u_{xy}\phi_x \\ & - 6u_x\left(\sum_{k\le m} a_{k,x}\partial_x^k\phi + \sum_{k\le m} a_k\partial_x^{k+1}\phi\right) - 6u_y\phi_{xx} \\ & - 6u\left(\sum_{k\le m} a_{k,xx}\partial_x^k\phi + 2\sum_{k\le m} a_{k,x}\partial_x^{k+1}\phi + \sum_{k\le m} a_k\partial_x^{k+2}\phi\right) \\ & - \Bigg(\sum_{k\le m} a_{k,xxxx}\partial_x^k\phi + 4\sum_{k\le m} a_{k,xxx}\partial_x^{k+1}\phi + 6\sum_{k\le m} a_{k,xx}\partial_x^{k+2}\phi \\ & \quad + 4\sum_{k\le m} a_{k,x}\partial_x^{k+3}\phi + \sum_{k\le m} a_k\partial_x^{k+4}\phi\Bigg).\end{aligned} \tag{3.5}$$

Differentiating both sides of (3.4) with respect to $t$ and $x$, we get

$$\begin{aligned}\phi_{ytx} = {} & a_{m,t}\partial_x^{m+1}\phi + \sum_{k\le m}(a_{k,tx} + a_{k-1,t})\partial_x^k\phi \\ & + \sum_{k\le m} a_k\partial_x^k\phi_{yy} - 6\sum_{k\le m} a_k\partial_x^{k+1}(u\phi_x) - \sum_{k\le m} a_k\partial_x^{k+4}\phi \\ & + \sum_{k\le m} a_{k,x}\partial_x^{k-1}\phi_{yy} - 6\sum_{k\le m} a_{k,x}\partial_x^k(u\phi_x) - \sum_{k\le m} a_{k,x}\partial_x^{k+3}\phi.\end{aligned} \tag{3.6}$$

Demanding $\phi_{ytx} = \phi_{txy}$, we get, after simplification,

$$\begin{aligned}
&a_{m,t}\partial_x^{m+1}\phi + \sum_{k\le m}(a_{k,tx}+a_{k-1,t})\partial_x^k\phi - 6\sum_{k\le m} a_k\partial_x^{k+1}(u\phi_x) + \sum_{k\le m} a_{k,x}\partial_x^{k-1}\phi_{yy} \\
&\quad -6\sum_{k\le m} a_{k,x}\partial_x^k(u\phi_x) + 3\sum_{k\le m} a_{k,x}\partial_x^{k+3}\phi \\
&\quad = \sum_{k\le m} a_{k,yy}\partial_x^k\phi + 2\sum_{k\le m} a_{k,y}\partial_x^k\phi_y - 6u_{xy}\phi_x \\
&\quad -6u_x\left(\sum_{k\le m} a_{k,x}\partial_x^k\phi + \sum_{k\le m} a_k\partial_x^{k+1}\phi\right) - 6u_y\phi_{xx} \\
&\quad -6u\left(\sum_{k\le m} a_{k,xx}\partial_x^k\phi + 2\sum_{k\le m} a_{k,x}\partial_x^{k+1}\phi + \sum_{k\le m} a_k\partial_x^{k+2}\phi\right) \\
&\quad -\left(\sum_{k\le m} a_{k,xxxx}\partial_x^k\phi + 4\sum_{k\le m} a_{k,xxx}\partial_x^{k+1}\phi + 6\sum_{k\le m} a_{k,xx}\partial_x^{k+2}\phi\right).
\end{aligned} \tag{3.7}$$

**Claim 3.1** (if $m \ge 3$, then $a_m = 0$)**.** Suppose $m \ge 3$ and $a_m \ne 0$. Using (3.4) for $\phi_y$ and $\phi_{yy}$ to equate the coefficients of the same order of partial derivatives of $\phi$ with respect to $x$ on both sides of (3.7), we get, for the coefficient of $\partial_x^{3m-1}\phi : a_{m,x}a_m^2 = 0$. Hence,

$$a_m = \text{constant} \ne 0. \tag{3.8}$$

Note that since $m \ge 3$, $\partial_x^{3m-1}\phi$ is the highest order of partial derivative of $\phi$ with respect to $x$. Substituting (3.8) into (3.7), we get

$$\begin{aligned}
&a_{m-1,t}\partial_x^{m}\phi + \sum_{k\le m-1}(a_{k,tx}+a_{k-1,t})\partial_x^k\phi - 6\sum_{k\le m} a_k\partial_x^{k+1}(u\phi_x) \\
&\quad + \sum_{k\le m-1} a_{k,x}\partial_x^{k-1}\phi_{yy} - 6\sum_{k\le m-1} a_{k,x}\partial_x^k(u\phi_x) + 3\sum_{k\le m-1} a_{k,x}\partial_x^{k+3}\phi \\
&\quad = \sum_{k\le m-1} a_{k,yy}\partial_x^k\phi + 2\sum_{k\le m-1} a_{k,y}\partial_x^k\phi_y - 6u_{xy}\phi_x \\
&\quad -6u_x\left(\sum_{k\le m-1} a_{k,x}\partial_x^k\phi + \sum_{k\le m} a_k\partial_x^{k+1}\phi\right) - 6u_y\phi_{xx} \\
&\quad -6u\left(\sum_{k\le m-1} a_{k,xx}\partial_x^k\phi + 2\sum_{k\le m-1} a_{k,x}\partial_x^{k+1}\phi + \sum_{k\le m} a_k\partial_x^{k+2}\phi\right) \\
&\quad -\left(\sum_{k\le m-1} a_{k,xxxx}\partial_x^k\phi + 4\sum_{k\le m-1} a_{k,xxx}\partial_x^{k+1}\phi + 6\sum_{k\le m-1} a_{k,xx}\partial_x^{k+2}\phi\right).
\end{aligned} \tag{3.9}$$

Again, using (3.4) for $\phi_y$ and $\phi_{yy}$ to equate the coefficients of the partial derivatives of $\phi$ with respect to $x$ on both sides of (3.9), we get, for the coefficient of $\partial_x^{3m-2}\phi$, $a_{m-1,x}a_m^2 = 0$. Hence $a_{m-1,x} = 0$, $a_{m-1} = \text{constant}$. Note that since $m \ge 3$, $\partial_x^{3m-2}\phi$ is the highest order of partial derivatives of $\phi$ with respect to $x$ in (3.9).

Repeating the above argument, we can show that, for the coefficient of $\partial_x^{3m-h}\phi$, $a_{m-h+1,x}a_m^2 = 0$ for $1 \le h < 2m-2$. Hence $a_{m-h+1,x} = 0$, $a_{m-h+1} =$ constant for $1 \le h < 2m-2$. When $h = 2m-2$, $\partial_x^{3m-h}\phi = \partial_x^{m+2}\phi$. Now, equating the coefficient of $\partial_x^{m+2}\phi$ on both sides of (3.9), we get $-6a_m u - a_{-m+3,x}a_m^2 = -6ua_m$. Hence $a_{-m+3,x} = 0$, $a_{-m+3} =$ constant. Now, equating the coefficient of $\partial_x^{m+1}\phi$ on both sides of (3.9), we get $-6a_m(m+1)u_x - 6a_{m-1}u = 6u_x a_m - 6ua_{m-1}$. Hence $6a_m(m+2)u_x = 0$.

Hence $a_m = 0$. But this is a contradiction to the assumption that $a_m \neq 0$.

Thus, if $m \ge 3$, then $a_m = 0$. Therefore, (3.4) becomes

$$\phi_y = \sum_{k\le 2} a_k \partial_x^k \phi. \tag{3.10}$$

Now with $m = 2$ in (3.7) and using (3.10) to equate the coefficients of $\partial_x^{(h)}\phi$ on both sides of (3.7) for $h \le 5$, we get, for $\partial_x^5\phi$,

$$(a_2^2+3)a_{2,x} = 0 \quad \text{(hence either } a_2^2+3 = 0 \text{ or } a_2 = \text{constant)}. \tag{3.11}$$

For $\partial_x^4\phi$,

$$(a_2^2+3)a_{1,x} = 0 \quad \text{(hence either } a_2^2+3 = 0 \text{ or } a_1 = \text{constant)}. \tag{3.12}$$

For $\partial_x^3\phi$,

$$12a_2u_x - 2a_{1,x}a_2a_1 - 6a_{1,xx} + 2a_{1,y}a_2 - (a_2^2+3)a_{0,x} = 0. \tag{3.13}$$

For $\partial_x^2\phi$,

$$\begin{aligned} a_{1,t} &- 18a_2u_{xx} - 6a_1u_x + 2a_{1,x}a_0a_2 + 2(a_{1,x})^2a_2 \\ &+ a_1^2a_{1,x} + 2a_1a_2a_{0,x} + 6a_{1,x}u - 2a_{1,y}a_1 \\ &+ 6u_y - 2a_{0,y}a_2 + 4a_{1,xxx} + 6a_{0,xx} + (a_2^2+3)a_{-1,x} = 0. \end{aligned} \tag{3.14}$$

In view of (3.11) and (3.12), we will take $a_2^2+3 = 0$ so that $a_1$ does not have to be a constant. Without loss of generality, let $a_2 = -(3)^{1/2}i$, where $i = (-1)^{1/2}$. From (3.13),

$$12a_2u_x = 2a_{1,x}a_2a_1 + 6a_{1,xx} - 2a_{1,y}a_2. \tag{3.15}$$

Hence

$$u_x = \left(\frac{1}{6}\right)a_{1,x}a_1 + \left(\frac{1}{(2a_2)}\right)a_{1,xx} - \left(\frac{1}{6}\right)a_{1,y}. \tag{3.16}$$

Thus

$$u = \left(\frac{1}{12}\right)a_1^2 + \left(\frac{1}{(2a_2)}\right)a_{1,x} - \left(\frac{1}{6}\right)\partial_x^{-1}a_{1,y} + \lambda \quad \text{for some constant } \lambda. \tag{3.17}$$

Substituting (3.17) into (3.1), we get

$$\begin{aligned} 0 &= u_t - \partial_x^{-1} u_{yy} + u_{xxx} + 6uu_x \\ &= M\left(a_{1,t} - \partial_x^{-1} a_{1,yy} + a_{1,xxx} + \left(\frac{1}{2}\right) a_1^2 a_{1,x} - (\partial_x^{-1} a_{1,y}) a_{1,x} + 6\lambda a_{1,x}\right), \end{aligned} \tag{3.18}$$

where $M = (1/6)a_1 + (1/(2a_2))\partial_x - (1/6)\partial_x^{-1}\partial_y$.

Thus (3.17) is a Miura-type transformation relating the Kadomtsev-Petviashvili equation, (3.1), to the modified Kadomtsev-Petviashvili equation

$$a_{1,t} - \partial_x^{-1} a_{1,yy} + a_{1,xxx} + \left(\frac{1}{2}\right) a_1^2 a_{1,x} - (\partial_x^{-1} a_{1,y}) a_{1,x} + 6\lambda a_{1,x} = 0. \tag{3.19}$$

Substituting (3.17) and (3.19) into (3.14), we get

$$K(a_0 - a_{1,x}) = 0, \tag{3.20}$$

where $K = 6\partial_x^2 + 2a_1 a_2 \partial_x + 2a_{1,x} a_2 - 2a_2 \partial_y$.

Thus $a_0 - a_{1,x} = 0$ is a solution to (3.20). Hence, we take

$$a_0 = a_{1,x}. \tag{3.21}$$

Now equating the coefficient of $\partial_x \phi$ on both sides of (3.7) with the use of (3.17), (3.19), and (3.21), we get $K(a_{-1}) = 0$. Again we take

$$a_{-1} = 0 \tag{3.22}$$

as the solution. Now equating the coefficient of $\phi$ on both sides of (3.7) with the use of (3.17), (3.19), (3.21), and (3.22), we get $K(a_{-2}) = 0$. Again, we take $a_{-2} = 0$ as the solution.

We now claim that we can take $a_{-n} = 0$ for all $n \geq 1$. Assume that $a_{-h} = 0$ for $1 \leq h \leq n+1$, where $n \geq 1$. We want to show that we can take $a_{-n-2} = 0$ as a solution.

Equating the coefficient of $\partial_x^{-n}\phi$ on both sides of (3.7) and using (3.17), (3.19), and (3.21) and the assumption that $a_{-h} = 0$ for $1 \leq h \leq n+1$, where $n \geq 1$, we get $K(a_{-n-2}) = 0$. Thus we can take $a_{-n-2} = 0$ as a solution. Since we have already shown that we can take $a_{-1} = 0$ and $a_{-2} = 0$, we can take $a_{-n} = 0$ for all $n \geq 1$. Hence (3.4) can be terminated at $a_0$, that is,

$$\phi_y = a_2 \phi_{xx} + a_1 \phi_x + a_0 \phi, \tag{3.23}$$

where $a_2 = -(3)^{1/2} i$, $a_0 = a_{1,x}$, and $a_1$ satisfies (3.17) and (3.19). Thus (3.2) and (3.23),

$$\begin{aligned} \phi_t &= \partial_x^{-1} \phi_{yy} - \phi_{xxx} - 6u\phi_x, \\ \phi_y &= a_2 \phi_{xx} + a_1 \phi_x + a_0 \phi, \end{aligned} \tag{3.24}$$

form a pair of inverse scattering problems for (3.1).

**Remark 3.2.** Substituting $u = (1/12)a_1^2 + (1/(2a_2))a_{1,x} - (1/6)\partial_x^{-1}a_{1,y} + \lambda$ into (3.2), and using (3.23), we get a pair of inverse scattering problems for the modified Kadomtsev-Petviashvili equation, (3.19).

**Remark 3.3.** Let $a_1 = 2(3)^{1/2}i\Psi_x/\Psi$. With $a_2 = -(3)^{1/2}i$, we get, from (3.17) and (3.19),

$$\begin{gathered}\Psi_{xx} + (3)^{-1/2}i\Psi_y + u\Psi = \lambda\Psi,\\ \Psi_t + 4\Psi_{xxx} + 6u\Psi_x + 3u_x\Psi - (3)^{1/2}(\partial_x^{-1}u_y)\Psi = 0.\end{gathered} \tag{3.25}$$

These are the Lax-pair equations found by Dryuma [4].

**4. Conclusion.** We have demonstrated a method to construct the inverse scattering problems for the integrable nonlinear evolution equations in the two-spatial dimension in which the temporal equation is the adjoint of the linearized equation. A Miura-type transformation and a modified equation are obtained as a byproduct of the process. An inverse scattering problem for the modified equation is also found. The applicability of this method for testing the integrability of nonlinear evolution equations in the two-spatial dimension is currently under study.

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H. H. Chen: Department of Physics and Astronomy, University of Maryland, College Park, MD 20742, USA

*E-mail address*: chenhh@umd.edu

J. E. Lin: Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA

*E-mail address*: jelin@gmu.edu