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REFINEMENT EQUATIONS FOR GENERALIZED TRANSLATIONS

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We study refinement equations which relate the dilation of a function with generalized translates of the function, consisting of convolutions against certain kernels including Cauchy and Gaussian densities; solutions are expressed in terms of solutions of the corresponding refinement equation involving ordinary translation.

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1. Introduction. Refinement equations (also known as scale equations or dilation equations) form the basis of typical wavelet constructions (see, e.g., [4]). Such equations relate a dilation of a function to a combination of its translates. Here, we will consider a refinement equation involving a generalized translation operator T consisting of convolution against a kernel function or density,

$$\phi\left(\frac{x}{2}\right) = \sum_{k\geq 0} 2c_k T^k \phi(x). \tag{1.1}$$

We will consider this equation in the case that $\sum c_k = 1$ and $c_k \geq 0$ for all k. This is therefore a generalization of the refinement equation

$$\phi\left(\frac{x}{2}\right) = \sum_{k\geq 0} 2c_k \phi(x-k). \tag{1.2}$$

We will consider the generalized translation operator T consisting of convolution against a Cauchy density function, and we will consider T consisting of convolution against a Gaussian density function. In the first case, we define the operator T by $Tf = h * f$, where $h(x) = g(x-1)$, where g is the Cauchy density function $g(x) = 1/(\pi(1+x^2))$. In the second case, we define $Tf = G(1,\sigma^2) * f$, where

$$G(\mu,\sigma^2;x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]. \tag{1.3}$$

Under certain weak conditions on the coefficients c_k (see below), the refinement equation (1.2) has a solution η consisting of a probability measure given by the weak-$*$ convergent convolution product

$$\eta = \mathop{*}_{j=1}^{\infty} \left(\sum_{k\geq 0} c_k \delta_{k/2^j}\right). \tag{1.4}$$

Our strategy for solving (1.1) is to locate a function $f(x,t)$ such that

$$\phi(x) = \int f(x,t)\,d\eta(t). \tag{1.5}$$

In Section 2 of this paper, we will do this for the refinement equation involving convolution against the Cauchy kernel, and in Section 3 we will do this for certain η for the refinement equation involving the Gaussian kernel. In the last section of this paper, we will define a general partial multiresolution analysis for generalized translations.

Now, (1.1) is an example of a more general sort of refinement equation

$$\phi\left(\frac{x}{2}\right) = 2\mu * \phi(x), \tag{1.6}$$

where μ is a finite regular Borel measure with $\int d\mu = 1$. Equations of this sort are known as continuous refinement equations, and are studied in many references, including [1, 3, 5, 6, 9, 10, 11, 13, 14, 15]. In particular, [9] gives simple conditions for such an equation to have distributional solutions, in great generality, while [5] uses probabilistic methods to study the convergence of subdivision algorithms solving such equations. Chui and Shi [1] give conditions for continuous refinement equations to have solutions, and they develop a continuous multiresolution analysis with associated dyadic wavelets for these equations. Rvachev [14] defines the "up-function," a smooth, compactly supported function solving the equation $2\phi(x/2) = \int_{x-1}^{x} \phi(y)\,dy$, and develops many applications. Kabaya-Imai and Iri [11] consider similar equations arising in estimates of roundoff errors in large calculations; and Dahmen and Micchelli consider other examples of equations generalizing the up-function equation. Goodman et al. [6] compute spectral radii for dilation-convolution operators, while Kabaya-Imai and Iri compute eigenvalues and eigenvectors for their operators.

Equation (1.6) may be solved iteratively, using the algorithm

$$\phi^{(n+1)}(x) = 2\left(\mu * \phi^{(n)}\right)(2x), \tag{1.7}$$

where $\phi^{(0)}$ is a suitable initial approximation such as the "box" function $\phi^{(0)} = 1_{[0,1)}$. Derfel et al. [5] use a theorem of Grincevičjus [8] concerning the convergence of certain series of random variables to prove the theorem that if $\int \sup\{1,\log|x|\}\,d\mu(x) < \infty$, then the iteration converges weak-$*$ to a probability measure. This is of course true if μ has compact support; the condition will also hold in the examples below. (A simple measure that does not meet this condition, and in fact for which the iteration diverges to zero, is $\mu = \sum_{n\ge 1} 1/(n(n+1))\delta_{2^{2^n}}$). This same condition is sufficient for η to be given by the convolution product given above in (1.4). (For more about such convolution products, see [7].) We use the iteration (1.7) to produce the plottings for Section 3.

2. The Cauchy density. Here, we consider (1.1) with generalized translation $Tf = h * f$, where $h(x) = g(x-1)$ and $g(x) = 1/(\pi(1+x^2))$. As mentioned above, our strategy for solving (1.1) for this generalized translation is to locate a function $f(x,t)$ such that $\phi(x) = \int f(x,t)\,d\eta(t)$, where η is a solution of (1.2). Indeed, we will show that

$$f(x,t)=\frac{g((x-t)/t)}{t}=\frac{1}{\pi}\frac{t}{t^2+(x-t)^2}\tag{2.1}$$

gives the desired result. We have the following theorem.

Theorem 2.1. *Equation (1.1) has solution*

$$\phi(x)=\int_0^n\frac{1}{\pi}\frac{t}{t^2+(x-t)^2}d\eta(t),\tag{2.2}$$

where η is the probability measure (1.4), where we assume that $c_k=0$ for $k>n$ in (1.2) (so the support of η is $[0,n]$).

Proof. First, we show that ϕ is well defined by (2.2). We will show that $\phi(x)$ exists (is a finite value) for every $x\neq 0$. Then, we will verify that $\phi\in L^1(\mathbb{R})$.

Now, if $x\neq 0$, the function $f(x,t)$ as defined by (2.1) is bounded as a function of t. Indeed, $f(x,t)$ reaches a maximum value of $(1+\sqrt{2})/(2\pi|x|)$ at $t=x/\sqrt{2}$ or $t=-x/\sqrt{2}$. Since η is a probability measure, then

$$|\phi(x)|\le\int_0^n\frac{1}{\pi}\frac{t}{t^2+(x-t)^2}d\eta(t)\le\int_0^n\frac{1+\sqrt{2}}{2\pi|x|}d\eta(t)\le\frac{1+\sqrt{2}}{2\pi|x|}.\tag{2.3}$$

So, $\phi(x)$ is finite for $x\neq 0$. We can then use Fubini's theorem to show $\phi\in L^1(\mathbb{R})$,

$$\|\phi\|_1=\iint f(x,t)d\eta(t)dx=\iint f(x,t)dx\,d\eta(t)=\int 1d\eta(t)=1.\tag{2.4}$$

We now are able to show that ϕ solves (1.1).

First, for $a>0$ define the dilation g_a of g by $g_a(x)=g(x/a)/a$. It turns out that $g_a*g_b=g_{a+b}$. (This follows from the fact that the Fourier transform of g_a is $\hat{g}_a(\xi)=e^{-|a\xi|}$.)

Next, we note a self-similarity property of the measure η: for any integrable function f,

$$\int f(2t)d\eta(t)=\int\left(\sum_{k\ge 0}c_kf(t+k)\right)d\eta(t).\tag{2.5}$$

This is because if we write $\tilde{\eta}$ for the dilation of η defined by $\tilde{\eta}(E)=\tilde{\eta}((1/2)E)$, then $\tilde{\eta}=(\sum_{k\ge 0}c_k\delta_k)*\eta$.

We now have

$$\begin{aligned}\frac{1}{2}\phi\left(\frac{x}{2}\right)&=\frac{1}{2}\int g_t\left(\left(\frac{x}{2}\right)-t\right)d\eta(t)=\int\frac{1}{2t}g\left(\frac{x-2t}{2t}\right)d\eta(t)\\&=\int\left(\sum_{k\ge 0}\frac{c_k}{t+k}g\left(\frac{x-(t+k)}{t+k}\right)\right)d\eta(t)=\int\sum_{k\ge 0}c_kg_{t+k}(x-t-k)d\eta(t).\end{aligned}\tag{2.6}$$

Then, we have

$$\begin{aligned}\sum_{k\geq 0} c_k (T^k\phi)(x) &= \int \sum_{k\geq 0} c_k \Big(\delta_k * \overbrace{g*\cdots*g}^{k\text{ factors}} * g_t * \delta_t\Big)(x)\,d\eta(t)\\ &= \int \sum_{k\geq 0} c_k (g_{t+k} * \delta_{t+k})(x)\,d\eta(t)\\ &= \int \sum_{k\geq 0} c_k g_{t+k}(x-t-k)\,d\eta(t) = \frac{1}{2}\phi\Big(\frac{x}{2}\Big).\end{aligned} \tag{2.7}$$

(Fubini's theorem is used in the first equality.) So, we have verified that ϕ satisfies (1.1). □

It may be observed that the Fourier transform of the solution ϕ to (1.1) satisfies the equation

$$\hat{\phi}(\xi) = \hat{\eta}(\xi - i|\xi|). \tag{2.8}$$

This suggests generalizing Theorem 2.1 to higher dimensions.

To that end, we will write $x \in \mathbb{R}^n$ as $x = (x_1, \ldots, x_n)$. We define addition and scalar multiplication as usual, and we will write $|x| = (|x_1|, \ldots, |x_n|)$. We will write $x \geq 0$ provided that $x_1 \geq 0, \ldots, x_n \geq 0$.

Let $\eta = *_{j=1}^{\infty}(\sum_{\alpha\geq 0} c_\alpha \delta_{2^{-j}\alpha})$, where $\alpha \in \mathbb{Z}^n$; we assume that $c_\alpha > 0$ and $\sum_\alpha c_\alpha = 1$. The measure η satisfies the refinement equation

$$\phi(2^{-1}x) = \sum_{\alpha\geq 0} 2c_\alpha \phi(x-\alpha). \tag{2.9}$$

Let

$$g_t(x) = g(t,x) = \left(\frac{t_1}{\pi(t_1^2 + x_1^2)}\right)\cdots\left(\frac{t_n}{\pi(t_n^2 + x_n^2)}\right). \tag{2.10}$$

Defining the Fourier transform on $\mathbb{R}^n$ as usual by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\xi\cdot x}dx$, we find that the Fourier transform of $g(t,x)$ as a function of x is $e^{-t\cdot|\xi|} = e^{-t_1|\xi_1|}\cdots e^{-t_n|\xi_n|}$. From this it follows that $g(t,x) * g(s,x) = g(t+s,x)$ for any $t,s \in \mathbb{R}^n$ with $s,t \geq 0$ (where the convolution is with respect to the second argument x).

Then, we may generalize the refinement equation (2.9) as

$$\phi(2^{-1}x) = \sum_{\alpha\geq 0} T^\alpha \phi(x), \tag{2.11}$$

where T^α is the operator $T^\alpha f = g_\alpha * \delta_\alpha * f$. It turns out that the solution of this equation is given by $\phi(x) = \int_{\mathbb{R}^n} g(t, t-x)\,d\eta(t)$; the proof is very similar to the proof of Theorem 2.1. In this setting, as expected, we have $\hat{\phi}(\xi) = \hat{\eta}(\xi - i|\xi|)$.

We may generalize Theorem 2.1 in a different direction, returning to (1.1) in a single dimension. In Theorem 2.1, we required the coefficients c_k to be nonnegative. If we no longer require this, we are no longer assured that η solving (1.2) is a measure. Instead, if $\sum_k c_k = 1$, and all but finitely many terms of this sum are zero, then η will be a distribution [2]. If we write $\langle f,\eta\rangle$ for the value of the distribution η on the test function f, we find that the solution of (1.1) (with the Cauchy density) is $\phi(x) = \langle g_t(x-t),\eta(t)\rangle$. We can no longer expect $\phi \in L^1(\mathbb{R})$; however ϕ will be smooth except possibly at $x = 0$ (where the function may be unbounded). In this context, we still have $\hat{\phi}(\xi) = \hat{\eta}(\xi - i|\xi|)$.

We conclude this section with two examples.

Example 2.2. The refinement equation (1.1) with $c_0 = c_1 = 1/2$ and $c_k = 0$ for $k > 1$ has the following solution: here, $\eta = *_{j=1}^{\infty}((1/2)\delta_0 + (1/2)\delta_{k/2^j})$ is just Lebesgue measure restricted to the unit interval, so

$$\begin{aligned}\phi(x) &= \int_0^1 \frac{1}{\pi}\frac{t}{t^2+(x-t)^2}dt \\ &= \frac{1}{8} + \frac{1}{4\pi}\ln\left(\frac{1+(x-1)^2}{x^2}\right) - \frac{1}{2\pi}\tan^{-1}\left(\frac{x-2}{x}\right).\end{aligned} \tag{2.12}$$

This may be likened to a Haar scaling function for the Cauchy density generalized refinement equation, although the graph bears no resemblance (this function is continuous except for an infinite discontinuity at 0).

Example 2.3. The refinement equation (1.1) with $c_0 = 1/4 = c_2$, $c_1 = 1/2$, and $c_k = 0$ for $k > 2$ has the following solution. This time, the measure η is the absolutely continuous measure with the "triangle" density function $f(t) = 1 - |t-1|$ for $0 \le t \le 2$ and $f(t) = 0$ for all other x. Therefore,

$$\begin{aligned}\phi(x) &= \int_0^2 \frac{1}{\pi}\frac{t(1-|t-1|)}{t^2+(x-t)^2}dt \\ &= \frac{1}{\pi}\tan^{-1}\left(\frac{x-2}{x}\right) - \frac{1}{\pi}\tan^{-1}\left(\frac{x-4}{x}\right) - \frac{x\ln|x|}{2\pi} \\ &\quad + \frac{(2-x)\ln(x^2-4x+8)}{4\pi} + \frac{(x-1)\ln(x^2-2x+2)}{2\pi}.\end{aligned} \tag{2.13}$$

This function is continuous and bounded.

3. The Gaussian density. Here, we consider (1.1) with generalized translation $Tf = G(1,\sigma^2) * f$, where G is the Gaussian density (1.3), with some choice of $\sigma > 0$. Again, our strategy for solving (1.1) for this generalized translation is to locate a function $f(x,t)$ such that $\phi(x) = \int f(x,t)d\eta(t)$. Indeed, we will show that $f(x,t) = G(t,\theta(t);x)$ for a certain function θ. However, we will only be able to show what gives a solution for (1.1) in the case $c_k = 0$ for $k > 1$.

To do this, we first note for real μ_1, μ_2 and $\sigma_1,\sigma_2 > 0$ that $G(\mu_1,\sigma_1^2) * G(\mu_2,\sigma_2^2) = G(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2) = G(\mu_1+\mu_2,c^2)$, where $c = \sqrt{\sigma_1^2+\sigma_2^2}$. This will allow us to determine the function θ.

So, we let $\phi(x) = \int G(t,\theta(t),x)\,d\eta(t)$, and we substitute this into (1.1). The left-hand side of that equation becomes

$$\begin{aligned}\phi\left(\frac{x}{2}\right) &= \int G\left(t,\theta(t);\frac{x}{2}\right)d\eta(t)\\ &= \int G(2t,4\theta(t);x)\,d\eta(t)\\ &= \int \sum_{k\ge 0} c_k G\left(t+k,4\theta\left(\frac{(t+k)}{2}\right);x\right)d\eta(t),\end{aligned} \tag{3.1}$$

where the last equality uses (2.5). The right-hand side of (1.1) becomes

$$\begin{aligned}\sum_{k\ge 0} c_k T^k\phi(x) &= \sum_{k\ge 0}\int \overbrace{G(1,\sigma^2;x) * \cdots * G(1,\sigma^2;x)}^{k \text{ factors}} * G(t,\theta(t);x)\,d\eta(t)\\ &= \sum_{k\ge 0}\int G(t+k,k\sigma^2+\theta(t);x).d\eta(t).\end{aligned} \tag{3.2}$$

Therefore, if $4\theta((t+k)/2) = k\sigma^2+\theta(t)$ for $k \ge 0$ and $t \in \operatorname{supp}(\eta)$, then ϕ solves (1.1). These equations are not compatible if $\operatorname{supp}(\eta) \notin [0,1]$, which occurs if $c_k \ne 0$ for $k > 1$, and which results in three or more equations. But if $c_k = 0$ for $k > 1$, then we only have the two equations $4\theta(t/2) = \theta(t)$ and $4\theta((t+1)/2) = \theta(t)+\sigma^2$, where $0 \le t \le 1$. These are equivalent to

$$\theta(t) = \begin{cases}\dfrac{1}{4}\theta(2t) & \text{if } 0 \le t \le \dfrac{1}{2},\\[2mm] \dfrac{1}{4}\sigma^2 + \dfrac{1}{4}\theta(2t-1) & \text{if } \dfrac{1}{2} < t \le 1.\end{cases} \tag{3.3}$$

There is a solution θ to this self-similarity condition:

$$\theta(t) = \sigma^2 \sum_{j\ge 0} \epsilon_j 4^{-j}, \tag{3.4}$$

where $t \in [0,1]$ has nonterminating binary expansion $t = \sigma^2\sum_{j\ge 0}\epsilon_j 2^{-j}$, where $\epsilon_j \in \{0,1\}$ for all j.

We have the following theorem.

Theorem 3.1. *The equation $\phi(x/2) = 2c_0\phi(x) + 2c_1 G(1,\sigma^2) * \phi(x)$, where $c_0 + c_1 = 1$ and $c_0, c_1 > 0$, has solution $\phi(x) = \int G(t,\theta(t),x)\,d\eta(t)$, where η solves $\phi(x/2) = 2c_0\phi(x) + 2c_1\phi(x-1)$, and where θ is given by (3.4).*

Proof. It remains to verify that $\phi(x) = \int G(t,\theta(t),x)d\eta(t)$ with this θ is well defined. Now, it happens that $\sigma^2 t^2/4 \le \theta(t) \le 4\sigma^2 t^2/3$ for all $t \in [0,1]$. Therefore, for such t,

$$G(t,\theta(t);x) = \frac{1}{\sqrt{2\pi\theta(t)}}\exp\left(\frac{-(x-t)^2}{2\theta(t)}\right) \le \frac{1}{\sqrt{\pi\sigma^2 t^2/2}}\exp\left(\frac{-(x-t)^2}{2\theta(t)}\right) \le \frac{1}{\sqrt{\pi\sigma^2 t^2/2}}\exp\left(\frac{-3(x-t)^2}{8\sigma^2 t^2}\right). \quad (3.5)$$

If $x \ne 0$, this is bounded. So, for $x \ne 0$, $\phi(x)$ is finite,

$$|\phi(x)| = \int_0^1 G(t,\theta(t),x)d\eta(t) \le \int_0^1 \frac{c}{|x|}d\eta(t) \le \frac{c}{|x|} \quad (3.6)$$

since η is a probability measure. It follows that $\phi \in L^1(\mathbb{R})$, using the same argument as in Theorem 2.1. Thus, ϕ is well defined, and the formal calculations above now verify that ϕ is a solution as claimed. □

We can say more about ϕ. Closer estimates of $G(t,\theta(t);x)$ can be made, which show that ϕ is continuous if $0 \le c_0 < 1/2$ and discontinuous (at $x = 0$) if $c_0 > 1/2$. (Recall $c_0 + c_1 = 1$.) Also, ϕ is differentiable if $c_0 < 1/4$ but not if $c_0 > 1/4$. (It turns out that $\phi(x) \sim x^\alpha$, where $\alpha = -\log_2(2c_0)$, for $0 < x < 1$.) Actually, we can conclude that ϕ is differentiable n times if $c_0 < 2^{-n}$, from [5, Theorem 11]. This requires the distribution $G(1,\sigma^2)$ to obey a finite moment condition, equation (3.2) of [5], which it satisfies. It is interesting to note that this moment condition is not satisfied for the Cauchy density considered in the previous section of this paper, so we are unable to give general conclusions about the smoothness properties of ϕ except for the explicit formulas in Examples 2.2 and 2.3.

Example 3.2. If $c_0 = 0$ and $c_1 = 1$ in Theorem 3.1, the function ϕ happens to be just $\phi = G(1,\sigma^2/3)$. Other examples of ϕ are plotted in Figures 4.1 and 4.2. These were generated using code written in C++, using the iteration (1.7), and plotted using the PiCTEX macro package. In Figure 4.1, $c_0 = 1/2$ and $\sigma = 0.05$. In Figure 4.2, $c_0 = 0.2$ and $\sigma = 0.05$. For both figures, 15 iterations of (1.7) were used. Both functions are plotted over the interval $[0,2]$ (for $x < 0$ and $x > 2$, the values of the functions are very small).

4. Multiresolution analyses. For a generalized translation, we may define a suitable notion of multiresolution analysis. Suppose ϕ is a function satisfying the refinement equation

$$\phi\left(\frac{x}{2}\right) = 2c_0\phi(x) + 2c_1\mu * \phi(x) \quad (4.1)$$

with $c_0 + c_1 = 1$ and $c_0, c_1 \ge 0$. We then define the space V_0 by

$$V_0 = \operatorname{span}\{\phi, \mu * \phi, \mu^2 * \phi, \mu^3 * \phi, \dots\}. \quad (4.2)$$

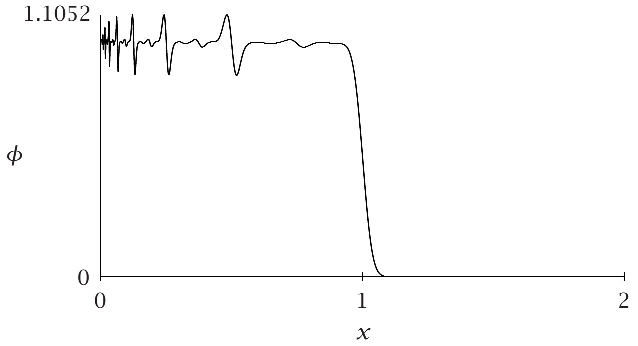

FIGURE 4.1

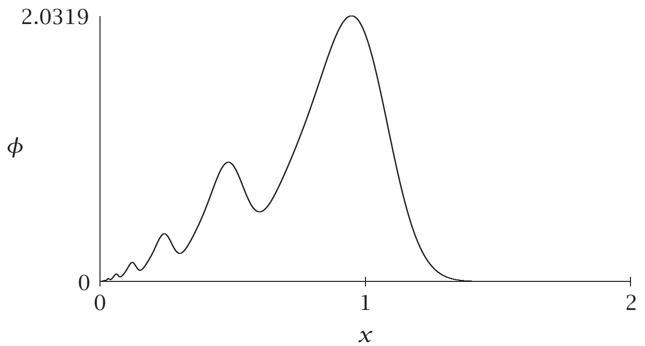

FIGURE 4.2

We would like to define (as usual) V_1 to be the dilation of V_0, but if we define V_1 in this way, then $V_0 \notin V_1$. Instead, let μ_1 be the dilation of μ (so $\mu_1(E) = \mu(2E)$ for Borel sets E), let $\phi_1(x) = \phi(2x)$ and let V_1 be defined by

$$V_1 = \operatorname{span}\{\mu^n * \phi_1, \mu^n * \mu_1 * \phi_1 : n \geq 0\}. \tag{4.3}$$

It then follows that $V_0 \subset V_1$.

This leads us to a definition of V_j for $j \geq 0$. We say $V_j = \operatorname{span}\{\phi_{j,k} : k \geq 0\}$, where $\phi_{j,k}$ is defined as follows. Suppose $k/2^j = n + \epsilon_1/2 + \epsilon_2/4 + \cdots + \epsilon_j/2^j$ is the binary expansion of $k/2^j$, so each $\epsilon_i \in \{0,1\}$ and n is a nonnegative integer. Let μ_j be the dilation of μ by 2^j (so $\mu_j(E) = \mu(2^j E)$ for Borel sets E), let $\phi_j(x) = \phi(2^j x)$, and for $0 \leq i \leq j$ let

$$\eta_i = \begin{cases} \delta_0 & \text{if } \epsilon_i = 0, \\ \mu_i & \text{if } \epsilon_i = 1. \end{cases} \tag{4.4}$$

Then, we define

$$\phi_{j,k} = \mu_0^n * \eta_1 * \eta_2 * \cdots * \eta_j * \phi_j. \tag{4.5}$$

(Note $\phi_j = \phi_{j,0}$.)

It then follows that $V_j \subseteq V_{j+1}$, since from (4.1) and (4.5) we may show that

$$c_0\phi_{j+1,2k} + c_1\phi_{j+1,2k+1} = \phi_{j,k}. \tag{4.6}$$

We will call the spaces V_j, so defined for $j \geq 0$, a *partial generalized multiresolution analysis*. We should note that if μ was just δ_1, then V_j would be the space spanned by $\phi(2^j x - k)$ (for $j \geq 0$ and $k \geq 0$).

A similar construction is given in [12], in a somewhat different context.

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