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ON POLYNOMIALS WITH SIMPLE TRIGONOMETRIC FORMULAS

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We show that the sequences of polynomials with zeros $\cot(m\pi/(n+2))$ and $\tan(m\pi/(n+2))$ are not orthogonal sequences with respect to any integral inner product. We give an algebraic formula for these polynomials, that is simpler than the formula originally derived by Cvijovic and Klinowski (1998). New sequences of polynomials with algebraic numbers as roots and closed trigonometric formulas are also derived by these methods.

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1. Introduction. It is easy to see that $\cot(m\pi/(n+2))$ and $\tan(m\pi/(n+2))$ are algebraic numbers, $n = 1,2,3,\ldots$, and $m = 1,2,\ldots,n+1$, unless $n+2$ is even and $m = (n+2)/2$, where the tangent is undefined. The harder problem of actually finding a polynomial of degree n or $n+1$ with integer coefficients having these numbers as roots was solved by Cvijovic and Klinowski [2], who showed that the cotangents above are roots of

$$C_{n+1}(x) = \sum_{m=0}^{[(n+1)/2]} (-1)^m \binom{n+2}{2m+1} x^{n-2m+1} \tag{1.1}$$

(here we use the degree of $C_{n+1}(x)$ as the subscript) and the tangents are roots of the reciprocal polynomial given by $K_{n+1}(x) = x^{n+1}C_{n+1}(1/x)$ (here the degree of $K_{2m} =$ the degree of $K_{2m+1} = 2m$ for all $m = 0,1,\ldots$). We noticed that the first sequence of polynomials, $\{C_n(x)\}$, has real roots and the root interlacing property, a property that sequences of real orthogonal polynomials are also known to have (for each $n \geq 1$, putting all the roots in ascending order, the roots of $C_{n+1}(x)$ alternate with those of $C_n(x)$). This led to our motivating question: is $\{C_n(x)\}$ a sequence of polynomials orthogonal with respect to some weighted integral inner product? We found that a change of variable gave a relation between Chebyshev polynomials and the above polynomials allowing a three-term recurrence formula for the $\{C_n(x)\}$ to be derived. While considering these formulas, we discovered an algebraic closed form for the polynomials $C_{n+1}(x)$ that allows the main results of [2] to be easily derived without the use of their expansion formula or this connection to the Chebyshev polynomials. It also suggested how to find such other polynomials having roots related to the remaining trigonometric functions.

2. Results

2.1. Is then $\{C_n(x)\}$ a sequence of polynomials orthogonal with respect to some weighted integral inner product? Recall that for an orthogonal sequence of polynomials $\{P_n(x)\}$, a three-term recurrence formula must hold of the type $P_{n+1}(x) = (a_n x - b_n)P_n(x) - c_n P_{n-1}(x)$ [1, page 178]. We exhibit a three-term recurrence formula between the $\{C_n(x)\}$ of a different type showing that these do not form an orthogonal sequence of polynomials. To do this, a simple closed form for these polynomials is given, which also makes the closed trigonometric form easy to compute. First, note that

$$C_{n+1}(x) = \Im(x+i)^{n+2} = \frac{(x+i)^{n+2} - (x-i)^{n+2}}{2i} \tag{2.1}$$

and has degree $n+1$. It follows that

$$\begin{aligned} 2iC_{n+1}(x) &= (x^2+2ix-1)(x+i)^n - (x^2-2ix-1)(x-i)^n \\ &= [2x(x+i) - (1+x^2)](x+i)^n - [2x(x-i) - (1+x^2)](x-i)^n. \end{aligned} \tag{2.2}$$

Thus,

$$C_{n+1}(x) = 2xC_n(x) - (1+x^2)C_{n-1}(x), \tag{2.3}$$

showing that these are not a system of orthogonal polynomials. It is also clear that $K_{n+1}(x)$ is not a sequence of orthogonal polynomials since, from $K_{n+1}(x) = x^{n+1}C_{n+1}(1/x)$, follows

$$K_{n+1}(x) = 2K_n(x) - (1+x^2)K_{n-1}(x). \tag{2.4}$$

The closed trigonometric forms given in [2] now easily follow by substituting $x = \cot(\theta)$ in (2.1). Since $\cot(\theta) + i = e^{i\theta}/\sin(\theta)$, it follows that

$$C_{n+1}(\cot(\theta)) = \frac{\sin((n+2)\theta)}{\sin^{n+2}(\theta)}. \tag{2.5}$$

This expression is zero when $\theta = m\pi/(n+2)$, $m = 1, \dots, n+1$, verifying that the roots of $C_{n+1}(x)$ are the cotangents of these θ. The relation between $K_{n+1}(x)$ and $C_{n+1}(x)$ produces the closed form

$$K_{n+1}(\tan(\theta)) = \frac{\sin((n+2)\theta)}{\sin(\theta)\cos^{n+1}(\theta)}, \tag{2.6}$$

giving the tangent expression for the roots of K_{n+1} mentioned above.

2.2. Naturally, one is next led to investigate

$$P_{n+2}(x) = \Re(x+i)^{n+2} \tag{2.7}$$

of degree $n+2$. Here, it is found that $P_{n+2}(\cot(\theta)) = \cos((n+2)\theta)/\sin^{n+2}(\theta)$ or

$$P_{n+2}(x) = \frac{\cos\left((n+2)\cot^{-1}(x)\right)}{\sin^{n+2}\left(\cot^{-1}(x)\right)} \tag{2.8}$$

having roots

$$\cot\left(\frac{(2m+1)}{(n+2)}\frac{\pi}{2}\right), \tag{2.9}$$

$m = 0,1,\dots,n+1$, and the roots of $Q_{n+2}(x) = x^{n+2}P_{n+2}(1/x)$ are $\tan(((2m+1)/(n+2))(\pi/2))$, $m = 0,1,\dots,n+1$ (unless n is odd, in which case $m = (n+1)/2$ does not define a root), giving two more sequences of polynomials with trigonometric formulas and roots. The three-term recurrence relation for $\{P_n(x)\}$ (and, of course, $\{Q_n(x)\}$) can be derived as above and is the same as that for $\{C_n(x)\}$ given in (2.3), except that the initial values are different: $P_0(x) = 1$, $P_1(x) = x$, while $C_0(x) = 1$, and $C_1(x) = 2x$. These seem to be previously unnoticed sequences of polynomials with simple formulas for the roots. (See [2].)

2.3. The well-known Chebyshev polynomials [1] are orthogonal polynomials and have the trigonometric forms given by $T_{n+1}(x) = \cos((n+1)\cos^{-1}(x))$, $-1 \le x \le 1$, and $U_{n+1}(x) = \sin((n+2)\cos^{-1}(x))/\sin(\cos^{-1}(x))$, $-1 < x < 1$. The roots of these are $\cos(((2m+1)/(n+1))(\pi/2))$, $m = 0,\dots,n$, and $\cos(m\pi/(n+2))$, $m = 1,\dots,n+1$, respectively. These are mentioned in [2], but not the related reciprocal polynomials. These also have simple root formulas and are given by $V_{n+1}(x) = x^{n+1}T_{n+1}(1/x)$ which can be written as

$$V_{n+1}(x) = \frac{\cos\left((n+1)\sec^{-1}(x)\right)}{\cos^{n+1}\left(\sec^{-1}(x)\right)}, \quad x < -1,\ x > 1, \tag{2.10}$$

with roots $\sec(((2m+1)/(n+1))(\pi/2))$, $m = 0,\dots,n$, where, if n is even, $m \ne n/2$, and $W_{n+1}(x) = x^{n+1}U_{n+1}(1/x)$ giving

$$W_{n+1}(x) = \frac{\sin\left((n+2)\sec^{-1}(x)\right)}{\sin\left(\sec^{-1}(x)\right)\cos^{n+1}\left(\sec^{-1}(x)\right)}, \quad x < -1,\ x > 1, \tag{2.11}$$

with roots $\sec(m\pi/(n+2))$, $m = 1,\dots,n+1$, and, again, if n is even, $m \ne (n+2)/2$.

2.4. This leaves the determination of polynomials having roots which are sines or cosecants of the above multiples of π or $\pi/2$. Suppose that $x = \sin(\theta)$. Then, $x = \cos(\pi/2-\theta)$, so this expression can be substituted into the Chebyshev polynomials to solve these remaining cases. For example,

$$T_{n+1}(x) = \cos\left(\frac{(n+1)\pi}{2} - (n+1)\sin^{-1}(x)\right) \tag{2.12}$$

has roots of the type $\sin(m\pi/(n+1))$, when n is even and, when n is odd, roots of the type $\sin(((2m+1)/(n+1))(\pi/2))$. Similar results hold for $U_{n+1}(x)$, with the roots also varying as n is even or odd. Finally, the related reciprocal polynomials also have closed trigonometric formulas and roots which are cosecants of these types.

2.5. We conclude with two relations between $C_{n+1}(x)$ and $U_{n+1}(x)$. If $y = \cos(\theta)$, $0 < \theta < \pi$, then $U_{n+1}(y) = (1-y^2)^{(n+1)/2}C_{n+1}(y/(1-y^2)^{1/2})$, while, if $x = \cot(\theta)$, then $C_{n+1}(x) = U_{n+1}(x/(1+x^2)^{1/2})(1+x^2)^{(n+1)/2}$. Equation (2.3) was originally derived from a relation like this and the three-term formula for the Chebyshev polynomials. Similar relations hold between the Chebyshev polynomials of the first kind, $T_{n+1}(x)$, and the polynomials $P_{n+2}(x)$ above.

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References

[1] K. E. Atkinson, *An Introduction to Numerical Analysis*, John Wiley & Sons, New York, 1978.
[2] D. Cvijovic and J. Klinowski, *Algebraic trigonometric numbers*, Nieuw Arch. Wisk. (4) **16** (1998), no. 1-2, 11–14.

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