IJMMS 2004:72, 3965–3974
PII. S0161171204204124
http://ijmms.hindawi.com
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FINITE ELEMENT LEAST-SQUARES METHODS FOR A COMPRESSIBLE STOKES SYSTEM

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Received 25 April 2002

The least-squares functional related to a *vorticity* variable or a *velocity flux* variable is considered for two-dimensional compressible Stokes equations. We show ellipticity and continuity in an appropriate product norm for each functional.

2000 Mathematics Subject Classification: 65F10, 65F30.

1. Introduction. Let Ω be a convex polygonal domain in $\mathbb{R}^2$. Consider the stationary compressible Stokes equations with *zero* boundary conditions for the *velocity* $\mathbf{u} = (u_1, u_2)^t$ and *pressure* p as follows:

$$\begin{aligned} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad &\text{in } \Omega, \\ \nabla\cdot\mathbf{u} + \boldsymbol{\beta}\cdot\nabla p = g \quad &\text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where the symbols Δ, ∇, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta\mathbf{u}$ is the vector of components Δu_i); the number μ is a viscous constant; $\mathbf{f}$ is a given vector function; $\boldsymbol{\beta} = (U, V)^t$ is a given C^1 function. The system (1.1) may be obtained by linearizing the steady-state barotropic compressible viscous Navier-Stokes equations without an ambient flow (see [8, 9] for more detail). Since the continuity equation is of hyperbolic type containing a convective derivative of p, we further assume that the boundary condition for the pressure is given on the inlet of the boundary where the characteristic function $\boldsymbol{\beta}$ points into Ω, that is,

$$p = 0 \quad \text{on } \Gamma_{\text{in}}, \tag{1.2}$$

where $\Gamma_{\text{in}} = \{(x, y) \in \partial\Omega \mid \boldsymbol{\beta}\cdot\mathbf{n} < 0\}$ with the outward unit normal $\mathbf{n}$ to $\partial\Omega$. Hence the boundary $\partial\Omega$ consists of Γ_{in} and Γ_{out} where $\Gamma_{\text{out}} = \{(x, y) \mid \boldsymbol{\beta}\cdot\mathbf{n} \geq 0\}$. There was a study on a mixed finite element theory for a compressible Stokes system (see, e.g., [8]), but there are a few trials dealing with a compressible Stokes system like (1.1) using least-squares method. Some papers focused on a H^{-1} least-squares method (see, e.g., [6, 9]). Least-squares approach was developed for the incompressible Stokes and Navier-Stokes equations in [1, 2, 7]. The purpose of this paper is to apply the philosophy of first-order system least-squares (FOSLS) methodology developed in [5] to a compressible stationary Stokes system. We consider two basic first-order systems. The first one is induced by a *vorticity* variable, and the second one is induced by a velocity *flux* variable which is further extended to the system's associated curl and trace equations. This extended system is not a system of first order but a mixture system of first- and second-order equations due to the continuity equation $\nabla\cdot\mathbf{u}+\boldsymbol{\beta}\cdot\nabla p = g$. In order to provide ellipticity for each functional, we assume the H^1 and H^2 regularity assumptions for the compressible Stokes equations. As usual in FOSLS approach, we first show that the H^{-1} and L^2 FOSLS functional is elliptic in the product norm $\|w\|+\|\mathbf{u}\|_1+\|q\|+\|p\|_{0,\boldsymbol{\beta}}$ for the functional involving vorticity variable and $\|\mathbf{U}\|+\|\mathbf{u}\|_1+\|p\|$ for the functional involving flux variable. We also show that the extended functional related to *velocity flux* variable is elliptic in the product norm $\|\mathbf{U}\|_1+\|\mathbf{u}\|_1+\|p\|_{1,\boldsymbol{\beta}}$. Then we provide the error estimates for using finite element methods. The outline of the paper is as follows. In Section 2, we discuss least-squares system and other preliminaries. The continuity and ellipticity of least-squares functionals are discussed in Section 3. These can be done by employing regularity estimates for (1.1). The finite element approximations are briefly discussed in Section 4.

2. Least-squares system for compressible Stokes equations, and other preliminaries. For the development of least-squares theory, we will adopt the notation introduced in [5] and introduce the necessary definitions in this section. A new independent variable related to the 4-vector function of gradients of the displacement vectors, u_i, $i = 1,2$ will be given. It will be convenient to view the original n-vector functions as column vectors and the new 4-vector functions as either block column vectors or matrices. The velocity variable $\mathbf{u} = (u_1,u_2)^t$ is a column vector with scalar components u_i, so that the gradient $\nabla\mathbf{u}^t$ is a matrix with columns ∇u_i. For a function $\mathbf{U}$ with 2-vector components $\mathbf{U}_i$

$$\mathbf{U} = \nabla\mathbf{u}^t = (\mathbf{U}_1,\mathbf{U}_2) = (U_{ij})_{2\times 2}, \tag{2.1}$$

which is a matrix with entries $U_{ij} = \partial u_j/\partial x_i$, $1\le i,j\le 2$. Then we can define the *trace* operator tr as

$$\operatorname{tr}\mathbf{U} = \sum_{i=1}^{n} U_{ii}. \tag{2.2}$$

Let, for $\mathbf{v}\in L^2(\Omega)^2$,

$$\begin{gathered}\nabla\times\mathbf{v} := \operatorname{curl}\mathbf{v} = \frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}, \qquad \nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y},\\ \nabla^{\perp}\mathbf{v}^t = (\nabla^{\perp}v_1,\nabla^{\perp}v_2) = \begin{pmatrix}\partial_y v_1 & \partial_y v_2\\ -\partial_x v_1 & -\partial_x v_2\end{pmatrix},\\ \mathbf{n}\times\mathbf{v} = -n_2v_1+n_1v_2.\end{gathered} \tag{2.3}$$

Define the curl as

$$\nabla\times\mathbf{U} = (\nabla\times\mathbf{U}_1,\nabla\times\mathbf{U}_2), \tag{2.4}$$

and the divergence as

$$(\nabla\cdot\mathbf{U})^t = (\nabla\cdot\mathbf{U}_1,\nabla\cdot\mathbf{U}_2)^t. \tag{2.5}$$

We also define the tangential operator $\mathbf{n}\times$ componentwise

$$\mathbf{n}\times\mathbf{U} = (\mathbf{n}\times\mathbf{U}_1,\mathbf{n}\times\mathbf{U}_2). \tag{2.6}$$

The inner products and norms on the block column vector functions are defined in the natural componentwise way; for example,

$$\|\mathbf{U}\|^2 = \sum_{i=1}^{2}\|\mathbf{U}_i\|^2 = \sum_{i,j=1}^{2}\|U_{ij}\|^2. \tag{2.7}$$

We use standard notations and definitions for the Sobolev spaces $H^s(\Omega)^n$, associated inner products $(\cdot,\cdot)_s$, and respective norms $\|\cdot\|_s$, $s\ge 0$. When $s=0$, $H^0(\Omega)^n$ is the usual $L^2(\Omega)^n$, in which case the norm and inner product will be denoted by $\|\cdot\|_0=\|\cdot\|$ and $(\cdot,\cdot)$, respectively. The space $H_0^s(\Omega)$ is the set of functions in $H^s(\Omega)$ vanishing on the boundaries. From now on, we will omit the superscript n and Ω if the dependence of vector norms on dimension is clear by context. We use $H_0^{-1}(\Omega)$ to denote the dual spaces of $H_0^1(\Omega)$ with norm defined by

$$\|\phi\|_{-1} = \sup_{\psi\in H_0^1(\Omega)}\frac{(\phi,\psi)}{\|\psi\|_1}. \tag{2.8}$$

Define the product spaces $H_0^s(\Omega)^2$ and $L^2(\Omega)^2$ in usual way with standard product norms. Let

$$H(\mathrm{div};\Omega) = \{\mathbf{v}\in L^2(\Omega)^2 : \nabla\cdot\mathbf{v}\in L^2(\Omega)\}. \tag{2.9}$$

Define a space

$$Q_k(\Omega) = \{q\in L^2(\Omega) : (\|q\|_k^2+\|\boldsymbol{\beta}\cdot\nabla q\|_k^2)^{1/2}<\infty\}, \tag{2.10}$$

where k is either 1 or 0, which is a Hilbert space with norm

$$\|q\|_{k,\boldsymbol{\beta}} = (\|q\|_k^2+\|\boldsymbol{\beta}\cdot\nabla q\|_k^2)^{1/2}. \tag{2.11}$$

We frequently use the notation constant C_Ω to denote that it depends on Ω only, but it may be a different constant. If a constant depends on another variable, we specify it in each place. Throughout this paper, we assume the following regularity.

Assumption 1. Assume that μ and $\boldsymbol{\beta}$ are such that (1.1) has a unique solution which satisfies the following a priori estimate:

$$\|\nabla\mathbf{u}^t\|_k^2+\|p\|_k^2 \le C_0(\mu,\Omega)(\|-\mu\Delta\mathbf{u}+\nabla p\|_{k-1}^2+\|\nabla\cdot\mathbf{u}+\boldsymbol{\beta}\cdot\nabla p\|_k^2), \tag{2.12}$$

where k is either 0 or 1; $C_0 := C_0(\mu,\Omega)$ is a constant depending on μ, $\boldsymbol{\beta}$, and Ω. Note that one may find (2.12) for $k=1$ in [10, Theorem 1.3] for $\boldsymbol{\beta}=(1,0)^t$ and one may get (2.12) for $k = 0$ by following the arguments in [10, Section 3]. In fact, using triangle inequality and the assumption (2.12), one may get the improved a priori estimates:

$$||\nabla \mathbf{u}^t||_k^2 + \|p\|_{k,\boldsymbol{\beta}}^2 \le C_0(\mu,\Omega)\left(\|-\mu\Delta\mathbf{u} + \nabla p\|_{k-1}^2 + \|\nabla\cdot\mathbf{u} + \boldsymbol{\beta}\cdot\nabla p\|_k^2\right), \tag{2.13}$$

where k is 1 or 0 and $C_0 := C_0(\mu,\Omega)$ is a constant depending on μ, $\boldsymbol{\beta}$, and Ω.

2.1. Velocity-vorticity-pressure formulation. Note that

$$\nabla^\perp(\nabla\times\mathbf{u}) = -\Delta\mathbf{u} + \nabla(\nabla\cdot\mathbf{u}). \tag{2.14}$$

As in [4] for Stokes equations, introducing the vorticity variable $w = \nabla\times\mathbf{u}$, the first equation of the compressible Stokes equations (1.1) using the second equation of (1.1) is

$$\mu\nabla^\perp w - \mu\nabla\cdot q + \nabla p = \mathbf{f}. \tag{2.15}$$

By setting $q = \nabla\cdot\mathbf{u}$, the equivalent first-order system is now

$$\begin{aligned} w - \nabla\times\mathbf{u} &= 0 \quad \text{in } \Omega,\\ q - \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega,\\ \mu\nabla^\perp w - \mu\nabla q + \nabla p &= \mathbf{f} \quad \text{in } \Omega,\\ q + \boldsymbol{\beta}\cdot\nabla p &= g \quad \text{in } \Omega,\\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega,\\ p &= 0 \quad \text{on } \Gamma_{\text{in}}. \end{aligned} \tag{2.16}$$

2.2. Velocity-flux-pressure formulation. As in [5] for Stokes equations, introducing the velocity flux variable $\mathbf{U} = \nabla\mathbf{u}^t$, the compressible Stokes equations (1.1) may be written as the following equivalent first-order system:

$$\begin{aligned} \mathbf{U} - \nabla\mathbf{u}^t &= \mathbf{0} \quad \text{in } \Omega,\\ -\mu(\nabla\cdot\mathbf{U})^t + \nabla p &= \mathbf{f} \quad \text{in } \Omega,\\ \nabla\cdot\mathbf{u} + \boldsymbol{\beta}\cdot\nabla p &= g \quad \text{in } \Omega,\\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega,\\ p &= 0 \quad \text{on } \Gamma_{\text{in}}. \end{aligned} \tag{2.17}$$

We consider the following extended equivalent system for (2.17):

$$\begin{aligned} \mathbf{U} - \nabla\mathbf{u}^t &= \mathbf{0} \quad \text{in } \Omega,\\ -\mu(\nabla\cdot\mathbf{U})^t + \nabla p &= \mathbf{f} \quad \text{in } \Omega,\\ \nabla\cdot\mathbf{u} + \boldsymbol{\beta}\nabla p &= g \quad \text{in } \Omega,\\ \nabla\times\mathbf{U} &= \mathbf{0} \quad \text{in } \Omega,\\ \nabla(\operatorname{tr}\mathbf{U}) + \nabla(\boldsymbol{\beta}\cdot\nabla p) &= \nabla g \quad \text{in } \Omega,\\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega,\\ \mathbf{n}\times\mathbf{U} &= \mathbf{0} \quad \text{on } \partial\Omega,\\ p &= 0 \quad \text{on } \Gamma_{\text{in}}. \end{aligned} \tag{2.18}$$

3. Least-squares functionals.

The main objective in this section is to establish ellipticity and continuity of least-squares functionals based on (2.16), (2.17), and (2.18) in appropriate Sobolev spaces.

3.1. Velocity, vorticity, and pressure.

The first-order least-squares functional corresponding to (2.16) is

$$G_0(w,\mathbf{u},q,p;\mathbf{f},g) = \|\mu\nabla^{\perp}w-\mu\nabla q+\nabla p-\mathbf{f}\|_{-1,0}^2 + \|q+\boldsymbol{\beta}\cdot\nabla p-g\|^2+\|w-\nabla\times\mathbf{u}\|^2+\|q-\nabla\cdot\mathbf{u}\|^2. \tag{3.1}$$

Define

$$M_0(w,\mathbf{u},q,p) = \|w\|^2+\|\mathbf{u}\|_1^2+\|q\|^2+\|p\|_{0,\boldsymbol{\beta}}^2, \tag{3.2}$$

and let

$$\mathcal{V}_0 = L^2(\Omega)\times H_0^1(\Omega)^2\times L^2(\Omega)\times Q_0(\Omega). \tag{3.3}$$

The FOSLS variational problem for the compressible Stokes equations corresponding to (2.16) is to minimize the quadratic functional G_0 over $\mathcal{V}_0$: find $(w,\mathbf{u},q,p)\in\mathcal{V}_0$ such that

$$G_0(w,\mathbf{u},q,p;\mathbf{f},g) = \inf_{(z,\mathbf{v},r,s)\in\mathcal{V}_0} G_0(z,\mathbf{v},r,s;\mathbf{f},g). \tag{3.4}$$

Theorem 3.1. *Under the assumption (2.12), there are two positive constants c and C, dependent on δ and Ω, such that for all $(w,\mathbf{u},q,p)\in\mathcal{V}_0$,*

$$cM_0(w,\mathbf{u},q,p)\le G_0(w,\mathbf{u},q,p;\mathbf{0},0)\le CM_0(w,\mathbf{u},q,p). \tag{3.5}$$

Proof. Upper bound in (3.5) is a simple consequence of the triangle inequality and Cauchy-Schwarz inequality. For any $(w,\mathbf{u},q,p)\in\mathcal{V}_0$, using (2.13), triangle inequality, and $(\cdot)$, we have

$$\begin{aligned}\|\nabla\mathbf{u}^t\|^2+\|p\|_{0,\boldsymbol{\beta}}^2 &\le C_0\big(\|-\mu\Delta\mathbf{u}+\nabla p\|_{-1,0}^2+\|\nabla\cdot\mathbf{u}+\boldsymbol{\beta}\cdot\nabla p\|^2\big)\\ &\le C_0\big(\|\mu\nabla^{\perp}w-\mu\nabla q+\nabla p\|_{-1,0}^2+\mu^2\|\nabla^{\perp}(w-\nabla\times u)\|_{-1,0}^2\\ &\qquad+\mu^2\|\nabla(\nabla\cdot\mathbf{u}-q)\|_{-1,0}^2+\|q+\boldsymbol{\beta}\cdot\nabla p\|^2\big)\\ &\le \hat{C}_0G_1(\mathbf{U},\mathbf{u},p;\mathbf{0},0),\end{aligned} \tag{3.6}$$

where $\hat{C}_0$ is a constant that depends on μ, $\boldsymbol{\beta}$, and Ω. Using (3.6), we have

$$(w,w) = (w-\nabla\times\mathbf{u},w)+(\nabla\times\mathbf{u},w)\le CG_0^{1/2}(w,\mathbf{u},q,p)\|w\|, \tag{3.7}$$

where C is a constant depending on Ω and the Poincare constant. Now, cancelling $\|w\|$ on both sides and squaring the remainder, we have

$$\|w\|^2\le CG_0(w,\mathbf{u},q,p;\mathbf{0},0), \tag{3.8}$$

where C is a constant depending on Ω and the Poincare constant. Now, using (3.6), we have

$$\begin{aligned}(q,q) &= (q-\nabla\cdot\mathbf{u},q)+(\nabla\cdot\mathbf{u},q)\\ &\le \|q-\nabla\cdot\mathbf{u}\|\,\|q\|+\|\nabla\cdot\mathbf{u}\|\,\|q\|\\ &\le CG_0^{1/2}(w,\mathbf{u},q,p)\|q\|,\end{aligned} \tag{3.9}$$

where C is a constant depending on Ω. Cancelling $\|q\|$ on both sides and squaring the remainder, we have

$$\|q\| \le CG_0(w,\mathbf{u},q,p). \tag{3.10}$$

Finally, combining (3.6), (3.8), and (3.10) yields the lower bound. This completes the proof. □

3.2. Velocity, flux, and pressure. The first-order least-squares functional corresponding to (2.17) is

$$\begin{aligned}G_1(\mathbf{U},\mathbf{u},p;\mathbf{f},g) &= \left\|-\mu(\nabla\cdot\mathbf{U})^t+\nabla p-\mathbf{f}\right\|_{-1,0}^2\\ &\quad+\|\nabla\cdot\mathbf{u}+\boldsymbol{\beta}\cdot\nabla p-g\|^2+\left\|\mathbf{U}-\nabla\mathbf{u}^t\right\|^2.\end{aligned} \tag{3.11}$$

The extended least-squares functional corresponding to (2.18) is

$$\begin{aligned}G_3(\mathbf{U},\mathbf{u},q,p;\mathbf{f},g) &= \left\|\mathbf{U}-\nabla\mathbf{u}^t\right\|^2+\left\|-\mu(\nabla\cdot\mathbf{U})^t+\nabla p-\mathbf{f}\right\|^2+\|\nabla\times\mathbf{U}\|^2\\ &\quad+\|\nabla\cdot\mathbf{u}+\boldsymbol{\beta}\cdot\nabla-g\|^2+\left\|\nabla\operatorname{tr}\mathbf{U}+\nabla(\boldsymbol{\beta}\cdot\nabla p)\right\|^2.\end{aligned} \tag{3.12}$$

Define

$$\begin{aligned}M_1(\mathbf{U},\mathbf{u},p) &= \|\mathbf{U}\|^2+\|\mathbf{u}\|_1^2+\|p\|_{0,\boldsymbol{\beta}}^2,\\ M_2(\mathbf{U},\mathbf{u},q,p) &= \|\mathbf{U}\|_1^2+\|\mathbf{u}\|_1^2+\|p\|_{1,\boldsymbol{\beta}}^2.\end{aligned} \tag{3.13}$$

Let

$$\mathbf{V}_0 = \{\mathbf{U}\in H^1(\Omega)^4 : \mathbf{n}\times\mathbf{U}=\mathbf{0}\text{ on }\partial\Omega\}. \tag{3.14}$$

Define

$$\begin{aligned}\mathcal{V}_1 &= L^2(\Omega)^4\times H_0^1(\Omega)^2\times Q_0(\Omega),\\ \mathcal{V}_2 &= \mathbf{V}_0\times H_0^1(\Omega)^2\times Q_1(\Omega).\end{aligned} \tag{3.15}$$

The least-squares variational problem for the compressible Stokes equations corresponding to (2.17) or (2.18) is to minimize the quadratic functional G_i over $\mathcal{V}_i$: find $(\mathbf{U},\mathbf{u},p)\in\mathcal{V}_i$ such that

$$G_i(\mathbf{U},\mathbf{u},p;\mathbf{f},g) = \inf_{(\mathbf{V},\mathbf{v},r)\in\mathcal{V}_i} G_i(\mathbf{V},\mathbf{v},r;\mathbf{f},g)\quad\text{for } i=1,2. \tag{3.16}$$

Theorem 3.2. *Under the assumption (2.12), there are two positive constants c and C, dependent on μ, $\boldsymbol{\beta}$, and Ω, such that for all $(\mathbf{U},\mathbf{u},p) \in \mathcal{V}_1$,*

$$cM_1(\mathbf{U},\mathbf{u},p) \le G_1(\mathbf{U},\mathbf{u},p;\mathbf{0},0) \le CM_1(\mathbf{U},\mathbf{u},p). \tag{3.17}$$

Proof. Upper bound in (3.17) is a simple consequence of the triangle inequality and Cauchy-Schwarz inequality. To limit arguments, it is enough to show that lower bound in (3.17) holds for $\tilde{\mathcal{V}} = H(\text{div};\Omega)^2 \times H_0^1(\Omega)^2 \times Q(\Omega)$. Using (2.12) and triangle inequality, we have

$$\begin{aligned}
&\|\nabla \mathbf{u}^t\|^2 + \|p\|_{0,\boldsymbol{\beta}}^2 \\
&\quad \le C_0\left(\|-\mu\Delta\mathbf{u} + \nabla p\|_{-1,0}^2 + \|\nabla\cdot\mathbf{u} + \boldsymbol{\beta}\cdot\nabla p\|^2\right) \\
&\quad \le C_0\left(\|-\mu(\nabla\cdot\mathbf{U})^t + \nabla p\|_{-1,0}^2 + \mu^2\|\nabla\cdot(\mathbf{U}-\nabla\mathbf{u}^t)^t\|_{-1,0}^2 + \|\nabla\cdot\mathbf{u} + \boldsymbol{\beta}\cdot\nabla p\|^2\right) \\
&\quad \le \hat{C}_0 G_1(\mathbf{U},\mathbf{u},p;\mathbf{0},0),
\end{aligned} \tag{3.18}$$

where $\hat{C}_0$ is a constant that depends on μ and Ω. Note that

$$(\mathbf{U},\mathbf{U}) = (\mathbf{U}-\nabla\mathbf{u}^t,\mathbf{U}) + (\nabla\mathbf{u}^t,\mathbf{U}) \le C\left(\|\mathbf{U}-\nabla\mathbf{u}^t\|\,\|\mathbf{U}\| + \|\mathbf{u}\|_1\|\mathbf{U}\|\right), \tag{3.19}$$

where C is a constant depending on Ω. Now cancelling $\|\mathbf{U}\|$ on both sides, squaring the remainder, and using (3.19), we have

$$\|\mathbf{U}\|^2 \le CG_1(\mathbf{U},\mathbf{u},p;\mathbf{0},0), \tag{3.20}$$

where C is a constant depending on μ, $\boldsymbol{\beta}$, and Ω. Finally, combining (3.19) and (3.20) yields the lower bound. This completes the proof. □

The following lemma is basically proved in [5, Lemma 3.2].

Lemma 3.3. *Let $\boldsymbol{\phi} = (\phi_1,\phi_2)^t$ and $\mathbf{q} = (q_1,q_2)^t$; if each $q_i \in H_0^1(\Omega)\cap H^2(\Omega)$ and each $\phi_i \in H^1(\Omega)$ is such that $\Delta\phi_i \in L^2(\Omega)$ and $\mathbf{n}\cdot\nabla\phi_i = 0$ on $\partial\Omega$, then*

$$|\nabla\cdot\mathbf{q} + \boldsymbol{\beta}\cdot\nabla p|_1^2 \le C_\Omega\left(|\nabla\cdot\mathbf{q} + \text{tr}\,\nabla^\perp\boldsymbol{\phi}^t + \boldsymbol{\beta}\cdot\nabla p|_1^2 + \|\Delta\boldsymbol{\phi}\|^2\right). \tag{3.21}$$

Proof. Note that $\text{tr}(\nabla^\perp\phi_1,\nabla^\perp\phi_2) = -\nabla\times\boldsymbol{\phi}$,

$$\begin{aligned}
|\nabla\cdot\mathbf{q} + \boldsymbol{\beta}\cdot\nabla p|_1^2 &\le 2\left(|\nabla\cdot\mathbf{q} - \nabla\times\boldsymbol{\phi} + \boldsymbol{\beta}\cdot\nabla p|_1^2 + |\nabla\times\boldsymbol{\phi}|_1^2\right) \\
&\le C\left(|\nabla\cdot\mathbf{q} + \text{tr}\,\nabla^\perp\boldsymbol{\phi}^t + \boldsymbol{\beta}\cdot\nabla p|_1^2 + |\boldsymbol{\phi}|_2\right) \\
&\le C\left(|\nabla\cdot\mathbf{q} + \text{tr}\,\nabla^\perp\boldsymbol{\phi}^t + \boldsymbol{\beta}\cdot\nabla p|_1^2 + \|\Delta\boldsymbol{\phi}\|\right),
\end{aligned} \tag{3.22}$$

where the constant C depends on Ω. □

Due to the above lemma, one may get the following theorem.

Theorem 3.4. *Under the assumption of (2.12), there are two positive constants c and C dependent on μ, $\boldsymbol{\beta}$, and Ω such that for all $(\mathbf{U},\mathbf{u},p) \in \mathcal{V}_2$,*

$$cM_2(\mathbf{U},\mathbf{u},q,p) \le G_2(\mathbf{U},\mathbf{u},q,p;\mathbf{0},0) \le CM_2(\mathbf{U},\mathbf{u},q,p). \tag{3.23}$$

The proof of Theorem 3.4 comes immediately by following techniques similar to those in [5].

4. Finite element approximations. In this section, we provide the finite element approximation of the minimization of the least-squares functionals G_0 only. Note that an obvious modification in this section also provides the finite element error analysis for the least-squares functionals G_1 and G_2. Let $T : H_0^{-1}(\Omega)^2 \to H_0^1(\Omega)^2$ be the solution operator ($\mathbf{u} = T\mathbf{f}$) for the following elliptic boundary value problem with zero boundary condition $-\Delta\mathbf{u}+\mathbf{u} = \mathbf{f}$ in Ω. It is well known that (see [3, Lemma 2.1])

$$(\mathbf{f},T\mathbf{f}) = \|\mathbf{f}\|_{-1}^2 = \sup_{\boldsymbol{\phi}\in H_0^1(\Omega)^2} \frac{(\mathbf{f},\boldsymbol{\phi})^2}{\|\boldsymbol{\phi}\|_1^2} \quad \forall \mathbf{f} \in H_0^{-1}(\Omega)^2. \tag{4.1}$$

Let $\mathcal{T}_h$ be a family of triangulations of Ω by standard finite element subdivisions of Ω into quasi-uniform triangles with $h = \max\{\operatorname{diam}(K) : K \in \mathcal{T}_h\}$.

Let $\mathcal{V}_{0,h}$ be a finite-dimensional subspace of $\mathcal{V}_0$ with an approximation property such that for $(w,\mathbf{u},q,p) \in \mathcal{V}_0$, there exists positive integers $l,m,n \ge 1$ and $s \ge 1$ satisfying

$$\begin{aligned}
\inf_{w_h\in\mathcal{U}_h} \{\|w-w_h\| + h\|w-w_h\|_1\} &\le Ch^r\|w\|_r,\\
\inf_{\mathbf{u}_h\in\mathcal{S}_{0,h}} \{\|\mathbf{u}-\mathbf{u}_h\| + h\|\mathbf{u}-\mathbf{u}_h\|_1\} &\le Ch^{s+1}\|\mathbf{u}\|_{s+1},\\
\inf_{q_h\in\mathcal{U}_h} \{\|q-q_h\| + h\|q-q_h\|_1\} &\le Ch^r\|q\|_r,\\
\inf_{p_h\in\mathcal{W}_h} \{\|p-p_h\| + h\|p-p_h\|_1\} &\le Ch^{k+1}\|p\|_{k+1},
\end{aligned} \tag{4.2}$$

where C is a positive integer. Then the finite element approximation of (3.4) is to find $(w_h,\mathbf{u}_h,q_h,p_h) \in \mathcal{V}_{0,h}$ which satisfies

$$G_0(w_h,\mathbf{u}_h,q_h,p_h;\mathbf{f},g) = \inf_{(z_h,\mathbf{v}_h,r_h,s_h)\in\mathcal{V}_{0,h}} G_0(z_h,\mathbf{v}_h,r_h,s_h;\mathbf{f},g). \tag{4.3}$$

From (4.1), we have

$$\begin{aligned}
G_0(w,\mathbf{u},q,p;\mathbf{0},0) = {} & (T(\mu\nabla^\perp w - \mu\nabla q + \nabla p), \mu\nabla^\perp w - \mu\nabla q + \nabla p)\\
& + (q + \boldsymbol{\beta}\cdot\nabla p, q + \boldsymbol{\beta}\cdot\nabla p) + (q - \nabla\cdot\mathbf{u}, q - \nabla\cdot\mathbf{u})\\
& + (w - \nabla\times\mathbf{u}, w - \nabla\times\mathbf{u}).
\end{aligned} \tag{4.4}$$

Theorem 4.1. *Suppose that the assumption in Theorem 3.1 holds. Assume that* $(w,\mathbf{u},q,p)\in\mathcal{V}_0$ *is the solution of the minimization problem for* G_1 *in (3.4) and* $(w_h,\mathbf{u}_h,q_h,p_h)$ *is the unique minimizer of* G_0 *over* $\mathcal{V}_{0,h}$. *Then*

$$\begin{aligned}&||w-w_h||^2+||\mathbf{u}-\mathbf{u}_h||_1^2+|q-q_h|^2+||p-p_h||_{0,\boldsymbol{\beta}}^2\\&\quad\le C\inf_{(z_h,\mathbf{v}_h,r_h,s_h)\in\mathcal{V}_{0,h}}\left(||w-z_h||^2+||\mathbf{u}-\mathbf{v}_h||_1^2+|q-r_h|^2+||p-s_h||_{0,\boldsymbol{\beta}}^2\right).\end{aligned}\tag{4.5}$$

Proof. For convenience, let

$$\begin{aligned}[w,\mathbf{u},q,p;z,\mathbf{v},r,s]=&\left(T(\mu\nabla^\perp w-\mu\nabla q+\nabla p),\mu\nabla^\perp z-\mu\nabla r+\nabla s\right)\\&+(q+\boldsymbol{\beta}\cdot\nabla p,r+\boldsymbol{\beta}\cdot\nabla s)+(w-\nabla\times\mathbf{u},z-\nabla\times\mathbf{v})\\&+(q-\nabla\cdot\mathbf{u},r-\nabla\cdot\mathbf{s}).\end{aligned}\tag{4.6}$$

Then, using (4.1), Theorem 3.1, the orthogonality of the error $(w-w_h,\mathbf{u}-\mathbf{u}_h,q-q_h,p-p_h)$ to $\mathcal{V}_{0,h}$, with respect to the above inner product, and the Schwarz inequality, we have the conclusion. □

From this theorem and approximate property of $\mathcal{V}_{0,h}$, we have

$$\begin{aligned}&||w-w_h||^2+||\mathbf{u}-\mathbf{u}_h||_1^2+||q-q_h||^2+||p-p_h||_{0,\boldsymbol{\beta}}^2\\&\quad\le C\left(h^{2l}\|w\|_l^2+h^{2m}\|\mathbf{u}\|_{m+1}^2+h^{2n}\|q\|_n^2+h^{2s}\|p\|_{s+1}^2\right),\end{aligned}\tag{4.7}$$

where $(w,\mathbf{u},q,p)\in(H^l(\Omega)\times H_0^m(\Omega)^2\times H^{n+1}(\Omega)^2\times H^s(\Omega))\cap\mathcal{V}_0$ is the solution of the minimization problem for G_0 in (3.4) and $(w_h,\mathbf{u}_h,q_h,p_h)$ is the unique minimizer of G_0 over $\mathcal{V}_{0,h}$.

Acknowledgment. This work was supported by the Korea Research Foundation Grant KRF-2002-070-C00014.

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